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OSCILLATORY AND NON OSCILLATORY CRITERIA FOR THE SYSTEMS OF TWO LINEAR FIRST ORDER TWO BY TWO DIMENSIONAL MATRIX ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The Riccati equation method is used for study the oscillatory and non oscillatory behavior of solutions of systems of two first order linear two by two dimensional matrix differential equations. An integral and an interval oscillatory criteria are obtained. Two non oscillatory criteria are obtained as well. On an example, one of the obtained oscillatory criteria is compared with some well known results.

1. Introduction

Let $P(t) \equiv \big(p_{jk}(t)\big)_{j,k=1}^2$, $Q(t) \equiv \operatorname{diag}\{q_1(t), q_2(t)\}$, $R(t) \equiv \big(r_{jk}(t)\big)_{j,k=1}^2$, $S(t) \equiv \big(s_{jk}(t)\big)_{j,k=1}^2$ be real valued continuous matrix functions on $[t_0;+\infty)$. Consider the matrix linear system

$$\begin{cases} \Phi' = P(t)\Phi + Q(t)\Psi\,; \\ \Psi' = R(t)\Phi + S(t)\Psi\,, \quad t \ge t_0\,. \end{cases} \tag{1.1}$$

Here $\Phi = \Phi(t) \equiv \big(\phi_{jk}(t)\big)_{j,k=1}^2$, $\Psi = \Psi(t) \equiv \big(\psi_{jk}(t)\big)_{j,k=1}^2$ are unknown continuously differentiable matrix functions on $[t_0;+\infty)$.

Remark 1.1. The general case $Q(t) \equiv S(t)\operatorname{diag}\{q_1(t), q_2(t)\}S^{-1}(t)$, $t \ge t_0$, where $S(t)$ is an invertible continuously differentiable on $[t_0;+\infty)$ matrix function, can be reduced to the case $Q(t) \equiv \operatorname{diag}\{q_1(t), q_2(t)\}$, $t \ge t_0$, of the system (1.1) by the linear transformation

$$\Phi = S(t)\Phi_1\,, \qquad \Psi = S(t)\Psi_1\,, \qquad t \ge t_0\,.$$

in (1.1).

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Definition 1.1. A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called oscillatory if $\det \Phi(t)$ has arbitrary large zeroes, otherwise it is called non oscillatory.

Definition 1.2. A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called oscillatory on the interval $[t_1; t_2]$, $(t_0 \le t_1 < t_2 < +\infty)$ if $\det \Phi(t)$ has at least one zero on $[t_1; t_2]$.

Definition 1.3. A solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called prepared (or preferred) if $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \ge t_0$, where $*$ is the transpose sign.

Definition 1.4. The system (1.1) is called oscillatory, if its all prepared solutions are oscillatory.

Definition 1.5. The system (1.1) is called oscillatory on the interval $[t_1; t_2]$, $(t_0 \le t_1 < t_2 < +\infty)$ if its all prepared solutions are oscillatory on the interval $[t_1; t_2]$.

Study of questions of oscillation and non oscillation of solutions of linear systems of matrix equations, in particular of the system (1.1), is an important problem of qualitative theory of differential equations and many works are devoted to them (see for example [1], [3], [10]–[14]). In most of cases in the works [1], [3], [10]–[14] and others on the matrix coefficients of the system are imposed conditions ensuring some symmetry property of corresponding matrix Riccati equation (the hamiltonian systems), namely if $Y(t)$ is a solution to corresponding Riccati equation then the transposed matrix function $Y^*(t)$ is a solution of the last one as well. In this work we study the conditions on the coefficients of the system (1.1), for which the last one has oscillatory and non oscillatory solutions. We impose conditions on the coefficients of the system (1.1) for which the hamiltonian structure of it can not be kept.

2. Auxiliary propositions

In this paragraph we prove two lemmas and represent a lemma and a theorem, proved in other works. They will be used in the next paragraph for proving oscillatory and non oscillatory criteria for the system (1.1).

In what follows the solutions of equations and systems of equations we will assume real valued. In the system (1.1) make a change

$$\Psi = Y(t)\Phi\,, \qquad t \ge t_0\,, \tag{2.1}$$

where $Y(t)$ is a continuously differentiable matrix function of dimension 2×2 on $[t_0; +\infty)$. We will get:

$$\begin{cases} \Phi' = [P(t) + Q(t)Y(t)]\Phi\,; \\ [Y'(t) + Y(t)Q(t)Y(t) + Y(t)P(t) - S(t)Y(t) - R(t)]\Phi = 0\,, \quad t \ge t_0\,. \end{cases} \tag{2.2}$$

Consider the matrix Riccati equation

$$Y' + YQ(t)Y + YP(t) - S(t)Y - R(t) = 0\,, \qquad t \ge t_0\,, \tag{2.3}$$

where $Y = \big(y_{jk}(t)\big)_{j,k=1}^{2}$. From (2.2) is seen that if $Y_1(t)$ is a solution of Eq. (2.3) on $[t_1;t_2)(t_0 \le t_1 < t_2 \le +\infty)$, then $(\Phi_1(t), Y_1(t)\Phi_1(t))$ is a solution to the system (1.1) on $[t_1;t_2)$, where $\Phi_1(t)$ is any solution to matrix equation

$$\Phi' = [P(t) + Q(t)Y_1(t)]\Phi\,, \qquad t \in [t_1;t_2)\,. \tag{2.4}$$

Obviously on the strength of (2.1) and (2.2) if $(\Phi(t), \Psi(t))$ is a solution of the system (1.1) and $\det\Phi(t) \ne 0$, $t \in [t_1;t_2)$, then $Y(t) \equiv \Psi(t)\Phi^{-1}(t)$ is a solution to Eq. (2.3) on $[t_1;t_2)$. Let $Y_0(t)$ be a solution to Eq. (2.3) on $[t_1;t_2)$.

Definition 2.1. We will say that $[t_1;t_2)$ is a maximum existence interval for $Y_0(t)$, if $Y_0(t)$ cannot be continued to the right of t_2 as a solution of Eq. (2.3).

Lemma 2.1. *Let $Y_0(t)$ be a solution of Eq.* (2.3) *on $[t_1;t_2)$, and let $t_2 < +\infty$. Then $[t_1;t_2)$ cannot be the maximum existence interval for $Y_0(t)$ provided the function $f(t) \equiv \int\limits_{t_1}^{t} \mathrm{tr}[Q(\tau)Y_0(\tau)]d\tau$, $t \in [t_1;t_2)$, is bounded from below on $[t_1;t_2)$.*

Proof. Let $\Phi_0(t)$ be a solution to the equation

$$\Phi' = [P(t) + Q(t)Y_0(t)]\Phi\,, \qquad t \ge t_0\,, \tag{2.5}$$

with $\Phi_0(t_1) \ne 0$. Then by Liouville formula

$$\det\Phi_0(t) = \det\Phi_0(t_1)\exp\Big\{\int\limits_{t_1}^{t} \mathrm{tr}\big[P(\tau) + Q(\tau)Y_0(\tau)\big]d\tau\Big\} \ne 0\,, \tag{2.6}$$

$t \in [t_1;t_2)$. Recall that for any solution $\Phi_0(t)$ of the linear matrix equation

$$\Phi' = A(t)\Phi\,, \qquad t \ge t_0\,,$$

where $A(t)$ is a square continuous matrix function, the Liuville's theorem states that (the Liuville's formula)

$$\det\Phi_0(t) = \det\Phi_0(t_0)\exp\Big\{\int\limits_{t_0}^{t} \mathrm{tr}(A(\tau))d\tau\Big\}$$

(see [9, p. 47, Theorem 1.2]). Let $(\widetilde{\Phi}(t), \widetilde{\Psi}(t))$ be the solution of the system (1.1) with $\widetilde{\Phi}(t_1) = \Phi_0(t_1)$, $\widetilde{\Psi}(t_1) = Y_0(t_1)\Phi_0(t_1)$. Then by (2.2)–(2.5) and the uniqueness theorem

$$\widetilde{\Phi}(t) = \Phi_0(t)\,, \qquad \widetilde{\Psi}(t) = Y_0(t)\Phi_0(t)\,, \quad t \in [t_1;t_2)\,. \tag{2.7}$$

From the conditions of the lemma and from (2.6) it follows that $|\det\Psi_0(t)| \ge \varepsilon$, $t \in [t_1;t_2)$, for some $\varepsilon > 0$. Then since $\det\widetilde{\Psi}_0(t)$ is a continuous function from (2.7) it follows that $\det\widetilde{\Phi}(t) \ne 0$, $t \in [t_1;t_3)$, for some $t_3 > t_2$. Therefore $\widetilde{Y}_0(t) \equiv \widetilde{\Psi}(t)\widetilde{\Phi}^{-1}(t)$ is a solution to Eq. (2.3) on $[t_1;t_3)$. By (2.7) we have $\widetilde{Y}_0(t) = Y_0(t)$, $t \in [t_1;t_2)$. Hence $[t_1;t_2)$ is not the maximum existence interval for $Y_0(t)$. The lemma is proved. $\square$

Let $a(t)$, $b(t)$, $c(t)$, $c_1(t)$ be continuously differentiable functions on $[t_0;+\infty)$. Consider the Riccati equations

$$y' + a(t)y^2 + b(t)y + c(t) = 0\,, \qquad t \geq t_0\,; \tag{2.8}$$

$$y' + a(t)y^2 + b(t)y + c_1(t) = 0\,, \qquad t \geq t_0\,; \tag{2.9}$$

Theorem 2.1. *Let Eq. (2.9) has the solution $y_1(t)$ on $[t_1;t_2)$ $(t_0 \leq t_1 < t_2 \leq +\infty)$, and let $a(t) \geq 0$, $c(t) \leq c_1(t)$, $t \in [t_1;t_2)$. Then for each $y_{(0)} \geq y_1(t_0)$ Eq.(2.8) has the solution $y_0(t)$ on $[t_1;t_2)$ with $y_0(t_0) = y_{(0)}$, and $y_0(t) \geq y_1(t)$, $t \in [t_1;t_2)$.*

A proof for a more general theorem is presented in [4] (see also [5]).

Let us write Eq. (2.3) in the expanded form. We have:

$$\begin{cases} y'_{11} + q_1(t)y_{11}^2 + a_{11}(t)y_{11} + q_2(t)y_{12}y_{21} + p_{21}(t)y_{12} - s_{12}(t)y_{21} - r_{11}(t) = 0\,; \\ y'_{22} + q_2(t)y_{22}^2 + a_{22}(t)y_{22} + q_1(t)y_{12}y_{21} + p_{12}(t)y_{21} - s_{21}(t)y_{12} - r_{22}(t) = 0\,; \\ y'_{12} + [q_1(t)y_{11} + q_2(t)y_{22} + a_{21}(t)]y_{12} + p_{12}(t)y_{11} - s_{12}(t)y_{22} - r_{12}(t) = 0\,; \\ y'_{21} + [q_1(t)y_{11} + q_2(t)y_{22} + a_{12}(t)]y_{21} + p_{21}(t)y_{22} - s_{21}(t)y_{11} - r_{21}(t) = 0\,, \end{cases} \tag{2.10}$$

where $a_{jk}(t) \equiv p_{jj}(t) - s_{kk}(t)$, $j,k = 1,2$, $t \geq t_0$. Denote:

$$I_k(\tau;t) \equiv \int\limits_\tau^t \exp\Big\{-\int\limits_s^t a_{kk}(\zeta)d\zeta\Big\} r_{kk}(s)ds\,, \quad t \geq \tau \geq t_0\,,\ k = 1,2\,.$$

Lemma 2.2. *Let the following conditions hold*

(A) *$q_k(t) \geq 0$, $k = 1,2$, $r_{12}(t) \geq 0$ (≤ 0), $r_{21}(t) \leq 0$ (≥ 0), $p_{21}(t) \geq 0$ (≤ 0), $s_{12}(t) \geq 0$ (≤ 0), $t \geq t_0$;*

(B) *there exist infinitely large sequences $\xi_{0,k} = t_0 < \xi_{1,k} < \cdots < \xi_{m,k} < \cdots$ $(k = 1,2)$ such that*

$$\int\limits_{\xi_{m,k}}^t \exp\Big\{\int\limits_{\xi_{m,k}}^\tau \big[a_{kk}(s) + q_k(s)I_k(\xi_{m,k};s)\big]ds\Big\} r_{kk}(\tau)d\tau \geq 0\,, \tag{2.11}$$
$$t \in [\xi_{m,k};\xi_{m+1,k})\,,\ \ m = 0,1,2,\ldots\,, \quad k = 1,2\,.$$

Then for each $y_{kk,0} > 0$, $k = 1,2$, $y_{12,0} \leq 0$ (≥ 0), $y_{21,0} \geq 0$ (≤ 0) Eq. (2.3) has the solution $Y_0(t) \equiv \big(y^0_{jk}(t)\big)_{j,k=1}^2$ on $[t_0;+\infty)$, satisfying the initial conditions $y^0_{jk}(t_0) = y_{jk,0}$, $j,k = 1,2$, and

$$\det Y_0(t) > 0\,, \quad t \geq t_0\,. \tag{2.12}$$

Proof. Show that

$$y^0_{kk}(t) > 0\,, \quad t \in [t_0;T)\,,\ k = 1,2\,, \tag{2.13}$$

where $[t_0;T)$ is the maximum existence interval for $Y_0(t)$. Suppose that it is not so. Then from the initial conditions is seen that

$$y^0_{kk}(t) > 0\,, \quad t \in [t_0;T_1)\,, \tag{2.14}$$

$$y^0_{11}(T_1)y^0_{22}(T_1) = 0\,, \tag{2.15}$$

for some $T_1 \in (t_0;T)$. By virtue of the third and fourth equations of the system (2.10) we have:

$$\begin{aligned} y^0_{12}(t) = \exp\Big\{ -\int\limits_{t_0}^{t} [q_1(\tau)y^0_{11}(\tau) + q_2(\tau)y^0_{22}(\tau) + a_{21}(\tau)]d\tau\Big\} \\ \times \Big[y^0_{12}(t_0) - \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\big[q_1(s)y^0_{11}(s) + q_2(s)y^0_{22}(s) + a_{21}(s)\big]ds\Big\} \\ \times \big(p_{12}(\tau)y^0_{11}(\tau) - s_{12}(\tau)y_{22}(\tau) - r_{12}(\tau)\big)d\tau\Big]\,, t \in [t_0;T); \end{aligned} \tag{2.16}$$

$$\begin{aligned} y^0_{21}(t) = \exp\Big\{ -\int\limits_{t_0}^{t} [q_1(\tau)y^0_{11}(\tau) + q_2(\tau)y^0_{22}(\tau) + a_{12}(\tau)]d\tau\Big\} \\ \times \Big[y^0_{21}(t_0) - \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\big[q_1(s)y^0_{11}(s) + q_2(s)y^0_{22}(s) + a_{12}(s)\big]ds\Big\} \\ \times \big(p_{21}(\tau)y^0_{22}(\tau) - s_{21}(\tau)y_{11}(\tau) - r_{21}(\tau)\big)d\tau\Big]\,, \quad t \in [t_0;T)\,. \end{aligned} \tag{2.17}$$

From here from the conditions of lemma and from (2.14) it follows that

$$y^0_{12}(t) \ge 0 \;\; (\le 0)\,, \qquad y^0_{21}(t) \le 0 \;\; (\ge 0)\,, \qquad t \in [t_0;T_1)\,. \tag{2.18}$$

Consider the Riccati equations

$$y' + q_k(t)y^2 + a_{kk}(t)y - r_{kk}(t) = 0\,, \qquad t \ge t_0\,, \tag{2.19}$$

$$y' + q_k(t)y^2 + a_{kk}(t)y + \mathcal{L}_k(y^0_{k,3-k}(t), y^0_{3-k,k}(t), t) = 0\,, \qquad t \ge t_0\,, \tag{2.20}$$

$k = 1,2$, where $\mathcal{L}_k(u,v,t) \equiv q_{3-k}(t)uv + p_{3-k,k}(t)u - s_{k,3-k}(t)v - r_{kk}(t)$, u, $v \in R$, $t \ge t_0$, $k = 1,2$. From the conditions (A) of lemma and from (2.18) it follows that

$$\mathcal{L}_k\big(y^0_{k,3-k}(t), y^0_{3-k,k}(t), t\big) \le -r_{kk}(t)\,, \qquad t \in [t_0;T_1)\,, \;\; k = 1,2\,. \tag{2.21}$$

Let $y_k(t)$ be the solution of Eq. (2.19) with $y_k(t_0) = y^0_{kk}(t_0) > 0$, $(k = 1,2)$. Then on the strength of Theorem 4.1 of work [5] from the conditions (B) of lemma it follows that $y_k(t)$ exists on $[t_0;T)$ and

$$y_k(t) > 0\,, \qquad t \in [t_0;T)\,, \;\; k = 1,2\,. \tag{2.22}$$

Obviously by (2.10) the function $y^0_{kk}(t)$ is a solution to Eq. on $[t_0;T)$, $(k = 1,2)$. Then by virtue of Theorem 2.1 from (2.21) and (2.22) it follows that $y^0_{kk}(t) \ge y_k(t) > 0$, $t \in [t_0;T_1]$, $k = 1,2$, which contradicts (2.15). The obtained contradiction proves (2.13). Show that $T = +\infty$. From the conditions $q_k(t) \ge 0$, $t \ge t_0$, $k = 1, 2$ (a part of (A)), and from (2.13) it follows that

$$\int_{t_0}^{t} \mathrm{tr}[Q(\tau)Y_0(\tau)]d\tau \ge 0\,, \qquad t \in [t_0; T)\,. \tag{2.23}$$

Suppose $T < +\infty$. Then by Lemma 2.1 from (2.23) it follows that $[t_0; T)$ is not the maximum existence interval for $y_0(t)$. The obtained contradiction shows that $T = +\infty$. From here, from the conditions (A) of lemma, from (2.13), (2.16) and (2.17) it follows (2.12). The lemma is proved. □

Remark 2.1. The conditions (B) of Lemma 2.2 are satisfied if in particular $r_{kk}(t) \ge 0$, $t \ge t_0$, $k = 1, 2$.

Lemma 2.3. *Let Eq.* (2.8) *has a solution on* $[t_1; +\infty)$ *for some* $t_1 \ge t_0$*, and let* $a(t) \ge 0$*,* $c(t) \ge 0$*,* $t \ge t_0$*,* $\int\limits_{t_0}^{+\infty} a(\tau) \exp\bigl\{-\int\limits_{t_0}^{\tau} b(s)ds\bigr\}d\tau = +\infty$*. Then Eq.* (2.8) *has a positive solution on* $[t_1; +\infty)$*.*

The proof is presented in [6].

3. Oscillatory and non oscillatory criteria

Denote:

$$F_k(t) \equiv \begin{cases} r_{kk}(t) - (p_{3-k,k}(t) - s_{k,3-k}(t))^2/(4q_{3-k}(t))\,, & q_{3-k}(t) \ne 0\,; \\ r_{kk}(t)\,, & q_{3-k}(t) = 0\,, \end{cases}$$

$t \ge t_0$, $k = 1, 2$. Let $j (\in \{1, 2\})$ be fixed. Consider the Riccati equation

$$y' + q_j(t)y^2 + a_{jj}(t)y - F_j(t) = 0\,, \qquad t \ge t_0\,. \tag{3.1}$$

The solutions $y(t)$ of this equation existing on some interval $[t_1; t_2)$ ($t_0 \le t_1 < t_2 \le +\infty$), are connected with the solutions $(\phi(t), \psi(t))$ of the system of scalar equations

$$\begin{cases} \phi' = p_{jj}(t)\phi + q_j(t)\psi\,; \\ \psi' = F_j(t)\phi + s_{jj}(t)\psi\,, \quad t \ge t_0\,, \end{cases} \tag{3.2}$$

by relations (see [7])

$$\phi(t) = \phi(t_1)\exp\Bigl\{\int_{t_1}^{t} \bigl[q_j(\tau)y(\tau) + p_{jj}(\tau)\bigr]d\tau\Bigr\}\,, \;\; \phi(t_1) \ne 0\,, \;\; \psi(t) = y(t)\phi(t)\,, \tag{3.3}$$

$t \in [t_1; t_2)$.

Definition 3.1. The system (3.2) is called oscillatory if for its each solution $(\phi(t), \psi(t))$ the function $\phi(t)$ has arbitrary large zeroes.

Definition 3.2. The system (3.2) is called oscillatory on the interval $[t_1; t_2]$ if for its each solution $(\phi(t), \psi(t))$ the function $\phi(t)$ has at least one zero on $[t_1; t_2]$.

Theorem 3.1. *Let the following conditions be satisfied:*
(I) $q_k(t) \ge 0$, $k = 1, 2$, *and if* $q_{3-j}(t) = 0$, *then* $p_{3-j,j}(t) = s_{j,3-j}(t)$, $t \ge t_0$;
(II) *the system* (3.2) *is oscillatory.*

Then the system (1.1) *is oscillatory.*

Proof. Let $(\Phi(t), \Psi(t))$ be a prepared solution to the system (1.1). Suppose that $(\Phi(t), \Psi(t))$ is not oscillatory. Then $\det\Phi(t) \ne 0$, $t \ge T$, for some $T \ge t_0$. Let $Y_0(t) \equiv \big(y^0_{jk}(t)\big)^2_{j,k=1} = \Psi(t)\Phi^{-1}(t)$, $t \ge T$. By (2.1) $Y_0(t)$ is a solution of Eq. (2.3) on $[T; +\infty)$. Then by (2.10) $y^0_{jj}(t)$ satisfies to the following Riccati equation

$$y' + q_j(t)y^2 + a_{jj}(t)y + \mathcal{L}_j(y^0_{j,3-j}(t), y^0_{3-j,j}(t), t) = 0\,, \qquad t \ge T \tag{3.4}$$

(the definition of $\mathcal{L}_j$ see below (2). Since $(\Phi(t), \Psi(t))$ is a prepared solution we have $Y_0(t) = Y_0^*(t)$, $t \ge T$. From here and from the conditions (I) of theorem it follows that

$$\mathcal{L}_j(y^0_{j,3-j}(t), y^0_{3-j,j}(t), t) \ge F_j(t)\,, \qquad t \ge T\,. \tag{3.5}$$

Consider the Riccati equation

$$y' + q_j(t)y^2 + a_{jj}(t)y - F_j(t) = 0\,, \qquad t \ge T\,. \tag{3.6}$$

Let $y_j(t)$ be its solution with $y_j(t) \ge y^0_{jj}(T)$. Then using Theorem 2.1 by applying (3.5) to the equations (3.4) and (3.6) we will conclude that $y_j(t)$ exists on $[T; +\infty)$. Therefore by (3.1)–(3.3) the functions

$$\phi_j(t) = \exp\Big\{ \int\limits_T^t \big[q_j(\tau)y(\tau) + p_{jj}(\tau)\big]d\tau \Big\}\,, \qquad \psi_j(t) = y_j(t)\phi_j(t)\,,\ t \ge T$$

form the solution $(\phi_j(t), \psi_j(t))$ of the system (3.2) on $[T; +\infty)$, which can be continued on $[t_0; +\infty)$ as a solution of the system (3.2). It is evident that $\phi_j(t)$ has no arbitrary large zeroes which contradicts (II). The theorem is proved. □

By analogy can be proved

Theorem 3.2. *Let the following conditions be satisfied:*
(I*) $q_k(t) \ge 0$, $k = 1, 2$, *and if* $q_{3-j}(t) = 0$, *then* $p_{3-j,j}(t) = s_{j,3-j}(t)$, $t \in [t_1; t_2]$ $(t_0 \le t_1 < t_2 < +\infty)$;
(II*) *the system* (3.2) *is oscillatory on the interval* $[t_1; t_2]$.
Then the system (1.1) *is oscillatory on the interval* $[t_1; t_2]$.

Remark 3.1. The restrictions (I) on $Q(t)$ in Theorem 3.1 means that $Q(t)$ is nonnegative definite meanwhile in the works [1], [3], [10]–[14] and others the corresponding coefficient is positive definite.

Remark 3.2. Suppose $p_{12}(t) = -s_{21}(t)$, $p_{12}(t) = -s_{21}(t)$, $a_{12}(t) = a_{21}(t)$, $r_{12}(t) = r_{21}(t)$, $t \ge t_0$. Then by (2.10), if $Y_0(t)$ is a solution of Eq. (2.3) on some interval $[t_0; t_1)$, then $Y_0^*(t)$ is a solution of Eq. (2.3) on $[t_0; t_1)$ too. On the strength of the uniqueness theorem from here it follows that if $Y_0(t_0) = Y_0^*(t_0)$, then $Y_0(t) = Y_0^*(t)$, $t \in [t_0; t_1)$. Therefore taking into account (2.1) we conclude that if $(\Phi(t), \Psi(t))$ is a solution of the system (1.1) with $\det \Phi(t_0) \neq 0$, $\Phi^*(t_0)\Psi(t_0) = \Psi^*(t_0)\Phi(t_0)$, then $\Phi^*(t)\Psi(t) = \Psi^*(t)\Phi(t)$, $t \in [t_0; t_1)$. Obviously the last equality will be satisfied on the whole interval $[t_0; +\infty)$, provided we additionally require that $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ be analytical functions on the some domain of complex plane containing the half line $[t_0; +\infty)$. From the given restrictions above on $P(t)$, $Q(t)$, $R(t)$ and $S(t)$ is seen that the system (1.1) can be not hamiltonian. So the system (1.1) can have prepared solution not only in the case when it is hamiltonian but also in the other cases.

Example 3.1. Consider the matrix equation

$$\Phi'' + K(t)\Phi = 0\,, \qquad t \geq t_0\,. \tag{3.7}$$

where $K(t) \equiv \begin{pmatrix} a_1 \sin \mu_1 t + a_2 \sin \mu_2 t & \frac{b \cos \mu t}{t^\alpha} \\ \frac{b \cos \mu t}{t^\alpha} & a_1 \sin \mu_1 t + a_2 \sin \mu_2 t \end{pmatrix}$, a_1, a_2, α, μ, μ_1, μ_2 are some real nonzero constants and $\alpha > 1$, μ_1/μ_2 is irrational. This equation is equivalent to the system (1.1) with $P(t) = S(t) \equiv 0$, $R(t) \equiv K(t)$, $Q(t) \equiv I$ where I is the identity matrix of dimension 2×2. Therefore for this equation the system (3.2) has the form

$$\begin{cases} \phi' = \psi\,; \\ \psi' = -(a_1 \sin \mu_1 t + a_2 \sin \mu_2 t)\phi\,, \quad t \geq t_0\,. \end{cases}$$

which is equivalent to the scalar equation

$$\phi'' + (a_1 \sin \mu_1 t + a_2 \sin \mu_2 t)\phi = 0\,, \qquad t \geq t_0\,.$$

This equation is oscillatory (see [8, Corollary 3.4]). Therefore the last system is oscillatory too. By virtue of Theorem 3.1 from here it follows that Eq. (3.7) is oscillatory. The eigenvalues $\lambda_\pm(t)$ of the matrix $K(t)$ are equal

$$\lambda_\pm(t) = a_1 \sin \mu_1 t + a_2 \sin \mu_2 t \, \pm \, \frac{|b \cos \mu t|}{t^\alpha}\,, \qquad t \geq t_0\,.$$

From here is seen that the Theorems 5 and 6 of work [3], and the Theorems 1, 2 and 3 of work [2] are not applicable to Eq. (3.7). The remaining theorems of these works and the results of works [1], [10]–[14] are not explicit for applying them to Eq. (3.7) (it is hard to guess can we apply them to Eq. (3.7)).

Corollary 3.1. *Let the conditions* (I) *of Theorem 3.1 be satisfied and let*

(III) $\int\limits_{t_0}^{+\infty} q_j(\tau) \exp\Big\{ - \int\limits_{t_0}^{\tau} a_{jj}(s)ds \Big\} d\tau = \int\limits_{t_0}^{+\infty} [-F_j(\tau)] \exp\Big\{ \int\limits_{t_0}^{\tau} a_{jj}(s)ds \Big\} d\tau = +\infty.$

Then the system (1.1) *is oscillatory.*

Proof. On the strength of Theorem 3.1 it is enough to show that the system (3.2) is oscillatory. Suppose that the system (3.2) is not oscillatory. Then by (3.1)–(3.3) Eq. (3.1) has a solution on $[t_1; +\infty)$ for some $t_1 \geq t_0$. Set $W(t) \equiv$

$-F_j(t)\exp\Big\{\int\limits_{t_1}^{t} a_{jj}(\tau)d\tau\Big\}$, $t \geq t_1$. In Eq. (3.1) make the change

$$y = z\exp\Big\{-2\int\limits_{t_1}^{t} a_{jj}(\tau)d\tau\Big\}, \qquad t \geq t_1 .$$

We will come to the equation

$$z' + U(t)z^2 + W(t) = 0\,, \qquad t \geq t_1\,, \tag{3.8}$$

where $U(t) \equiv q_j(t)\exp\big\{-\int\limits_{t_1}^{t} a_{jj}(\tau)d\tau\big\}$. Show that

$$\int\limits_{t_1}^{+\infty} U(\tau)\exp\Big\{\int\limits_{t_1}^{t} U(s)ds\int\limits_{t_1}^{s} W(\zeta)d\zeta\Big\}d\tau = +\infty\,. \tag{3.9}$$

On the strength of (III) we have: $\int\limits_{t_1}^{t} W(\tau)d\tau = -\int\limits_{t_1}^{t} F_j(\tau)\exp\big\{\int\limits_{t_1}^{t} a_{jj}(s)ds\big\}d\tau \geq 0$, $t \geq t_2$, for some $t_2 \geq t_1$. By (III) from here it follows (3.9). In Eq. (3.8) make the change

$$z = u - \int\limits_{t_1}^{t} W(\tau)d\tau\,, \qquad t \geq t_1 .$$

We will get

$$u' + U(t)u^2 - 2U(t)\int\limits_{t_1}^{t} W(\tau)d\tau u + U(t)\Big[\int\limits_{t_1}^{t} W(\tau)d\tau\Big]^2 = 0\,, \qquad t \geq t_1 . \tag{3.10}$$

Since by assumption Eq. (3.1)has a solution on $[t_1;+\infty)$, from the above substitutions is seen that Eq. (3.10) has a solution on $[t_1;+\infty)$ as well. By virtue of Lemma 2.3 from here from (3.9) and from the inequalities $q_j(t) \geq 0$, $U(t)\big[\int\limits_{t_1}^{t} W(\tau)d\tau\big]^2 \geq 0$, $t \geq t_1$ it follows that Eq. (3.10) has a positive solution $u_0(t)$ on $[t_1;+\infty)$. Then $z_0(t) \equiv u_0(t) - \int\limits_{t_1}^{t} W(\tau)d\tau$ is a solution to Eq. (3.8) such that

$$z_0(t) > \int\limits_{t_1}^{t} W(\tau)d\tau\,, \qquad t \geq t_1 . \tag{3.11}$$

From (3.8) it follows that

$$z_0(t) = z_0(t_1) - \int\limits_{t_1}^{t} U(\tau)z_0^2(\tau)d\tau - \int\limits_{t_1}^{t} W(\tau)d\tau\,, \qquad t \geq t_1 . \tag{3.12}$$

From here and from (3.11) we have:

$$0 \le \int\limits_{t_1}^{t} U(\tau) z_0^2(\tau) d\tau < z_0(t_1)\,, \qquad t \ge t_1\,. \tag{3.13}$$

$(z_0(t_1) = u_0(t_1) > 0)$. Taking into account (III) from here we will get: $\big[z_0(t_1) - \int\limits_{t_1}^{t} U(\tau) z_0^2(\tau) d\tau - \int\limits_{t_1}^{t} W(\tau) d\tau\big]^2 \ge 1$, $t \ge T$, for some $T \ge t_0$. From here and from (3.12) it follows that $z_0^2(t) \ge 1$, $t \ge T$. Therefore by (III) $\int\limits_{T}^{+\infty} U(\tau) z_0^2(\tau) d\tau \ge \int\limits_{T}^{+\infty} U(\tau) d\tau = +\infty$, which contradicts (3.13). The corollary is proved. □

Corollary 3.2. *Let the conditions* (I*) *of Theorem 3.2 be satisfied and let*

(IV) $\int\limits_{t_1}^{t_2} \min\big[q_j(\tau) \exp\{-\int\limits_{t_1}^{\tau} a_{jj}(s) ds\}, -F_j(\tau) \exp\{\int\limits_{t_1}^{\tau} a_{jj}(s) ds\}\big] d\tau \ge \pi.$

Then the system (1.1) *is oscillatory on the interval* $[t_1; t_2]$.

Proof. On the strength of Theorem 3.2 it is enough to show that the system (3.2) is oscillatory on the interval $[t_1; t_2]$. In (3.2) make the changes

$$\begin{cases} \phi = \exp\{\int\limits_{t_1}^{t} p_{jj}(\tau) d\tau\} \rho \sin\theta\,; \\ \psi = \exp\{\int\limits_{t_1}^{t} s_{jj}(\tau) d\tau\} \rho \cos\theta \qquad t \ge t_0\,. \end{cases} \tag{3.14}$$

We will get:

$$\begin{cases} \rho' \sin\theta + \theta' \rho \cos\theta = Q_j(t) \rho \cos\theta\,; \\ \rho' \cos\theta - \theta' \rho \sin\theta = R_j(t) \rho \sin\theta, \;\; t \ge t_0\,, \end{cases} \tag{3.15}$$

where $Q_j(t) \equiv \exp\{-\int\limits_{t_1}^{t} a_{jj}(\tau) d\tau\} q_j(t)$, $R_j(t) \equiv \exp\{\int\limits_{t_1}^{t} a_{jj}(\tau) d\tau\} F_j(t)$, $t \ge t_0$ (the function $a_{jj}(t)$ is defined below (2.10)). This system is equivalent to the system (3.2) in the sense that to each nontrivial solution $(\phi(t), \psi(t))$ of the system (3.2) corresponds the solution $(\rho(t), \theta(t))$ of the system (3.15) with $\rho(t) > 0$, $t \ge t_0$ defined by (3.14). Let us multiply the first equation of the system (3.15) on $\cos\theta$ and the second one multiply on $\sin\theta$ and subtract from the first obtained equation the second one. We will get:

$$\theta' \rho = \rho \big[Q_j(t) \cos^2\theta - R_j(t) \sin^2\theta\big]\,, \qquad t \ge t_0\,. \tag{3.16}$$

Let $(\phi_0(t), \psi_0(t))$ be a nontrivial solution of the system (3.2) and let $(\rho_0(t), \theta_0(t))$ be the corresponding (by (3.14)) to $(\phi_0(t), \psi_0(t))$ solution of the system (3.15).

Then $\rho_0(t) \neq 0,\ \ t \geq t_0$, and therefore by (3.16) the following equality takes place

$$\begin{aligned}\theta_0'(t) &= Q_j(t)\cos^2\theta_0(t) - R_j(t)\sin^2\theta_0(t)\\ &= \frac{1}{2}\big[Q_j(t) - R_j(t) + (Q_j(t) + R_j(t))\cos 2\theta_0(t)\big]\,,\end{aligned}$$

$t \geq t_0$. From here it follows

$$\theta_0'(t) \geq \frac{1}{2}\big[Q_j(t) - R_j(t) - |Q_j(t) + R_j(t)|\big] = \min\{Q_j(t), -R_j(t)\}\,, \qquad t \geq t_0\,.$$

Let us integrate this inequality from t_1 to t_2 Taking into account the conditions of the corollary we will get:

$$\theta_0(t_2) - \theta_0(t_1) \geq \int\limits_{t_1}^{t_2} \min\{Q_j(\tau), -R_j(\tau)\}d\tau \geq \pi\,.$$

Due to (3.14) from here it follows that $\phi_0(t)$ has at least one zero on $[t_1; t_2]$. The corollary is proved. □

Remark 3.3. Let $t_0 \leq \eta_1 < \zeta_1 < \ldots \eta_m < \zeta_m \ldots$ be a infinitely large sequence and let the following conditions be satisfied:
(IV$_m$) $q_k(t) \geq 0$, and if $q_{3-j}(t) = 0$, then $p_{3-j}(t) = s_{j,3-j}(t)$, $t \in [\eta_m; \zeta_m]$, $k = 1, 2$;
$\int\limits_{\eta_m}^{\zeta_m} \min\big[q_j(\tau)\exp\{-\int\limits_{\eta_m}^{\tau} a_{jj}(s)ds\}, -F_j(\tau)\exp\{\int\limits_{\eta_m}^{\tau} a_{jj}(s)ds\}\big]d\tau \geq \pi$, $m = 1, 2, \ldots$.
Then on the strength of Corollary 3.2 the system (1.1) is oscillatory. From the conditions (IV$_m$) $m = 1, 2, \ldots$ is seen that outside of the set $\bigcup\limits_{m=1}^{+\infty} [\eta_m; \zeta_m]$ the functions $q_1(t)$ and $q_2(t)$ can take values of arbitrary sign and therefore the nonnegative definiteness of $Q(t)$ on $[t_0; +\infty)$ can be broken.

Remark 3.4. Let $P(t) = S(t) \equiv 0,\ \ Q(t) = -R(t) \equiv I$, $t \geq 0$, where I is the identity matrix of dimension 2×2. It is evident that in this particular case the conditions (I*) of Corollary 3.2 are satisfied on the arbitrary interval $[t_1; t_2](\subset [0; +\infty))$ and the condition (IV) is fulfilled only if $t_2 - t_1 \geq \pi$. It also is evident that for this case $(\Phi_0(t), \Psi_0(t))$, where $\Phi_0(t) \equiv \mathrm{diag}\{\sin t, \sin t\}$, $\Psi_0(t) \equiv \mathrm{diag}\{\cos t, \cos t\}$, is a prepared solution to the system (1.1). This solution is not oscillatory on $[\varepsilon; \pi - \varepsilon]$ for each $\varepsilon \in (0; \pi)$. Therefore in the inequality (IV) we may not replace π by a number less than π.

Example 3.2. Consider the system

$$\begin{cases}\Phi' = K_1(t)\Psi\,;\\ \Psi' = V_1(t)\Phi\,, \quad t \geq t_0\,,\end{cases} \tag{3.17}$$

where

$$\begin{aligned}K_1(t) &\equiv \mathrm{diag}\,\big\{\max\{\sin t, 0\}, \max\{\sin t, 0\}\big\}\,,\\ V_1(t) &\equiv \mathrm{diag}\,\big\{\min\{\sin t, 0\}, \min\{\sin t, 0\}\big\}\,,\end{aligned}$$

$t \geq t_0$. Obviously for this system the conditions (I*) of Corollary 3.2 are not fulfilled for all $[t_1; t_2](\subset [t_0; +\infty))$. Therefore Corollary 3.2 cannot be used to establish oscillatory behavior of the system (3.17). It is easy to verify that for the system (3.17) the conditions of Corollary 3.1 are fulfilled. Therefore the system (3.17) is oscillatory.

Example 3.3. Consider the system

$$\begin{cases} \Phi' = K_2(t)\Psi\,; \\ \Psi' = -K_2(t)\Phi\,, \quad t \geq 0\,, \end{cases} \tag{3.18}$$

where $K_2(t) \equiv \mathrm{diag}\{\lambda \sin t, \lambda \sin t\}$, $t \geq 0$, $\lambda \geq \frac{\pi}{2}$. Obviously the conditions (I) of Corollary 3.1 for this system are not fulfilled. Therefore it cannot be applied to the system (3.18). It is not difficult to verify that for $t_1 = 2\pi m$, $t_2 = \pi(2m+1)$ the conditions of Corollary 3.2 are fulfilled for all $m = 1, 2, \ldots$. Taking into account Remark 3.3 from here we conclude that the system (3.18) is oscillatory.

Theorem 3.3. *Let the conditions of Lemma 2.2 be satisfied. Then for each solution* $(\Phi(t), \Psi(t)) \equiv \big((\phi_{jk}(t))_{j,k=1}^2, (\psi_{jk}(t))_{j,k=1}^2\big)$ *of the system* (1.1) *with* $\det\Phi(t_0) \neq 0$, $y_{11}^0 \equiv \frac{\psi_{11}(t_0)\phi_{22}(t_0) - \psi_{12}(t_0)\phi_{21}(t_0)}{\det\Phi(t_0)} > 0$, $y_{22}^0 \equiv \frac{\psi_{22}(t_0)\phi_{11}(t_0) - \psi_{21}(t_0)\phi_{12}(t_0)}{\det\Phi(t_0)} > 0$, $y_{12}^0 \equiv \frac{\psi_{12}(t_0)\phi_{11}(t_0) - \psi_{11}(t_0)\phi_{12}(t_0)}{\det\Phi(t_0)} \geq 0 \quad (\leq 0)$, $y_{21}^0 \equiv \frac{\psi_{21}(t_0)\phi_{22}(t_0) - \psi_{22}(t_0)\phi_{21}(t_0)}{\det\Phi(t_0)} \leq 0$ (≥ 0), *the equality*

$$\mathrm{sign}\,[\det\Phi(t)] = \mathrm{sign}\,[\det\psi(t)] \neq 0\,, \qquad t \geq t_0\,. \tag{3.19}$$

takes place. Therefore $(\Phi(t), \Psi(t))$ *is non oscillatory.*

Proof. On the strength of Lemma 2.2 Eq. (2.3) has the solution $Y_0(t) \equiv \big(y_{jk}(t)\big)_{j,k=1}^2$, on $[t_0; +\infty)$ with $y_{jk}(t_0) = y_{jk}^0$, $j, k = 1, 2$, and

$$\det Y_0(t) > 0\,, \qquad t \geq t_0\,. \tag{3.20}$$

Since by (2.4) $\Phi(t)$ is a solution to the matrix equation

$$\Phi' = [P(t) + Q(t)y_0(t)]\Psi\,, \qquad t \geq t_0\,,$$

according to Liouville formula we have

$$\det\Phi(t) = \det\Phi(t_0)\exp\Big\{\int\limits_{t_0}^{t} \mathrm{tr}[P(\tau) + Q(\tau)Y_0(\tau)]d\tau\Big\} \neq 0\,, \qquad t \geq t_0\,. \tag{3.21}$$

By (2.1) the equality $\Psi(t) = Y_0(t)\Phi(t)$, $t \geq t_0$, holds. From here from (3.20) and (3.21) it follows (3.19). The theorem is proved. $\square$

Denote:

$$\widetilde{I}_1(\tau;t) \equiv \int\limits_{\tau}^{t} \exp\Big\{ - \int\limits_{s}^{t} a_{jj}(\zeta)d\zeta\Big\} F_j(s)ds\,,$$

$$\widetilde{I}_2(\tau;t) \equiv - \int\limits_{\tau}^{t} \exp\Big\{ - \int\limits_{s}^{t} a_{3-j,3-j}(\zeta)d\zeta\Big\} F_{3-j}(s)ds\,, \qquad t \ge \tau \ge t_0\,.$$

Theorem 3.4. *Let the following conditions be satisfied:*

(C) $q_j(t) \ge 0$, $q_{3-j}(t) \le 0$ *and if* $q_j(t) = 0$ *then* $p_{j,3-j}(t) = s_{3-j,j}(t)$, *if* $q_{3-j}(t) = 0$ *then* $p_{3-j,j}(t) = s_{j,3-j}(t)$, $t \ge t_0$;

(D) *there exists infinitely large sequences* $\xi_{0,k} = t_0 < \xi_{1,k} < \cdots < \xi_{m,k} < \cdots$, $k = 1,2$ *such that*

(D_1) $\int\limits_{\xi_{m,1}}^{t} \exp\{ \int\limits_{\xi_{m,1}}^{\tau} \big[a_{jj}(s) + q_j(s)\widetilde{I}_1(\xi_{m,1};s)\big]ds\} F_j(\tau)d\tau \ge 0$, $t \in [\xi_{m,1};\xi_{m+1,1})$,

(D_2) $\int\limits_{\xi_{m,2}}^{t} \exp\{ - \int\limits_{\xi_{m,2}}^{\tau} \big[a_{3-j,3-j}(s) + q_{3-j}(s)\widetilde{I}_2(\xi_{m,2};s)\big]ds\} F_{3-j}(\tau)d\tau \le 0$, $t \in [\xi_{m,2};\xi_{m+1,2})$, $m = 1,2,\ldots$.

Then for each prepared solution $(\Phi(t),\Psi(t)) \equiv \big((\phi_{jk}(t))_{j,k=1}^{2}, (\psi_{jk}(t))_{j,k=1}^{2}\big)$ *of the system* (1.1) *with* $\det\Phi(t_0) \ne 0$,

$$y_{11}^0 \equiv \frac{\psi_{11}(t_0)\phi_{22}(t_0) - \psi_{12}(t_0)\phi_{21}(t_0)}{\det\Phi(t_0)} \ge 0\,,$$

$$y_{22}^0 \equiv \frac{\psi_{22}(t_0)\phi_{11}(t_0) - \psi_{21}(t_0)\phi_{12}(t_0)}{\det\Phi(t_0)} \le 0 \quad \textit{the inequality}$$

$$\det\Phi(t) \ne 0\,, \qquad t \ge t_0\,, \tag{3.22}$$

takes place. Therefore $(\Phi(t),\Psi(t))$ *is non oscillatory. Moreover if* $y_{11}^0 > 0$, $y_{22}^0 < 0$, *then*

$$\operatorname{sign}\det\Phi(t) = -\operatorname{sign}\det\Psi(t) \ne 0\,, \qquad t \ge t_0\,. \tag{3.23}$$

Proof. Let $Y(t) \equiv \big(y_{jk}(t)\big)_{j,k=1}^{2}$ be the solution of Eq. (2.3) with $Y(t_0) = \Psi(t_0)\Phi^{-1}(t_0)$, where $(\Phi(t),\Psi(t))$ is a prepared solution to the system (1.1), satisfying the conditions of the theorem, and let $[t_0;T)$ be the maximum existence interval for $Y(t)$. Show that

$$T = +\infty\,. \tag{3.24}$$

By (2.10) $y_{jj}(t)$ and $y_{3-j,3-j}(t)$ are solutions to the equations

$$y' + q_j(t)y^2 + a_{jj}(t)y + \mathcal{L}_j(y_{j,3-j}(t), y_{3-j,j}(t), t) = 0\,, \tag{3.25}$$

$$y' - q_{3-j}(t)y^2 + a_{3-j,3-j}(t)y + \mathcal{L}_{3-j}(y_{3-j,j}(t), y_{j,3-j}(t), t) = 0\,, \tag{3.26}$$

for each $t \in [t_0; T)$, respectively. From the conditions (C) it follows that the following inequalities are satisfied:

$$\mathcal{L}_j(X, X, t) \le F_j(t)\,, \qquad \mathcal{L}_{3-j}(X, X, t) \le -F_{3-j}(t)\,, \qquad X \in R,\ \ t \ge t_0\,. \tag{3.27}$$

(for $q_j(t) \neq 0$ $(q_{3-j}(t) \neq 0)$ the $F_j(t)$ $(F_{3-j}(t))$ is the maximum for the quadratic trinomial $\mathcal{L}_j(X, X, t)$ $(\mathcal{L}_{3-j}(X, X, t))$ of variable $X \in R$). Show that

$$\det \Phi(t) \neq 0\,, \qquad t \in [t_0; T). \tag{3.28}$$

By (2.4) $\Phi(t)$ is a solution to the matrix equation

$$\Phi' = [P(t) + Q(t)Y(t)]\Phi\,, \qquad t \in [t_0; T)\,.$$

By virtue of Liouville formula from the condition $\det \Phi(t_0) \neq 0$ of theorem it follows (3.28). Therefore by (2.1) and the uniqueness theorem $Y(t) = \Psi(t)\Phi^{-1}(t)$, $t \in [t_0; T)$. Then since $(\Phi(t), \Psi(t))$ is prepared we have $Y(t) = Y^*(t)$, $t \in [t_0; T)$. Hence

$$y_{12}(t) = y_{21}(t)\,, \qquad t \in [t_0; T)\,. \tag{3.29}$$

Let $y_1(t)$ and $y_2(t)$ be the solutions to the equations

$$y' + q_j(t)y^2 + a_{jj}(t)y + F_j(t) = 0\,, \qquad t \ge t_0\,, \tag{3.30}$$

$$y' - q_{3-j}(t)y^2 + a_{3-j,3-j}(t)y - F_{3-j}(t) = 0\,, \qquad t \ge t_0\,, \tag{3.31}$$

respectively with $y_1(t_0) = y_2(t_0) = 0$. By virtue of Theorem4.1 of work [5] from (C), (D_1) and (D_2) it follows that $y_1(t)$, $y_2(t)$ exist on $[t_0; +\infty)$ and are nonnegative for all $t \ge t_0$ Moreover if $y_k(t_0) > 0$, $k = 1, 2$ then $y_k(t) > 0$, $t \ge t_0$, $k = 1, 2$. Using Theorem 2.1 to the pairs (3.25), (3.30) and (3.26), (3.32) taking into account (3.27) from here we will get:

$$y_{11}(t) \ge y_1(t) \ge 0\,, \qquad y_{22}(t) \le -y_2(t) \le 0\,, \qquad t \in [t_0; T)\,, \tag{3.32}$$

and if $y_{11}^0 = y_{11}(t_0) > 0$, $y_{22}^0 = y_{22}(t_0) < 0$, then

$$y_{11}(t) > 0\,, \qquad y_{22}(t) < 0\,, \qquad t \in [t_0; T)\,. \tag{3.33}$$

Suppose $T < +\infty$. Then from (C) and (3.32) it follows that the function $f(t) \equiv \int\limits_{T_0}^{t} \mathrm{tr}[Q(\tau)Y(\tau)]d\tau$, $t \in [t_0; T)$ is bounded from below on $[t_0; T)$. By Lemma 2.1 from here it follows that $[t_0; T)$ is not the maximum existence interval for $Y(t)$. The obtained contradiction proves (3.24). From (3.24) and (3.28) it follows (3.22), and from (3.24), (3.28) and (3.33) it follows (3.23). The theorem is proved. □

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