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TWO-SPINOR TETRAD AND LIE DERIVATIVES OF EINSTEIN-CARTAN-DIRAC FIELDS

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Abstract. An integrated approach to Lie derivatives of spinors, spinor connections and the gravitational field is presented, in the context of a previously proposed, partly original formulation of a theory of Einstein-Cartan-Maxwell-Dirac fields based on "minimal geometric data": the needed underlying structure is determined, via geometric constructions, from the unique assumption of a complex vector bundle $S \rightarrowtail M$ with 2-dimensional fibers, called a 2-*spinor bundle*. Any further considered object is assumed to be a dynamical field; these include the gravitational field, which is jointly represented by a soldering form (the *tetrad*) relating the tangent space $\mathrm{T}M$ to the 2-spinor bundle, and a connection of the latter (*spinor connection*). The Lie derivatives of objects of all considered types, with respect to a vector field $x \colon M \to \mathrm{T}M$, turn out to be well-defined without making any special assumption about x, and fulfill natural mutual relations.

Introduction

Lie derivatives of 4-spinors on curved spacetime have been studied by Kosmann [32] and others [15, 19, 35] by exploiting structure groups and their representations in order to extend to spinors the notion of transport of tensor fields by the local 1-parameter group associated with a vector field. A somewhat different approach by Penrose and Rindler [43] recovers the Lie derivative of a 2-spinor from the requirement that it is related to the usual Lie derivative through the Leibnitz rule. A recent article [25] examined the relations among various approaches.

One key point about this topic is the "soldering" of the spinor and spacetime geometries. Usually, the soldering is implicitely contained in the formalism; we propose to make it explicit by means of a partly original formulation of tetrad gravity, described in previous papers [2, 3], which yields an integrated treatment of Einstein-Cartan-Maxwell-Dirac fields starting from minimal geometric data.

The so-called *tetrad* formalism [11, 12, 21, 22, 23, 24, 26, 46, 49, 50] could be just regarded as using an orthonormal spacetime frame in order to describe gravitation. A geometric refinement can be achieved by assuming a vector bundle $H \rightarrowtail M$ whose

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4-dimensional fibers are endowed with a Lorentz metric, and defining the tetrad as a *soldering form* between H and the tangent bundle of M. This extra assumption, apparently contrary to Ockam's razor principle, can be actually turned into a free benefit since H can be derived, by a geometric construction, from a complex vector bundle $S \rightarrowtail M$ with 2-dimensional fibers. The same S naturally yields the bundle W of 4-spinors together with the Dirac map $H \to \operatorname{End} W$, and any other structure needed for the aforementioned integrated field theory (length units included). The tetrad itself replaces the spacetime metric g, and indeed it can be regarded as a "square root" of g, while a connection of S naturally splits into "gravitational" and "electromagnetic" contributions. The spacetime connection, on the other hand, is no more regarded as a fundamental field but rather as a byproduct. The underlying 2-spinor formalism is compatible with the Penrose-Rindler formalism [42, 43], with a few adjustments.

After an essential account of the above said setting we address the notions of Lie derivatives of all involved fields with respect to a vector field $x \colon M \to \mathrm{T}M$. We find that by explicitely taking the soldering form into account we are able to give a natural definition of Lie derivative of a spinor of any type without imposing any constraint on x (such as being Killing or conformal Killing). The usual notion of Lie derivative of a Dirac spinor can be recovered as a special case.

Furthermore, as a consequence, the Lie derivative of the tetrad itself turns out to be naturally defined too, and that exactly takes care of any discrepances with usual approaches.

Then we address Lie derivatives of connections. We recall that the notion of the Lie derivative of a linear connection of the tangent space of a manifold has been known for a long time (see e.g. Yano [53], §I.4). The main use of that notion in the literature deals with energy tensors in General Relativity [31, 41], possibly in the disguised form of "deformations" of the spacetime connection [20, 33]. Exploiting that concept, and using the Lie derivative of spinors, we introduce the Lie derivative of the spinor connection, by a procedure that uses the soldering form and the corresponding natural decompositions of the spaces of endomorphisms of S and H. We state the fundamental relations among the various considered operations and write down the basic coordinate formulas.

Finally we discuss the notion of a deformed theory of Einstein-Cartan-Dirac fields.

1. Einstein-Cartan-Maxwell-Dirac fields using "minimal geometric data"

The next four sections deal with purely algebraic constructions, whose only ingredient is a 2-dimensional complex vector space S. Afterwards we'll consider a vector bundle $S \rightarrowtail M$ over a real 4-dimensional manifold M, where our constructions can be performed fiberwise yielding various associated bundles and natural maps. Any needed topological constraint will be implicitely assumed to hold.

Though most of the material in the next six sections can be found in previous work [2, 3, 4], this not-to-short summary may be appropriate in order to make the article's context precise.

1.1. **Unit spaces and their rational powers.** Though physical scales (or "dimensions") are often dealt with in an informal way, a mathematically rigorous treatment introduced in 1995 after an idea of M. Modugno [7, 30] has been exploited by various authors [28, 29, 38, 48, 51, 52], particularly in the context of the "Covariant Quantum Mechanics" program. The basic concepts are *unit spaces* and their rational powers. We summarize the main notions involved, observing that while most of these look like a straightforward reformulation of standard multilinear algebra, a thorough discussion of certain finer points is not trivial. Eventually, however, the ensuing formalism turns out to allow a natural handling of similar but differently scaled spaces, and also provides useful indications about geometric constructions appropriate for physical theories.

A *semi-vector space* is defined to be a set equipped with an internal addition map and multiplication by positive reals, fulfilling the usual axioms of vector spaces except those properties which involve opposites and the zero element. In particular, any vector space is a semi-vector space, as well as the subset of all linear combinations over $\mathbb{R}^+$ of n independent assigned vectors. If A and B are semi-vector spaces, the notion of a *semi-linear* map $f\colon A \to B$ is defined in an obvious way; we then obtain the semi-vector space $\mathrm{sLin}(A,B)$ of all semi-linear maps $A \to B$. In particular, the (*semi-*)*dual space* of a semi-vector space A is defined to be the semi-vector space $A^* \equiv \mathrm{sLin}(A,\mathbb{R}^+)$.

A semi-vector space $\mathbb{U}$ is called a *positive space*, or a *unit space*, if the multiplication $\mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ is a transitive left action of the group $(\mathbb{R}^+,\cdot\,)$ on $\mathbb{U}$. Thus a positive space has no zero element. Moreover if $b \in \mathbb{U}$ then any other element $u \in \mathbb{U}$ can be written as $u^0\,b$ with $u^0 \in \mathbb{R}^+$, namely a positive space can be regarded as a 1-dimensional semi-vector space.

If $\mathbb{U}$ and $\mathbb{V}$ are positive spaces, then the semi-vector space $\mathrm{sLin}(\mathbb{U},\mathbb{V})$ turns out to be a positive space. In particular, $\mathrm{sLin}(\mathbb{U},\mathbb{U})$ is naturally isomorphic to $\mathbb{R}^+$, since any semi-linear map $f\colon \mathbb{U} \to \mathbb{U}$ is of the type $f\colon u \mapsto r\,u$ with $r \in \mathbb{R}^+$.

Let V and W be arbitrary real vector spaces of finite dimension. A map $\mathbb{U} \times V \to W$ which is semi-linear with respect to the first factor and linear with respect to the second factor is called *sesqui-linear*.

A (*left*) *tensor product* of a unit space $\mathbb{U}$ and a vector space V is defined to be a vector space $\mathbb{U}\otimes V$ along with a sesqui-linear map $\otimes\colon \mathbb{U}\times V \to \mathbb{U}\otimes V$ fulfilling the following universal property: if $f\colon \mathbb{U}\times V \to W$ is a sesqui-linear a map, then there is a unique linear map $\tilde f\colon \mathbb{U}\otimes V \to W$ such that $f = \tilde f \circ \otimes$. It can be proved [30] that the tensor product indeed exists, is unique up to a distinguished linear isomorphism and is linearly generated by the image of the map $\otimes$. In particular we obtain the *universal vector extension* $\mathbb{R}\otimes\mathbb{U}$ of $\mathbb{U}$, which turns out to be the disjoint union $\mathbb{R}\otimes\mathbb{U} = \mathbb{U}_- \cup \{0\} \cup \mathbb{U}_+$, where $\mathbb{U}_+ \equiv \{1\otimes u : u \in \mathbb{U}\}$ and $\mathbb{U}_- \equiv \{(-1)\otimes u : u \in \mathbb{U}\}$ are positive spaces.

If $\{\mathsf{b}_i\} \subset V$ is a basis of V and $\mathsf{b} \in \mathbb{U}$, then it is not difficult to prove that $\{\mathsf{b}\otimes\mathsf{b}_i\}$ is a basis of $\mathbb{U}\otimes V$. Thus $\dim(\mathbb{U}\otimes V) = \dim V$. The right semi-tensor product $V\otimes\mathbb{U}$ can be defined similarly, and turns out to be naturally isomorphic to $\mathbb{U}\otimes V$; thus we identify $v\otimes u \in V\otimes\mathbb{U}$ with $u\otimes v$, getting the number-like behavior of elements in positive spaces. If $u \in \mathbb{U}$ then the unique $u^{-1} \in \mathbb{U}^*$ such that $\langle u^{-1}, u\rangle = 1$, namely the *dual* element of u, can be regarded as the *inverse* of u.

A (*semi-*)*tensor product* of positive spaces $\mathbb{U}$ and $\mathbb{V}$ is a positive space $\mathbb{U}\otimes\mathbb{V}$ along with a semi-bilinear map $\otimes : \mathbb{U}\times\mathbb{V}\to\mathbb{U}\otimes\mathbb{V}$, fulfilling the universal property which is formally expressed as before. While the uniqueness of the semi-tensor product is easily established, the proof of its existence, requiring the universal vector extensions of $\mathbb{U}$ and $\mathbb{V}$, is somewhat more involved [30]. Eventually one gets natural semi-linear isomorphisms

$$\mathbb{R}^+\otimes\mathbb{U}\cong\mathbb{U}\otimes\mathbb{R}^+\cong\mathbb{U}\,, \qquad \mathbb{R}\otimes(\mathbb{U}\otimes\mathbb{V})\cong(\mathbb{R}\otimes\mathbb{U})\otimes(\mathbb{R}\otimes\mathbb{V})\,,$$

$$\mathbb{V}\otimes\mathbb{U}^*\cong\mathrm{sLin}(\mathbb{U},\mathbb{V})\,, \qquad \mathrm{Tr}\colon \mathbb{U}\otimes\mathbb{U}^*\to\mathbb{R}^+\,.$$

The semi-tensor product can be easily generalized to any number of factors, and it turns out to be associative. By setting

$$\mathbb{U}^n\equiv\underbrace{\mathbb{U}\otimes\ldots\otimes\mathbb{U}}_{n\text{ factors}},\quad \mathbb{U}^0\equiv\mathbb{R}^+\,,\quad \mathbb{U}^{-1}\equiv\mathbb{U}^*\,,$$

we get the notion of integer power $\mathbb{U}^q$ for any $q\in\mathbb{Z}$. This notion can be extended to rational powers by the following construction.

We say that a function $f\colon\mathbb{U}\to\mathbb{R}^+$ is of *degree* $\alpha\in\mathbb{R}$ if

$$f(r\,u)=r^\alpha\,f(u)\ \forall r\in\mathbb{R}^+,\ u\in\mathbb{U}\,.$$

The set $\mathrm{F}^\alpha(\mathbb{U})$ of all such functions turns out to be a positive space. Note that each element in $\mathrm{F}^\alpha(\mathbb{U})$ is determined by the value it takes on any fixed element in $\mathbb{U}$. Conversely, each $u\in\mathbb{U}$ determines a distinguished element $f_u\in\mathrm{F}^\alpha(\mathbb{U})$ by the rule $f_u(u)=1$. In particular, $\mathrm{F}^0(\mathbb{U})\cong\mathbb{R}^+$ and $\mathrm{F}^1(\mathbb{U})\cong\mathbb{U}^*$. If $n\in\mathbb{N}$ then $\mathrm{F}^n(\mathbb{U})\cong(\mathbb{U}^*)^n\equiv\mathbb{U}^{-n}$. A natural semi-linear isomorphism $\mathrm{F}^{-1}(\mathbb{U})\cong\mathbb{U}^{**}\cong\mathbb{U}$ is determined by the identification of $f\in\mathrm{F}^{-1}(\mathbb{U})$ with u^{-1}, where $u\in\mathbb{U}$ is characterized by $f(u)=1$. More generally, $\mathrm{F}^{-\alpha}(\mathbb{U})\cong\mathrm{F}^\alpha(\mathbb{U}^*)$. We'll use the convenient shorthand

$$\mathbb{U}^\alpha\equiv\mathrm{F}^{-\alpha}(\mathbb{U})\,.$$

If $u\in\mathbb{U}$ then u^α is defined to be the unique element in $\mathbb{U}^\alpha$ such that $u^\alpha(u^{-1})=1$.

If $n\in\mathbb{N}$ then we have a natural semi-linear isomorphism

$$(\mathbb{U}^{1/n})^n\equiv\underbrace{\mathbb{U}^{1/n}\otimes\ldots\otimes\mathbb{U}^{1/n}}_{n\text{ factors}}\leftrightarrow\mathbb{U}\ :\ u^{1/n}\otimes\ldots\otimes u^{1/n}\leftrightarrow u\,.$$

Then we are led to regard the positive space $\mathbb{U}^{1/n}\equiv\mathrm{F}^{1/n}(\mathbb{U}^*)$ as the *n-root* of $\mathbb{U}$. We find that *rational powers* of positive spaces behave quite naturally, since for any p, $q\in\mathbb{Q}$ one has

$$(\mathbb{U}^q)^p\cong\mathbb{U}^{p\,q}\,,\quad \mathbb{U}^p\otimes\mathbb{U}^q\cong\mathbb{U}^{p+q}\,.$$

In particular, $(\mathbb{U}^q)^*\cong(\mathbb{U}^*)^q$.

In many physical theories it is convenient to assume the space $\mathbb{T}$ of *time units*, the space $\mathbb{L}$ of *length units*, and the space $\mathbb{M}$ of *mass units*, and construct any other needed scale space as $\mathbb{S}=\mathbb{T}^{d_1}\otimes\mathbb{L}^{d_2}\otimes\mathbb{M}^{d_3}$ with $d_i\in\mathbb{Q}$. Two sections $\sigma\colon M\to E$ and $\sigma'\colon M\to\mathbb{S}\otimes E$ of differently scaled vector bundles can be compared by means of a *coupling constant* $s \in \mathbb{S}$. In an unscaled frame, the components of an $\mathbb{S}$-scaled section are valued into $\mathbb{S} \otimes \mathbb{R}$. In particular we have the speed of light $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$. Together, these determine isomorphisms $\mathbb{T} \cong \mathbb{L}$ and $\mathbb{M} \cong \mathbb{L}^{-1}$, so we can actually reduce all scale spaces to powers of $\mathbb{L}$; this amounts to the so-called *natural* system, corresponding to the setting $c = \hbar = 1$. Then, in particular, a *mass* is an element $m \in \mathbb{L}^{-1}$ and, in Einstein spacetime (M, g), we identify the bundle $\mathbb{M} \otimes \mathbb{L}^2 \otimes \mathbb{T}^{-1} \otimes \mathrm{T}^*M$ of 4-momenta with T^*M.

1.2. **Two-spinor algebra and Lorentzian geometry.** Let V be a finite-dimensional complex vector space. We denote by V^* and $\overline{V^*}$ the associated *dual* and *anti-dual* spaces, namely the complex vector spaces of all linear and anti-linear maps $f\colon V \to \mathbb{C}$, respectively (the latter fulfilling $f(cv) = \bar{c}\, f(v)$, $c \in \mathbb{C}$). Then we obtain a natural *conjugation map*

$$V^* \to \overline{V^*}\colon f \mapsto \bar{f}\,,$$

$$\bar{f}(v) \equiv \overline{f(v)}\ , \quad v \in V\,.$$

Accordingly we consider a further complex vector space associated with V, that is the *conjugate space* $\overline{V}$, defined to be the dual space of $\overline{V^*}$. We actually obtain the natural isomorphism

$$\overline{V} \equiv \overline{V^*}^* \cong \overline{V^{**}}\,,$$

since the conjugate $\bar{v} \in \overline{V^{**}}$ of any $v \in V \cong V^{**}$ can be regarded as the anti-linear map

$$V^* \to \mathbb{C}\colon f \mapsto \bar{v}(f) \equiv \overline{f(v)} \equiv \bar{f}(v)\,.$$

Summarizing, conjugation determines anti-isomorphisms $V \leftrightarrow \overline{V}$ and $V^* \leftrightarrow \overline{V}^*$. Using conjugation together with transposition we also obtain an antilinear involution of $V \otimes \overline{V}$, determining a decomposition into the direct sum of the *real* eigenspaces corresponding to eigenvalues ± 1, namely

$$V \otimes \overline{V} = \mathrm{H}(V \otimes \overline{V}) \oplus \mathrm{i}\,\mathrm{H}(V \otimes \overline{V})\,;$$

these are respectively called the *Hermitian* and *anti-Hermitian* subspaces of $V \otimes \overline{V}$.

A basis (b_α) of V yields the conjugate basis $(\bar{\mathsf{b}}_{\dot\alpha})$ of $\overline{V}$. The conjugate $\bar{v} \in \overline{V}$ of an element $v = v^\alpha\, \mathsf{b}_\alpha \in V$ has then the basis expression $\bar{v} = \bar{v}^{\dot\alpha}\, \bar{\mathsf{b}}_{\dot\alpha}$, where the components $\bar{v}^{\dot\alpha}$ are the complex conjugate[1] of the v^α. Hermitian tensors $w \in V \otimes \overline{V}$ are characterized by the condition that their components, in any basis, fulfill $\bar{w}^{\dot\alpha\alpha} = w^{\alpha\dot\alpha}$.

The notion of Hermitian decomposition is the source of much of the rich algebraic structure which can be extracted from a 2-dimensional complex vector space S without any further assumption – the relation to a formalism more familiar to physicists can be seen by using a basis (ξ_A) of S. We distinguish a few steps.

[1]Distinguishing conjugate components by dotted indices is a handy, commonly used notation, somewhat analogous to distinguishing covariant and contravariant indices by their position.

• We start by observing that the antisymmetric subspace $\wedge^2 S \subset S\otimes S$ is a 1-dimensional complex vector space, so that the Hermitian subspace of $\wedge^2 S\otimes\wedge^2\overline{S}$ is a real 1-dimensional vector space; this has a distinguished orientation, positively and negatively oriented elements being of the type $\pm w\otimes\bar w$, $w\in\wedge^2 S$. We denote the positively oriented semispace by $\mathbb{L}^2$, namely we write

$$\mathrm{H}(\wedge^2 S\otimes\wedge^2\overline{S})\equiv\mathbb{R}\otimes\mathbb{L}^2\subset\wedge^2 S\otimes\wedge^2\overline{S}\,.$$

Now $\mathbb{L}^2$ has the square root semispace $\mathbb{L}\equiv(\mathbb{L}^2)^{1/2}$, which will be identified with the space of *length units*. In the ensuing field theory, $\mathbb{L}$ turns out to be the natural target for the *dilaton* field. The choice of a basis of S determines a basis in each of the associated spaces and, in particular, a length unit $l\in\mathbb{L}$.

• We introduce the new 2-dimensional space $U\equiv\mathbb{L}^{-1/2}\otimes S$, which has the induced basis $\bigl(\zeta_A\bigr)=\bigl(l^{-1/2}\,\xi_A\bigr)$. This is our 2-*spinor space*. Now since $U^*\cong\mathbb{L}^{1/2}\otimes S^*$, the 1-dimensional complex space $\wedge^2 U$ turns out to be naturally endowed with a Hermitian metric, namely the identity element in

$$\mathbb{L}^2\otimes\mathbb{L}^{-2}\cong\mathbb{L}^2\otimes\mathrm{H}\bigl[(\wedge^2\overline{S}^*)\otimes(\wedge^2 S^*)\bigr]\cong\mathrm{H}\bigl[(\wedge^2\overline{U}^*)\otimes(\wedge^2 U^*)\bigr]\,.$$

Hence any two normalised elements in $\wedge^2 U^*$ are related by a phase factor. The chosen basis determines one such element, namely $\varepsilon=\varepsilon_{AB}\,\zeta^A\wedge\zeta^B$, where $\varepsilon_{AB}=\delta^1_A\,\delta^2_B-\delta^1_B\,\delta^2_A$ are the antisymmetric Ricci coefficients and $\bigl(\zeta^A\bigr)$ denotes the dual basis of U^*. Each normalised $\varepsilon\in\wedge^2 U^*$ yields the isomorphism $\varepsilon^\flat\colon U\to U^*: u\mapsto u^\flat\equiv\varepsilon(u,\_\_)$, with the coordinate expression $u_B\equiv(u^\flat)_B=\varepsilon_{AB}\,u^A$. The dual construction also yields $\varepsilon^\#=\varepsilon^{AB}\,\zeta_A\wedge\zeta_B$ and the inverse isomorphism $U^*\to U$.

• We'll be specially involved with the Hermitian subspace

$$H\equiv\mathrm{H}(U\otimes\overline{U})\subset U\otimes\overline{U}\,.$$

This is a 4-dimensional real vector space, which turns out to be naturally endowed with a Lorentz metric. Actually if $\varepsilon\in\wedge^2 U^*$ is normalized then $\varepsilon\otimes\bar\varepsilon\in\wedge^2 U^*\otimes\wedge^2\overline{U}^*$ is independent of the phase factor in ε; thus it is a natural object, which can be regarded as the bilinear form g on $U\otimes\overline{U}$ characterized by

$$g(u\otimes\bar v,r\otimes\bar s)=\varepsilon(u,r)\,\bar\varepsilon(\bar v,\bar s)=\varepsilon_{AB}\,\bar\varepsilon_{A^{\boldsymbol\cdot}B^{\boldsymbol\cdot}}\,u^A r^B\bar v^{A^{\boldsymbol\cdot}}\bar s^{B^{\boldsymbol\cdot}}\,.$$

Alongside with the induced basis $\bigl(\zeta_A\otimes\bar\zeta_{A^{\boldsymbol\cdot}}\bigr)$ of $U\otimes\overline{U}$ we also consider the basis $\bigl(\tau_\lambda\bigr)$, defined in term of the Pauli matrices as

$$\tau_\lambda\equiv\tfrac{1}{\sqrt2}\,\sigma_\lambda^{AA^{\boldsymbol\cdot}}\,\zeta_A\otimes\bar\zeta_{A^{\boldsymbol\cdot}}\,,\qquad\lambda=0,1,2,3\,.$$

A straightforward computation then shows that this is an orthonormal basis of H, with squares $(+1,-1,-1,-1)$. Null elements in H are of the form $\pm u\otimes\bar u$ with $u\in U$. One assigns a time-orientation in H by letting *future-pointing* null elements be of the type $+u\otimes\bar u$.

1.3. **Two-spinors and Dirac spinors.** Our step-by-step algebraic constructions continue by considering the 4-dimensional complex vector space $W\equiv U\oplus\overline{U}^*$. This can be naturally regarded as the space of 4-spinors, as we can exhibit a natural linear map $\gamma\colon U\otimes\overline{U}\to\mathrm{End}(W)$ whose restriction to the Minkowski space H turns out to be a Clifford map. It is characterized by

$$\gamma(r\otimes\bar{s})(u,\chi)=\sqrt{2}\big(\langle\bar{\lambda},\bar{s}\rangle\,p\,,\langle r^\flat,u\rangle\,\bar{s}^\flat\,\big)\,,\qquad u,p,r,s\in U\,,\ \chi\in\overline{U}^*\,,$$

an expression which is independent of the phase factor in the normalized ε yielding $r^\flat\in U^*$ and $\bar{s}^\flat\in\overline{U}^*$. The usual Weyl representation can be recovered by using the basis

$$\big(\omega_\alpha\big)\equiv(\zeta_1\,,\zeta_2\,,-\bar{\zeta}^1,-\bar{\zeta}^2)\,,$$

where ζ_1 is a simplified notation for $(\zeta_1,0)$, and the like: setting $\gamma_\lambda\equiv\gamma(\tau_\lambda)\in\mathrm{End}\,W$, $\lambda=0,1,2,3$, the matrices $\big(\gamma_\lambda{}^\alpha{}_\beta\big)$ in this basis turn out to be the Weyl matrices. By a suitable basis transformation one also recovers the Dirac representation, which is associated with the choice of an *observer*|that is, in the present algebraic context, the choice of a time-like direction.

Next we observe that the conjugate space of W is $\overline{W}=\overline{U}\oplus U^*$, whence by inverting the order of the two sectors we obtain the dual space $U^*\oplus\overline{U}=W^*$. Let's explicitely denote this switching map, which is obviously an isomorphism, as

$$\mathrm{s}:\overline{W}\to W^*\colon(\bar{u},\lambda)\mapsto(\lambda,\bar{u})\,.$$

If $\psi\equiv(u,\bar{\lambda})\in W$ then applying the conjugation anti-isomorphism to it we get

$$\bar{\psi}=(\bar{u},\lambda)\in\overline{W}\quad\Rightarrow\quad\mathrm{s}(\bar{\psi})\in W^*\,.$$

This $\mathrm{s}(\bar{\psi})$ is exactly the object which is traditionally denoted as $\bar{\psi}$, namely the *Dirac adjoint* of ψ. When no confusion arises, we may as well adopt that notation as a shorthand. The map $W\to W^*\colon\psi\mapsto\bar{\psi}$ can be regarded as associated with a Hermitian scalar product on W, which turns out to have signature $(++--)$, as one sees immediately in the Dirac representation.

We end this section with a few remarks about aspects of our presentation which are different from the usual 2-spinor and 4-spinor formalisms.

- No complex symplectic form on U is fixed. The 2-form ε, yielding the isomorphisms $\varepsilon^\flat$ and $\varepsilon^\#$, is unique up to a phase factor.
- No Hermitian scalar product on U is assigned; such assignment amounts to the choice of a timelike element in $H^*\subset U^*\otimes\overline{U}^*$, hence it essentially amounts to the choice of an "observer" in the Minkowski space H.
- Consequently there is no fixed complex symplectic form nor positive Hermitian structure on the 4-spinor space W as well. The usual mapping $\psi\mapsto\psi^\dagger$ is related to a positive Hermitian structure associated with an observer (while Dirac adjunction is observer-independent). Charge conjugation is related to the choice of ε (namely of a phase factor).

1.4. **Endomorphism decomposition in spinor and Minkowski spaces.** The discussion contained in this section is prerequisite to laying out a precise pattern of relations among Lie derivatives of spinor fields and other objects, with respect to a vector field $x\colon M\to\mathrm{T}M$ on the spacetime manifold. These derivatives turn out to be related to the tensor field

$$\nabla x\colon M\to\mathrm{T}^*M\otimes\mathrm{T}M\,,$$

that can be regarded as a fibered endomorphism of $\mathrm{T}M$. Endomorphisms of H and U will then enter the picture via a soldering form (§1.7).

In the vector space $\mathrm{End}\,H \equiv H\otimes H^*$ of all linear endomorphisms of H we have the Lorentz metric transposition $\mathrm{End}\,H \to \mathrm{End}\,H\colon K\mapsto K^\dagger$, where $(K^\dagger)^\lambda{}_\mu = g^{\lambda\nu}\,K^\rho{}_\nu\,g_{\rho\mu}$. The subspace of all endomorphisms which are antisymmetric with respect to this operation is the Lie subalgebra[2] $\mathfrak{so}(H,g)$. We obtain a natural vector space decomposition

$$\mathrm{End}\,H = \mathfrak{so}(H,g)\oplus\mathbb{R}1\!\!1_H\oplus\mathcal{S}_0H\,,$$

where $\mathbb{R}1\!\!1_H$ is the subspace generated by the identity of H and $\mathcal{S}_0H$ is the space of all trace-free symmetric endomorphisms. Indeed, any $K\in\mathrm{End}\,H$ can be uniquely decomposed as

$$K=\tfrac{1}{2}\big(K-K^\dagger\big)+\tfrac{1}{4}\,\mathrm{Tr}K\,1\!\!1_H+\big(\tfrac{1}{2}\,(K+K^\dagger)-\tfrac{1}{4}\,\mathrm{Tr}K\,1\!\!1_H\big)\,.$$

In particular we have a projection

$$\mathrm{p}:\mathrm{End}\,H\to\mathfrak{so}(H,g)\oplus\mathbb{R}1\!\!1_H : K\mapsto\tfrac{1}{2}\big(K-K^\dagger\big)+\tfrac{1}{4}\,\mathrm{Tr}K\,1\!\!1_H\,,$$

whose target space is a Lie-subalgebra of $\mathrm{End}\,H$ (while its complementary space $\mathcal{S}_0H$ is not closed with respect to the commutator).

Similarly, the vector space $\mathrm{End}\,U\equiv U\otimes U^*$ of all $\mathbb{C}$-linear endomorphisms of U has the natural decomposition

$$\mathrm{End}\,U=\mathfrak{sl}(U)\oplus\mathbb{C}1\!\!1_U=\mathfrak{sl}(U)\oplus\mathbb{R}1\!\!1_U\oplus\mathrm{i}\mathbb{R}1\!\!1_U$$

where $\mathfrak{sl}(U)$ is the Lie subalgebra of all trace-free endomorphisms, as any $k\in\mathrm{End}\,U$ can be uniquely decomposed as $k=(k-\frac{1}{2}\,\mathrm{Tr}\,k\,1\!\!1_U)+\frac{1}{2}\,\mathrm{Tr}\,k\,1\!\!1_U$, and the trace can be further decomposed into its real and imaginary parts. Now recalling $H\subset U\otimes\overline{U}$ we introduce $\mathbb{R}$-linear maps $\pi\colon\ \mathrm{End}\,H\to\mathrm{End}\,U$ and $\imath\colon\ \mathrm{End}\,U\to\mathrm{End}\,H$ as follows. The former is defined via traces and can be best expressed in component form as

$$(\pi K)^A{}_B=\tfrac{1}{2}\,K^{AA^\cdot}{}_{BA^\cdot}-\tfrac{1}{8}\,K^{CC^\cdot}{}_{CC^\cdot}\,\delta^A{}_B\,.$$

The latter is defined as $\imath k\equiv k\otimes\bar{1\!\!1}+1\!\!1\otimes\bar{k}$, and is expressed in component form as

$$(\imath k)^{AA^\cdot}{}_{BB^\cdot}=k^A{}_B\,\delta^{A^\cdot}{}_{B^\cdot}+\delta^A{}_B\,\bar{k}^{A^\cdot}{}_{B^\cdot}\,.$$

The following statements are then easily checked:

- $\pi 1\!\!1_H=\frac{1}{2}\,1\!\!1_U\,,\ \imath 1\!\!1_U=2\,1\!\!1_H\,.$
- π and p have the same kernel: the symmetric traceless sector of $\mathrm{End}\,H$.
- The kernel of $\imath$ is the imaginary part of the identity sector.
- Both p and $\imath$ are valued onto $\mathfrak{so}(H,g)\oplus\mathbb{R}1\!\!1_H$.
- The restriction of π to $\mathfrak{so}(H,g)\oplus\mathbb{R}1\!\!1_H$ and the restriction of $\imath$ to $\mathfrak{sl}(U)\oplus\mathbb{R}1\!\!1_U$ are inverse Lie-algebra isomorphisms.

[2] $\mathrm{End}\,H$ together with the ordinary commutator is a Lie algebra.

Remark. The definition of $\imath$ is crafted in such a way that the action of $\imath k$ on isotropic elements $u\otimes\bar{u}\in H$ is determined by the Leibnitz rule $\imath k(u\otimes\bar{u}) = ku\otimes\bar{u}+u\otimes\bar{k}\bar{u}$. This is perhaps the most relevant aspect of this matter in relation to Lie derivatives of spinors. Also note that this expression is closely related to the decomposition $\Phi_{AA^{\cdot}BB^{\cdot}} = \phi_{AB}\,\bar{\varepsilon}_{A^{\cdot}B^{\cdot}} + \varepsilon_{AB}\,\bar{\phi}_{A^{\cdot}B^{\cdot}}$, valid for a Minkowski space 2-tensor Φ whose symmetric part is proportional to the Lorentz metric.

We now introduce a further map, the $\mathbb{R}$-linear inclusion[3]

$$\varkappa\colon\ \mathrm{End}\,U \to \mathrm{End}\,W : k \mapsto \varkappa(k) \equiv (k, -\bar{k}^*)\,,$$

where $\bar{k}^* : \overline{U}^* \to \overline{U}^*$ is the conjugate transpose of k.

The composition $\pi\circ\varkappa\colon\ \mathrm{End}\,H \to \mathrm{End}\,W$ can be then expressed in terms of the components of the Dirac map as

$$\varkappa(\pi K) = \tfrac{1}{8}K_{\lambda\mu}\,(\gamma^\lambda\gamma^\mu - \gamma^\mu\gamma^\lambda) + \tfrac{1}{8}K^\nu{}_\nu\,\gamma_5\,,$$

where $\mathrm{i}\,\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ is the element in the Dirac algebra corresponding to the natural volume form of H.

A diagram of the mutual relations among the introduced maps may be useful:

$$\begin{tikzcd}
\mathrm{End}\,H \arrow[r, "\pi"] \arrow[d, "\mathrm{p}"'] & \mathfrak{sl}(U)\oplus\mathbb{R}1\!\!1_U \subset \mathrm{End}\,U \arrow[r, "\varkappa"] \arrow[dl, "\imath"] & \mathrm{End}\,W \\
\mathfrak{so}(H,g)\oplus\mathbb{R}1\!\!1_H & &
\end{tikzcd}$$

1.5. **Spinor bundles and connections.** We consider a vector bundle $S \rightarrowtail M$ with complex 2-dimensional fibers over the real manifold M. For the moment we make no special assumption about M, including dimension, nor we assume any special relation between S and the tangent space $\mathrm{T}M$: that relation is mediated by a soldering form, which will be introduced as a subsequent step in §1.7.

The constructions of §1.2–1.4 now yield bundles $\mathbb{L}$, U, H and W over M, with smooth natural structures; a local frame (ξ_A) of S yields the associated frames of the other bundles. Moreover we'll use local coordinates (x^a) on M. We ramark that the fibers of H, in particular, are endowed with a Lorentz metric.

A $\mathbb{C}$-linear connection Γ of $S \rightarrowtail M$, called a 2-*spinor connection*, is expressed by coefficients $\Gamma_a{}^A{}_B : M \to \mathbb{C}$. Their complex conjugates are the coefficents $\bar{\Gamma}_a{}^{A^{\cdot}}{}_{B^{\cdot}}$ of the induced *conjugate connection* $\bar{\Gamma}$ of $\overline{S} \rightarrowtail M$, characterized by the rule $\nabla\bar{s} = \overline{\nabla s}$. By a standard argument we also get the corresponding *dual connection* of $S^* \rightarrowtail M$ and *anti-dual connection* of $\overline{S}^* \rightarrowtail M$. Moreover Γ yields linear connections of all bundles associated with S. If we fix a reference connection B (a 'gauge') then $\Gamma - \mathrm{B}$ is a tensor field valued in $\mathrm{T}^*M\otimes\mathrm{End}\,S$; hence, with proper care, we can describe

[3] A detailed examination in two-spinor terms of Lie groups and Lie algebras involved in spinor and Minkowsky space geometries can be found in a previous work [8].

the relations among the various connections in terms of bundle endomorphisms using the notions exposed in §1.4, with obvious extensions of the needed operations. In particular:

- the induced connection of $\wedge^2 S$ is denoted as $\hat{\digamma} \equiv \operatorname{Tr} \digamma$, with coefficients $\hat{\digamma}_a = \digamma_a{}^A{}_A$;
- the induced connection of $\mathbb{L}$ has the coefficients $G_a \equiv \frac{1}{2}(\digamma_a{}^A{}_A + \bar{\digamma}_a{}^{A^\bullet}{}_{A^\bullet})$, namely $\nabla_a l = -G_a\, l$, and can be regarded as the "real part" of $\hat{\digamma}$;
- the induced connection of $S \otimes \overline{S}$ is denoted as $\imath\digamma$, with coefficients
$$(\imath\digamma)_a{}^{AA^\bullet}{}_{BB^\bullet} = \digamma_a{}^A{}_B\, \delta^{A^\bullet}{}_{B^\bullet} + \delta^A{}_B\, \bar{\digamma}_a{}^{A^\bullet}{}_{B^\bullet} \;;$$
- the induced connection of U is denoted as $\tilde{\digamma}$, with coefficients $\tilde{\digamma}_a{}^A{}_B = \digamma_a{}^A{}_B - \frac{1}{2}\, G_a\, \delta^A{}_B$;
- the induced connection of H is denoted as $\tilde{\Gamma} \equiv \imath\tilde{\digamma}$, with coefficients
$$\tilde{\Gamma}_a{}^{AA^\bullet}{}_{BB^\bullet} = \tilde{\digamma}_a{}^A{}_B\, \delta^{A^\bullet}{}_{B^\bullet} + \delta^A{}_B\, \bar{\tilde{\digamma}}_a{}^{A^\bullet}{}_{B^\bullet} = (\imath\digamma)_a{}^{AA^\bullet}{}_{BB^\bullet} - G_a\, \delta^A{}_B\, \delta^{A^\bullet}{}_{B^\bullet}\,.$$

Above, induced connections are expressed in the frames induced by (ξ_A). Conversely
$$\begin{aligned}\tilde{\digamma}_a{}^A{}_B &= \pi(\imath\digamma)_a{}^A{}_B = \tfrac{1}{2}\,(\imath\digamma)_a{}^{AA^\bullet}{}_{BA^\bullet} - \tfrac{1}{8}\,(\imath\digamma)_a{}^{CC^\bullet}{}_{CC^\bullet}\,\delta^A{}_B \\ &= \tfrac{1}{2}\,\tilde{\Gamma}_a{}^{AA^\bullet}{}_{BA^\bullet} = \pi(\imath\tilde{\digamma})_a{}^A{}_B\,.\end{aligned}$$

With regard to the latter expression, in particular, we note that $\tilde{\Gamma}_a{}^{AA^\bullet}{}_{AA^\bullet} = 0$. Furthermore $\tilde{\Gamma}$ turns out to be a *metric connection*, preserving the Lorentz fiber structure of H; the notion of torsion, on the other hand, needs a soldering form, and will be introduced later (§1.7).

It's not difficult to check that similar relations hold among the curvature tensor R of $\digamma$ and the curvature tensors of the induced connections. In particular
$$R_{ab}{}^{AA^\bullet}{}_{BB^\bullet} \equiv (\imath R)_{ab}{}^{AA^\bullet}{}_{BB^\bullet} = R_{ab}{}^A{}_B\, \delta^{A^\bullet}{}_{B^\bullet} + \delta^A{}_B\, \bar{R}_{ab}{}^{A^\bullet}{}_{B^\bullet}\,,$$
$$\begin{aligned}\tilde{R}_{ab}{}^A{}_B &= R_{ab}{}^A{}_B + \tfrac{1}{2}\,\partial_{[a}G_{b]}\,\delta^A{}_B \\ &= \tfrac{1}{2}\,R_{ab}{}^{AA^\bullet}{}_{BA^\bullet} - \tfrac{1}{8}\,R_{ab}{}^{CC^\bullet}{}_{CC^\bullet}\,\delta^A{}_B = \tfrac{1}{2}\,\tilde{R}_{ab}{}^{AA^\bullet}{}_{BA^\bullet}\,.\end{aligned}$$

We also consider the induced connection $\varkappa(\tilde{\digamma})$ on the 4-spinor bundle $W \equiv U \oplus \overline{U}^*$. This can be then expressed in terms of the components of the Dirac map as
$$\varkappa(\tilde{\digamma}) = \tfrac{1}{8}\tilde{\Gamma}_a{}^\lambda{}_\mu\,(\gamma_\lambda\gamma^\mu - \gamma^\mu\gamma_\lambda)\,,$$
since $\tilde{\Gamma}_a{}^\lambda{}_\lambda = 0$, where the components of $\tilde{\Gamma}$ are now expressed in the Pauli frame (τ_λ).

Last but not least we consider the induced connection Y of $\wedge^2 U$, whose fibers (§1.2) have a natural Hermitian structure. Indeed Y preserves that structure, and its coefficients can be written as $\mathrm{i}Y_a$ where $Y_a = \frac{1}{2\mathrm{i}}(\digamma_a{}^A{}_A - \bar{\digamma}_a{}^{A^\bullet}{}_{A^\bullet})$ is the imaginary part of $\hat{\digamma}_a$; namely $\hat{\digamma}_a = G_a + \mathrm{i}Y_a$. In particular, we get $\nabla_a\varepsilon = \mathrm{i}Y_a\,\varepsilon$.

1.6. **Breaking of dilaton symmetry.** In a general theory of fields that are sections of the various bundles derived from S one has to allow a *dilaton* field, that in our context can be described as a section $M \to \mathbb{L}$. In the literature this issue has been considered under various angles [1, 13, 14, 17, 21, 27, 34, 39, 40, 44, 45, 47], though a conclusive approach seems to be still lacking. One intriguing possibility is that the dilaton be closely related to the Higgs field. My own ideas about such speculations have been expressed in two papers [6, 9].

In the sequel, however, we'll work for simplicity in a "conservative" setup in which the dilaton symmetry is broken by some mechanism we do not worry about here. This enables a formulation, sketched in §1.8, of a theory of Einstein-Cartan-Maxwell--Dirac fields which is based on geometric constructions that use S with the only added assumption of such symmetry breaking.

A weak form of this assumption, sufficient for our present purposes, can be expressed as the requirement that the connection G induced on $\mathbb{L}$ has vanishing curvature. Hence one can always find local charts such that $G_a = 0$, and this amounts to gauging away the conformal symmetry.

In practice we may wish to simplify certain arguments by making the stronger assumption that the bundle $\mathbb{L} \rightarrowtail M$ be trivial, that is a global product. This means that we actually regard $\mathbb{L}$ just a semi-vector space, the space of length units. In a natural unit setting, coupling constants now arise as elements in $\mathbb{L}$.

1.7. **Two-spinor soldering form.** Henceforth we assume that M is a real 4-dimensional manifold, and consider sections

$$\Theta\colon M \to \mathbb{L} \otimes H \otimes \mathrm{T}^*M\,.$$

Note that such Θ can be seen as a linear morphism $\mathrm{T}M \to \mathbb{L} \otimes H$, and, if it is non-degenerate, as a *scaled soldering form*, since it relates the spacetime geometry to the algebraic fiber structures associated with the spinor bundle. If one fixes an orthonormal frame of H then Θ can be regarded as the assignement of an orthonormal spacetime frame, whence the term *tetrad* used in physics works.

We write the coordinate expression of Θ as

$$\Theta = \Theta^\lambda_a\,\tau_\lambda \otimes \mathrm{dx}^a = \Theta^{AA^{\cdot}}_a\,\zeta_A \otimes \bar\zeta_{A^{\cdot}} \otimes \mathrm{dx}^a\,,$$

where the coefficients Θ^λ_a and $\Theta^{AA^{\cdot}}_a$ are $\mathbb{L}$-valued, i.e. have the physical dimensions of a *length*.

Given a soldering form, the geometric structure of the fibers of H yields a similar, *scaled* structure on the fibers of $\mathrm{T}M$. Namely if we now denote by $\tilde g$, $\tilde\eta$ and $\tilde\gamma$ the Lorentz metric, the $\tilde g$-normalized volume form and the Dirac map of H, we get similar spacetime objects

$$g \equiv \Theta^*\tilde g = \tilde g_{\lambda\mu}\,\Theta^\lambda_a\,\Theta^\mu_b\,\mathrm{dx}^a \otimes \mathrm{dx}^b = \varepsilon_{AB}\varepsilon_{A^{\cdot}B^{\cdot}}\,\Theta^{AA^{\cdot}}_a\,\Theta^{BB^{\cdot}}_b\,\mathrm{dx}^a \otimes \mathrm{dx}^b\,,$$

$$\eta \equiv \Theta^*\tilde\eta = \det\Theta\,\mathrm{d}^4\mathrm{x} \equiv \det\Theta\,\mathrm{dx}^0 \wedge \mathrm{dx}^1 \wedge \mathrm{dx}^2 \wedge \mathrm{dx}^3\,,$$

$$\gamma \equiv \tilde\gamma \circ \Theta = \Theta^\lambda_a\,\gamma_\lambda \otimes \mathrm{dx}^a \equiv \gamma_a \otimes \mathrm{dx}^a\,.$$

The first relation above shows that Θ can be regarded as a "square root of the metric".

The components of the inverse morphism $\overleftarrow{\Theta}\colon M \to \mathbb{L}^{-1}\otimes H^*\otimes \mathrm{T}M$ can be expressed as

$$\overleftarrow{\Theta}{}^b_\mu \equiv \Theta^b_\mu \equiv \Theta^\lambda_a\,\tilde{g}_{\lambda\mu}\,g^{ab}\,.$$

Moreover a Pauli frame yields the orthonormal scaled spacetime frame and dual frame

$$\big(\Theta_\lambda\big) \equiv \overleftarrow{\Theta}(\tau_\lambda) = \big(\Theta^a_\lambda\,\partial\mathsf{x}_a\big)\,, \qquad \big(\Theta^\lambda\big) \equiv \Theta^*(\tau^\lambda) = \big(\Theta^\lambda_a\,\mathrm{d}\mathsf{x}^a\big)\,.$$

If $\digamma$ is a 2-spinor connection then a non-degenerate tetrad $\Theta\colon \mathrm{T}M \to \mathbb{L}\otimes H$ yields a unique connection Γ on $\mathrm{T}M$, characterized by the condition that Θ be covariantly constant with respect to the couple $(\Gamma,\tilde{\Gamma})$. Indeed the condition $\nabla\Theta = 0$ reads

$$\partial_b\Theta^\lambda_a + \Gamma^c_{b\,a}\,\Theta^\lambda_c - \tilde{\Gamma}^{\,\lambda}_{b\,\mu}\,\Theta^\mu_a = 0\,,$$

while the components of Γ in the orthonormal frame $\big(\Theta_\lambda\big)$ coincide with the components of $\tilde{\Gamma}$ in the associated Pauli frame: $\Gamma_{a\,\mu}^{\,\lambda} = \tilde{\Gamma}_{a\,\mu}^{\,\lambda}$. Moreover Γ is metric, $\nabla[\Gamma]g = 0$. The curvature tensors of Γ and $\tilde{\Gamma}$ are similarly related by $R_{ab}{}^{\lambda}{}_{\mu} = \tilde{R}_{ab}{}^{\lambda}{}_{\mu}$, that is

$$R_{ab}{}^{c}{}_{d} = \tilde{R}_{ab}{}^{\lambda}{}_{\mu}\,\Theta^c_\lambda\,\Theta^\mu_d\,.$$

The Ricci tensor and the scalar curvature can be expressed as

$$R_{ad} = R_{ab}{}^{b}{}_{d} = \tilde{R}_{ab}{}^{\lambda}{}_{\mu}\,\Theta^b_\lambda\,\Theta^\mu_d\,, \qquad R_a{}^a = \tilde{R}_{ab}{}^{\lambda\mu}\,\Theta^b_\lambda\,\Theta^a_\mu\,.$$

In general Γ will have non-vanishing torsion, which can be expressed as

$$\Theta^\lambda_c\,T^c{}_{ab} = \partial_{[a}\Theta^\lambda_{b]} + \Theta^\mu_{[a}\,\tilde{\Gamma}_{b]}{}^{\lambda}{}_{\mu}\,.$$

1.8. **Einstein-Cartan-Maxwell-Dirac fields.** The field theory we are going to sketch, as presented in previous papers [2, 3, 4], is based on "minimal geometric data" in the sense that the unique such datum is a vector bundle $S \rightarrowtail M$, with complex 2-dimensional fibers and real 4-dimensional base manifold. The basic idea is to assume no further background structure: all other bundles and fixed geometric objects are derived from S using only geometrical constructions. Any needed bundle section which is not determined by S is assumed to be a field. A natural Lagrangian can then be written, yielding a field theory which turns out to be essentially equivalent to a classical theory of Einstein-Cartan-Maxwell-Dirac fields, provided that one makes the further assumption that the dilatonic symmetry is broken as described in §1.6; otherwise we deal with a more general theory. Accordingly we regard $\mathbb{L}$ as a fixed semi-vector space, whose unique role consists of taking care of physical dimensions in a natural unit setting ($\hbar = c = 1$).

The fields are assumed to be the soldering form Θ, the 2-spinor connection $\digamma$, the electromagnetic field F and the electron field ψ. The gravitational field is represented both by Θ and $\tilde{\digamma}$, the latter being regarded as the gravitational part of $\digamma$. If Θ is non-degenerate then one obtains, as in the standard metric-affine approach [16, 21], essentially the Einstein equation and the equation for torsion; the metricity of the spacetime connection is a further consequence. The theory, however, is non-singular also if Θ is degenerate. Also note that in this approach the spacetime metric g and the spacetime connection Γ are not independent fields, but rather byproducts of the formalism. Thus we cannot just require the torsion to vanish.

The Dirac field $\psi \equiv (u, \chi)$ is a section $M \to \mathbb{L}^{-3/2} \otimes W$, representing a particle with one-half spin, mass $m \in \mathbb{L}^{-1}$ and charge $q \in \mathbb{R}$.

We assign the role of the electromagnetic potential to another sector of $\digamma$, namely the induced Hermitian connection Y of $\wedge^2 U$, whose coefficients we denote as $\mathrm{i}Y_a$. Locally one also writes $Y_a \equiv qA_a$ where A is a 1-form.

The electromagnetic field is represented by a spacetime 2-form F or, equivalently, by a section $\tilde{F}\colon M \to \mathbb{L}^{-2} \otimes \wedge^2 H^*$ related to it by $F \equiv \Theta^* \tilde{F}$. The relation between Y and F follows as one of the field equations.

The total Lagrangian density $\mathcal{L} = (\ell_{\text{grav}} + \ell_{\text{em}} + \ell_{\text{Dir}})\, \mathrm{d}^4\mathsf{x}$ is the sum of gravitational, electromagnetic and Dirac terms. These can be written in coordinate-free form, but the coordinate expressions are perhaps more readable without special explanations. We have

$$\ell_{\text{grav}} = \tfrac{1}{8\Bbbk}\, \varepsilon_{\lambda\mu\nu\rho}\, \varepsilon^{abcd}\, \tilde{R}_{ab}{}^{\lambda\mu}\, \Theta^{\nu}_{c}\, \Theta^{\rho}_{d}\,,$$

$$\ell_{\text{em}} = -\tfrac{1}{4}\, \varepsilon^{abcd}\, \varepsilon_{\lambda\mu\nu\rho}\, \partial_a Y_b\, \tilde{F}^{\lambda\mu}\, \Theta^{\nu}_{c}\Theta^{\rho}_{d} + \tfrac{1}{4}\, \tilde{F}^{\lambda\mu}\, \tilde{F}_{\lambda\mu}\, \det\Theta\,,$$

$$\ell_{\text{Dir}} = \tfrac{\mathrm{i}}{\sqrt{2}}\, \breve{\Theta}^{a}_{AA^{\bullet}} \big(\nabla_{\!a} u^A\, \bar{u}^{A^{\bullet}} - u^A\, \nabla_{\!a} \bar{u}^{A^{\bullet}} + \varepsilon^{AB} \bar{\varepsilon}^{A^{\bullet}B^{\bullet}} (\bar{\chi}_B\, \nabla_{\!a} \chi_{B^{\bullet}} - \nabla_{\!a} \bar{\chi}_B\, \chi_{B^{\bullet}})\big) \\ - m\, (\bar{\chi}_A u^A + \chi_{A^{\bullet}} \bar{u}^{A^{\bullet}})\, \det\Theta\,,$$

where

$$\breve{\Theta}^{a}_{AA^{\bullet}} \equiv \tfrac{1}{\sqrt{2}}\, \sigma^{\lambda}{}_{AA^{\bullet}} \breve{\Theta}^{a}_{\lambda} \equiv \tfrac{1}{\sqrt{2}}\, \sigma^{\lambda}{}_{AA^{\bullet}} \big(\tfrac{1}{3!}\, \varepsilon^{abcd}\, \varepsilon_{\lambda\mu\nu\rho}\, \Theta^{\mu}_{b}\Theta^{\nu}_{c}\Theta^{\rho}_{d}\big)$$

and $\Bbbk \in \mathbb{L}^2$ is Newton's gravitational constant.

The main results obtained by writing down the Euler-Lagrange equations deriving from $\mathcal{L}$ can be summarized as follows.

- The Θ-sector corresponds (in the non-degenerate case) to the Einstein equations.
- The $\tilde{\Gamma}$-sector gives the equation for torsion. Hence one sees that the spinor field is a source for torsion, and that in this context a possible torsion-free theory is not natural.
- The F-sector reads $F = 2\, \mathrm{d}Y$ in the non-degenerate case, and of course this yields the first Maxwell equation $\mathrm{d}F = 0$.
- The Y-sector reduces, in the non-degenerate case, to the 2nd Maxwell equation $\tfrac{1}{2} *\mathrm{d}{*F} = j$, where the 1-form j is the *Dirac current*.
- The $\bar{u}$- and $\bar{\chi}$-sectors yield the *Dirac equation* $(\mathrm{i}\,\nabla\!\!\!\!/ - m + \tfrac{\mathrm{i}}{2}\, \gamma^a T^b{}_{ab})\psi = 0$ for $\psi \equiv (u, \chi)$.
- The u- and χ-sectors yield the Dirac equation for the Dirac adjoint $\bar{\psi} = (\bar{\chi}, \bar{u})$.

2. Lie derivatives in spinor geometry and tetrad gravity

In the literature, the notion of Lie derivative of a spinor field is mainly studied in terms of matrix group representations. We propose a somewhat different approach by looking for natural constructions in the involved bundles, requiring the Leibnitz rule to hold whenever it is applicable. For a Dirac spinor we will find a straightforward extension of the known notion, that reduces to it in the case of a Killing vector field and vanishing torsion. Furthermore we will introduce Lie derivatives of a soldering form and of a spinor connection, and study the mutual relations among these operations. We stress that these constructions and results do not need field equations, so that the considered fields (spinor, tetrad, spinor connection) are, in general, mutually independent; the condition $\nabla\Theta = 0$, on the other hand, is related to the spacetime connection, an induced auxiliary object.

For any vector field $x\colon M \to \mathrm{T}M$ we will use the shorthand notation

$$\Xi \equiv \overleftarrow{\Theta}\lrcorner(\nabla x + x\lrcorner T)\lrcorner\Theta : M \to H \otimes H^* ,$$

with the coordinate expression

$$\Xi^{\lambda}{}_{\mu} = (\nabla_a x^b + x^c\, T^b{}_{ca})\, \overleftarrow{\Theta}^a_{\mu}\, \Theta^{\lambda}_b \,.$$

Moreover we set (§1.4) $\hat{\Xi} \equiv \mathrm{p}\Xi$, which has the coordinate expression

$$\hat{\Xi}^{\lambda}{}_{\mu} = \tfrac{1}{2}\big(\Xi^{\lambda}{}_{\mu} - (\Xi^{\dagger})^{\lambda}{}_{\mu}\big) + \tfrac{1}{4}\,\Xi^{\nu}{}_{\nu}\,\delta^{\lambda}{}_{\mu} \,.$$

Finally we set

$$\xi \equiv \pi(\Xi) = \pi(\hat{\Xi}) : M \to \operatorname{End} U ,$$

$$\Rightarrow \quad \xi^{A}{}_{B} = \tfrac{1}{2}\,\Xi^{AA^{\boldsymbol{\cdot}}}{}_{BA^{\boldsymbol{\cdot}}} - \tfrac{1}{8}\,\Xi^{CC^{\boldsymbol{\cdot}}}{}_{CC^{\boldsymbol{\cdot}}}\,\delta^{A}{}_{B} \,.$$

2.1. **Lie derivative of spinors.** We start by looking for a natural definition of Lie derivative of sections $w\colon M \to H$ with respect to a vector field $x\colon M \to \mathrm{T}M$. We observe that $\overleftarrow{\Theta}w$ is a vector field on M, and so is the Lie bracket $[x, \overleftarrow{\Theta}w] \equiv \mathrm{L}_x(\overleftarrow{\Theta}w)$. Then we obtain the section $\Theta[x, \overleftarrow{\Theta}w] : M \to H$. By a straightforward coordinate calculation one finds

$$\Theta[x, \overleftarrow{\Theta}w] = \nabla_x w^{\lambda} - \Xi\lrcorner w ,$$

where $\nabla_x w$ denotes the covariant derivative of w with respect to a connection $\tilde{\Gamma}$ of $H \rightarrowtail M$, possibly determined by a spinor connection (§1.5). We remark that Ξ is defined in terms of ∇x, which in turn is defined in terms of the spacetime connection characterized by $\nabla\Theta = 0$; this is how connections enter Lie derivatives of spinors.

We now face the following issue: in order to recover $\Theta[x, \overleftarrow{\Theta}w]$ from an operation performed on 2-spinors by means of the Leibnitz rule, $\Xi\colon M \to \operatorname{End} H$ must be valued into the sub-bundle $\mathfrak{so}(H, g) \oplus \mathbb{R}1\!\!1_H$ (§1.4). In the literature this is a known issue, which is dealt with by using various arguments. Actually the discussion turns out to be somewhat involved if the underlying use of the soldering form is not explicitely stated. In our context, remembering the discussion in §1.4, we immediately see the naturaleness of just replacing Ξ by $\hat{\Xi}$, as well as of the analogous definition of Lie derivative of a 2-spinor in terms of $\xi \equiv \pi(\Xi)$. We set:

Definition 1. The *Lie derivatives* of a section $w : M \to H$, of a section $u\colon M \to U$ and of a section $\chi\colon M \to \overline{U}^*$ with respect to a vector field $x\colon M \to \mathrm{T}M$ are defined to be

$$\begin{aligned}
\mathrm{L}_X w &\equiv \nabla_X w - \hat{\Xi}(w)\,, \\
\mathrm{L}_X u &\equiv \nabla_X u - \xi(u)\,, \\
\mathrm{L}_X \chi &\equiv \nabla_X \chi + \bar{\xi}^*(\chi)\,.
\end{aligned}$$

Moreover the Lie derivatives of the conjugate objects are defined in the obvious way as

$$\mathrm{L}_X \bar{u} \equiv \overline{\mathrm{L}_X u} = \nabla_X \bar{u} - \bar{\xi}(\bar{u})\,, \qquad \mathrm{L}_X \bar{\chi} \equiv \overline{\mathrm{L}_X \chi} = \nabla_X \bar{\chi} + \xi^*(\bar{\chi})\,.$$

It is then easy to check that the natural Leibnitz rules are fulfilled, namely:

Proposition 1. *We have*

$$\mathrm{L}_X(u \otimes \bar{u}) = (\mathrm{L}_X u) \otimes \bar{u} + u \otimes \mathrm{L}_X \bar{u}\,, \qquad X.\langle \bar{\chi}, u\rangle = \langle \mathrm{L}_X \bar{\chi}, u\rangle + \langle \bar{\chi}, \mathrm{L}_X u\rangle\,,$$

and the like.

We find the coordinate expressions

$$\mathrm{L}_X u^A = X^a\,(\partial_a u^A - \Gamma_a{}^A{}_B\, u^B) - \xi^A{}_B\, u^B\,, \qquad \mathrm{L}_X \chi_{\dot{A}} = X^a\,(\partial_a \chi_{\dot{A}} + \bar{\Gamma}_a{}^{\dot{B}}{}_{\dot{A}}\, \chi_{\dot{B}}) + \xi^{\dot{B}}{}_{\dot{A}}\, \chi_{\dot{B}}\,.$$

Now the notion of Lie derivative of a 4-spinor $\psi \equiv (u, \chi) \colon M \to W \equiv U \oplus \overline{U}^*$ naturally follows, yielding

$$\begin{aligned}
\mathrm{L}_X \psi &= \nabla_X \psi - \big(\xi\,, -\bar{\xi}^*\big)\psi = \nabla_X \psi - \varkappa(\pi\Xi) \\
&= \nabla_X \psi - \tfrac{1}{8}\,\Xi_{\lambda\mu}\,(\gamma^\lambda\gamma^\mu - \gamma^\mu\gamma^\lambda)\psi + \tfrac{1}{8}\,\Xi^\nu{}_\nu\,\gamma_5\psi\,.
\end{aligned}$$

Remark. When the torsion vanishes and X is a Killing vector field (then $\Xi^\nu{}_\nu = 0$) one essentially gets the usual Lie derivative of Dirac spinors [32], though a careful reader may notice an opposite sign in the second term. The standard expression can be recovered by exchanging the roles of the bundles U and $\overline{U}^*$, so that the difference can be eventually ascribed to conventions affecting representations of the involved Lie algebras. Similarly one sees that our expression for $\mathrm{L}_X u^A$ is the same as that in Penrose-Rindler [43], §6.6, when the torsion vanishes and X is a conformal Killing vector field.

Remark. The Fermi transport of spinors can be introduced by an analogous construction [5] starting from the Fermi transport of world-vectors.

2.2. **Lie derivative of a soldering form.**

Though the notion of Lie derivative of spinors proposed in §2.1 is well-defined for any vector field X, it is actually independent of the symmetric trace-free part of $\Xi \equiv \nabla X + X \rfloor T$. However that part has not merely disappeared from view, but is related to a natural definition of Lie derivative of the soldering form that follows from requiring the validity of the Leibnitz rule.

Proposition 2. *Let* $\Theta\colon M \to \mathbb{L}\otimes H\otimes \mathrm{T}^*M$ *be a soldering form and* $x\colon M\to \mathrm{T}M$ *a vector field. Then there exists a unique section*

$$\mathrm{L}_x\overleftarrow{\Theta}\colon M\to \mathbb{L}^{-1}\otimes H^*\otimes \mathrm{T}M$$

such that for any section $w\colon M\to H$ *one has*

$$\mathrm{L}_x(\overleftarrow{\Theta}w) = (\mathrm{L}_x\overleftarrow{\Theta})w + \overleftarrow{\Theta}(\mathrm{L}_x w)\,,$$

namely

$$\mathrm{L}_x\overleftarrow{\Theta} = \overleftarrow{\Theta}\lrcorner(\hat{\Xi}-\Xi)\,.$$

Proof. Remembering (§2.1) $\Theta[x,\overleftarrow{\Theta}w]=\nabla_x w-\Xi(w)$, the required Leibniz rule turns out to be equivalent to

$$\begin{aligned}\Theta\lrcorner\mathrm{L}_x\overleftarrow{\Theta}(w) &= \Theta\big(\mathrm{L}_x(\overleftarrow{\Theta}w)\big)-\mathrm{L}_x w = \nabla_x w-\Xi(w)-\big(\nabla_x w-\hat{\Xi}(w)\big)\\ &=(\hat{\Xi}-\Xi)(w)\,.\end{aligned}$$

□

Since $\Theta\lrcorner\overleftarrow{\Theta}=1\!\!1_H$ we now obtain the definition of $\mathrm{L}_x\Theta$ by requiring

$$0=\mathrm{L}_x(\Theta\lrcorner\overleftarrow{\Theta})=(\mathrm{L}_x\Theta)\lrcorner\overleftarrow{\Theta}+\Theta\lrcorner\mathrm{L}_x\overleftarrow{\Theta}\,.$$

Definition 2. The *Lie derivatives* of a soldering form Θ and its inverse $\overleftarrow{\Theta}$ are

$$\mathrm{L}_x\Theta=(\Xi-\hat{\Xi})\lrcorner\Theta\,,\qquad \mathrm{L}_x\overleftarrow{\Theta}=\overleftarrow{\Theta}\lrcorner(\hat{\Xi}-\Xi)\,.$$

In particular we obtain the coordinate expression

$$\mathrm{L}_x\Theta^\lambda_a=(\Xi^\lambda{}_\mu-\hat{\Xi}^\lambda{}_\mu)\,\Theta^\mu_a=\tfrac12\,(\Xi^\lambda{}_\mu+\Xi_\mu{}^\lambda)\,\Theta^\mu_a-\tfrac14\,\Xi^\nu{}_\nu\,\Theta^\lambda_a\,.$$

Remark. The above expression seems not to contain the derivatives of the components of Θ; these are actually contained in the torsion, which is contained in Ξ. In fact we can recover our result by a straightforward coordinate calculation from

$$\mathrm{L}_x\Theta^\lambda_a=x^b\,\partial_b\Theta^\lambda_a+\Theta^\lambda_b\,\partial_a x^b-x^b\,\tilde{\Gamma}_b{}^\lambda{}_\mu\,\Theta^\mu_a-\hat{\Xi}^\lambda{}_\mu\,\Theta^\mu_a$$

and then using $\partial_b\Theta^\lambda_a=\tilde{\Gamma}_b{}^\lambda{}_\mu\,\Theta^\mu_a-\Gamma_b{}^c{}_a\,\Theta^\lambda_c$ which is the coordinate expression of $\nabla\Theta=0$.

2.3. **Lie derivative of a spinor connection.** The Lie derivative of a linear connection of the tangent bundle of a manifold is a known notion [53]. Let Γ be an arbitrary linear connection of $\mathrm{T}M\rightarrowtail M$; then its Lie derivative along a vector field x is the tensor field

$$\mathrm{L}_x\Gamma\colon M\to \mathrm{T}^*M\otimes\mathrm{T}M\otimes\mathrm{T}^*M$$

characterized by the requirement that the identity

$$\mathrm{L}_x\Gamma\lrcorner z=\nabla\mathrm{L}_x z-\mathrm{L}_x\nabla z$$

holds for any vector field $z\colon M\to\mathrm{T}M$. Its coordinate expression turns out to be

$$\begin{aligned}\mathrm{L}_x\Gamma_a{}^b{}_c&=-\nabla_a\nabla_c x^b-\nabla_a(x^d\,T^b{}_{dc})+x^d R_{adc}{}^b\\ &=-\nabla_a\Xi^b{}_c+x^d R_{adc}{}^b\,.\end{aligned}$$

This notion can be applied in particular to the Riemannian spacetime connection, and as such it appears in the literature mainly in considerations related to energy tensors [31, 41], possibly in a somewhat disguised form [20, 33].

The Lie derivatives of linear connections of $U \rightarrowtail M$ and of $H \rightarrowtail M$ can be obtained by extending the above construction, since we avail of the notions of Lie derivatives of sections of these bundles. We preliminarly remark that a linear connection of any vector bundle $E \rightarrowtail M$ can be regarded as a section of an affine bundle over M whose derived vector bundle [10] is $\mathrm{T}^*M \otimes \operatorname{End} E \equiv \mathrm{T}^*M \otimes E \otimes E^* \rightarrowtail M$. Accordingly the Lie derivative of a linear connection, when it is well-defined, is valued into such vector bundle.

Moreover we note that if we avail of a notion of Lie derivative of sections $\sigma : M \to E$ with respect to a vector field on M, then the Leibnitz rule yields the Lie derivative of $\nabla\sigma \colon M \to \mathrm{T}^*M \otimes E$, where the covariant derivative is related to any connection.

Proposition 3. *Let* $\digamma$ *and* $\tilde{\Gamma}$ *be linear connections of* $U \rightarrowtail M$ *and* $H \rightarrowtail M$*, respectively, and* $x \colon M \to \mathrm{T}M$ *a vector field. Then:*

- *there exists a unique section*

$$\mathrm{L}_x \digamma \colon M \to \mathrm{T}^*M \otimes \operatorname{End} U$$

such that for every section $u \colon M \to U$ *one has*

$$\mathrm{L}_x \digamma(u) = \nabla \mathrm{L}_x u - \mathrm{L}_x \nabla u \ ,$$

namely

$$\mathrm{L}_x \digamma = -\nabla\xi - x \lrcorner R$$

where R *is the curvature tensor of* $\digamma$ *(§1.5);*

- *there exists a unique section*

$$\mathrm{L}_x \tilde{\Gamma} \colon M \to \mathrm{T}^*M \otimes \operatorname{End} H$$

such that for every section $w \colon M \to H$ *one has*

$$\mathrm{L}_x \tilde{\Gamma}(w) = \nabla \mathrm{L}_x w - \mathrm{L}_x \nabla w \,,$$

namely

$$\mathrm{L}_x \tilde{\Gamma} = -\nabla\hat{\Xi} - x \lrcorner \tilde{R}$$

where $\tilde{R}$ *is the curvature tensor of* $\tilde{\Gamma}$*.*

Proof. Coordinate calculations yield

$$(\nabla \mathrm{L}_x u - \mathrm{L}_x \nabla u)_a{}^A = (-\nabla_{\!a}\xi^A{}_B + x^d\, R_{ad}{}^A{}_B)\, u^B \,,$$

$$(\nabla \mathrm{L}_x w - \mathrm{L}_x \nabla w)_a{}^\lambda = (-\nabla_{\!a}\hat{\Xi}^\lambda{}_\mu + x^d\, \tilde{R}_{ad}{}^\lambda{}_\mu)\, w^\mu \,.$$

$\square$

Remark. For an arbitrary vector field x we have $\mathrm{L}_x \tilde{\Gamma}_a{}^\lambda{}_\lambda = -\nabla_{\!a}\hat{\Xi}^\lambda{}_\lambda \neq 0$, so that the "deformed connection" $\tilde{\Gamma} + \mathrm{L}_x\tilde{\Gamma}$ needs not be metric.

The linear connection $\tilde{\Gamma}$ in the above proposition is possibly unrelated to a spinor connection. However:

Proposition 4. *Let* $\tilde{\Gamma}$ *be the connection of* $H \rightarrowtail M$ *induced by the spinor connection* $\digamma$*. Then we have*

$$\mathrm{L}_X \digamma = \pi(\mathrm{L}_X \tilde{\Gamma}) \,, \qquad \mathrm{L}_X \tilde{\Gamma} = \imath(\mathrm{L}_X \digamma) \,.$$

Proof. It is straightforwardly checked by means of the relation between Ξ and ξ and the analogous relation (§1.5) between R and $\tilde{R}$. □

In coordinates, the above stated relations read

$$\mathrm{L}_X \digamma_{a\;B}^{\;A} = \tfrac{1}{2}\, \mathrm{L}_X \tilde{\Gamma}_{a\;\;B\dot{A}}^{\;A\dot{A}} - \tfrac{1}{8}\, \mathrm{L}_X \tilde{\Gamma}_{a\;\;C\dot{C}}^{\;C\dot{C}}\, \delta^{A}{}_{B} \,,$$

$$\mathrm{L}_X \tilde{\Gamma}_{a\;\;B\dot{B}}^{\;A\dot{A}} = \mathrm{L}_X \digamma_{a\;B}^{\;A}\, \delta^{\dot{A}}{}_{\dot{B}} + \delta^{A}{}_{B}\, \mathrm{L}_X \bar{\digamma}_{a\;\dot{B}}^{\;\dot{A}} \,.$$

We now recall that $\digamma$ yields connections of $\overline{U}$, U^*, $\overline{U}^*$. The Lie derivatives of all these are naturally defined by straightforward extensions of the above procedure, and their coordinate expressions are easily checked to be in the same mutual relations. Moreover we get the Lie derivative of the 4-spinor connection $\varkappa\tilde{\digamma}$ of W. By straightforward computations one finds:

Proposition 5. *We have*

$$\mathrm{L}_X (\varkappa\tilde{\digamma})_a = \tfrac{1}{8}\, \mathrm{L}_X \tilde{\Gamma}_{a\;\mu}^{\;\lambda}\, (\gamma_\lambda \gamma^\mu - \gamma^\mu \gamma_\lambda) + \tfrac{1}{8}\, \mathrm{L}_X \tilde{\Gamma}_{a\;\lambda}^{\;\lambda}\, \gamma_5 \,,$$

where $\mathrm{L}_X \tilde{\Gamma}_{a\;\lambda}^{\;\lambda} = -\nabla_a \hat{\Xi}^{\lambda}{}_{\lambda}$, $\mathrm{i}\, \gamma_5 \equiv \gamma_0\, \gamma_1\, \gamma_2\, \gamma_3$.

Our notion of Lie derivative of 2-spinors also naturally yields the Lie derivatives of the curvature tensors of $\digamma$ and $\tilde{\Gamma}$. We obtain the coordinate expressions

$$\mathrm{L}_X R_{ab\;B}^{\;\;A} = X^c\, \partial_c R_{ab\;B}^{\;\;A} + \partial_a X^c\, R_{cb\;B}^{\;\;A} + \partial_b X^c\, R_{ac\;B}^{\;\;A} + [R_{ab}\,, X^a \digamma_a + \xi\,]^{A}{}_{B} \,,$$

$$\mathrm{L}_X \tilde{R}_{ab\;\mu}^{\;\;\lambda} = X^c\, \partial_c \tilde{R}_{ab\;\mu}^{\;\;\lambda} + \partial_a X^c\, \tilde{R}_{cb\;\mu}^{\;\;\lambda} + \partial_b X^c\, \tilde{R}_{ac\;\mu}^{\;\;\lambda} + [\tilde{R}_{ab}\,, X^a \tilde{\Gamma}_a + \hat{\Xi}]^{\lambda}{}_{\mu} \,,$$

where $[R_{ab}\,, \xi\,]^{A}{}_{B} \equiv R_{ab\;C}^{\;\;A}\, \xi^{C}{}_{B} - \xi^{A}{}_{C}\, R_{ab\;B}^{\;\;C}$ and the like (brackets denote commutators of fiber endomorphisms).

Then it is not difficult to check that the algebraic relation between these two objects is essentially the same as the relation between $\digamma$ and $\tilde{\Gamma}$. Moreover let us regard $\digamma' \equiv \digamma + \mathrm{L}_X \digamma$ as a "deformed" spinor connection; then its curvature tensor turns out to be $R' = R + \mathrm{L}_X R$ up to terms which are of second order in the Lie derivatives. A similar statement holds true for the curvature of the deformed connection $\tilde{\Gamma}' \equiv \tilde{\Gamma} + \mathrm{L}_X \tilde{\Gamma}$.

Remark. For the reader who is familiar with the Frölicher-Nijenhuis bracket of tangent-valued forms [18, 36, 37], we can recast part of the above results in a convenient way. We first observe that if $E \rightarrowtail M$ is any vector bundle then an $\mathrm{End}E$-valued r-form $M \to \wedge^r \mathrm{T}^* M \otimes \mathrm{End}E$ can be regarded as a vertical-valued form on E. A linear connection can also be regarded as a tangent-valued 1-form, and its curvature tensor as a vertical-valued 2-form. Moreover a vector field on E is a tangent-valued 0-form. In particular, both ξ and

$$X \lrcorner \digamma = X^a\, \partial \mathsf{x}_a + X^a\, \digamma_{a\;B}^{\;A}\, \zeta^B\, \zeta_A$$

are vector fields on U. Indeed, the latter is the *horizontal prolongation* of x through the connection $\mathrm{Ғ}$. Similarly, $\hat{\Xi} \equiv \mathrm{p}\Xi$ and $x \lrcorner \tilde{\Gamma}$ can be regarded as vector fields on H. A computation then yields

$$\mathrm{L}_X \mathrm{Ғ} = [\![\, x \lrcorner \mathrm{Ғ} + \xi \,,\, \mathrm{Ғ} \,]\!]\,, \qquad \mathrm{L}_X R = [\![\, x \lrcorner \mathrm{Ғ} + \xi \,,\, R \,]\!],$$

$$\mathrm{L}_X \tilde{\Gamma} = [\![\, x \lrcorner \tilde{\Gamma} + \hat{\Xi} \,,\, \tilde{\Gamma} \,]\!]\,, \qquad \mathrm{L}_X \tilde{R} = [\![\, x \lrcorner \tilde{\Gamma} + \hat{\Xi} \,,\, \tilde{R} \,]\!]\,.$$

Furthermore Θ can be regarded as a vertical valued 1-form on H, while a 2-spinor $u \colon M \to U$ can be regarded as a section $U \to \mathrm{V}U$. Then we also find

$$\mathrm{L}_X u = [\![\, x \lrcorner \mathrm{Ғ} + \xi \,,\, u \,]\!]\,, \qquad \mathrm{L}_X \Theta = [\![\, x \lrcorner \tilde{\Gamma} + \hat{\Xi} \,,\, \Theta \,]\!]\,.$$

2.4. **Deformed tetrad gravity.** Consider arbitrarily deformed objects $\Gamma' \equiv \Gamma + \vartriangle\Gamma$, $\tilde{\Gamma}' \equiv \tilde{\Gamma} + \vartriangle\tilde{\Gamma}$, $\Theta' \equiv \Theta + \vartriangle\Theta$, where the generic deformations $\vartriangle\Gamma$, $\vartriangle\tilde{\Gamma}$ and $\vartriangle\Theta$ are sections of the same vector bundles as the respective Lie derivatives. Then up to first-order terms in the deformations we get

$$\begin{aligned}
\nabla'_{\!c} \Theta'^{\lambda}_{a} &= \partial_c \Theta'^{\lambda}_{a} + \Gamma'^{\;b}_{ca}\, \Theta'^{\lambda}_{b} - \tilde{\Gamma}'^{\;\lambda}_{c\;\mu}\, \Theta'^{\mu}_{a} \\
&= \partial_c(\Theta^{\lambda}_{a} + \vartriangle\Theta^{\lambda}_{a}) + (\Gamma^{b}_{ca} + \vartriangle\Gamma^{b}_{ca})\,(\Theta^{\lambda}_{b} + \vartriangle\Theta^{\lambda}_{b}) - (\tilde{\Gamma}^{\lambda}_{c\,\mu} + \vartriangle\tilde{\Gamma}^{\lambda}_{c\,\mu})\,(\Theta^{\mu}_{a} + \vartriangle\Theta^{\mu}_{a}) \\
&\cong \nabla_{\!c}\Theta^{\lambda}_{a} + \nabla_{\!c}(\vartriangle\Theta)^{\lambda}_{a} + \vartriangle\Gamma^{b}_{ca}\, \Theta^{\lambda}_{b} - \vartriangle\tilde{\Gamma}^{\lambda}_{c\,\mu}\, \Theta^{\mu}_{a}\,.
\end{aligned}$$

Since $\nabla\Theta = 0$, the above relation can be written as

$$\nabla'_{\!c} \Theta'^{\lambda}_{a} \cong \nabla_{\!c}(\vartriangle\Theta)^{\lambda}_{a} + \vartriangle\Gamma^{b}_{ca}\, \Theta^{\lambda}_{b} - \vartriangle\tilde{\Gamma}^{\lambda}_{c\,\mu}\, \Theta^{\mu}_{a}\,.$$

Now we consider the special case when the deformations are Lie derivatives along a vector field $x \colon M \to \mathrm{T}M$, namely

$$\begin{aligned}
\vartriangle\Gamma^{a}_{c\,b} &\equiv \mathrm{L}_X \Gamma^{a}_{c\,b} = -\nabla_{\!c}\Xi^{b}{}_{a} + x^d\, R_{cd}{}^{b}{}_{a}\,, \\
\vartriangle\tilde{\Gamma}^{\lambda}_{c\,\mu} &\equiv \mathrm{L}_X \tilde{\Gamma}^{\lambda}_{c\,\mu} = -\nabla_{\!c}\hat{\Xi}^{\lambda}{}_{\mu} + x^d\, R_{cd}{}^{\lambda}{}_{\mu}\,, \\
\vartriangle\Theta^{\lambda}_{a} &\equiv \mathrm{L}_X \Theta^{\lambda}_{a} = \Theta^{\mu}_{a}\,(\Xi - \hat{\Xi})^{\lambda}{}_{\mu}\,.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\nabla'_{\!c} \Theta'^{\lambda}_{a} &\cong \nabla_{\!c}(\Xi - \hat{\Xi})^{\lambda}{}_{\mu}\, \Theta^{\mu}_{a} + (x^d\, R_{cd}{}^{b}{}_{a} - \nabla_{\!c}\Xi^{b}{}_{a})\, \Theta^{\lambda}_{b} + (\nabla_{\!c}\hat{\Xi}^{\lambda}{}_{\mu} - x^d\, R_{cd}{}^{\lambda}{}_{\mu})\, \Theta^{\mu}_{a} \\
&= x^d\,(R_{cd}{}^{b}{}_{a}\, \Theta^{\lambda}_{b} - x^d\, R_{cd}{}^{\lambda}{}_{\mu}\, \Theta^{\mu}_{a}) = 0\,,
\end{aligned}$$

so that the deformed soldering form Θ' is covariantly constant with respect to the deformed connections Γ' and $\tilde{\Gamma}'$.

In the gravitational field theory formulation sketched in §1.8 the gravitational field is represented by the couple $(\Theta, \mathrm{Ғ})$ while the spacetime connection Γ is a byproduct, characterized by the condition $\nabla\Theta = 0$. Hence the above result can be interpreted as saying that a deformed couple $(\Theta', \mathrm{Ғ}')$ yields the deformed spacetime connection $\Gamma' \equiv \Gamma + \mathrm{L}_X\Gamma$, where the deformation is the Lie derivative of Γ in the usual sense.

2.5. **A remark on possible extensions.** The various connections induced by a 2-spinor connection Ƒ on the bundles constructed from S can be regarded as "pieces" into which Ƒ can be naturally decomposed. In particular, the imaginary part iY of $\hat{\text{Ƒ}}$ is the induced *Hermitian connection* of $\wedge^2 U$.

We note that Y does not enter the fields $\hat{\Xi}$ and ξ derived form ∇x , hence its contribution to the Lie derivatives of spinors, and the other related Lie derivatives, is limited to the covariant derivative ∇_x. We may say that the internal geometry of $\wedge^2 U$ is not soldered to spacetime geometry|this observation is also relevant in the construction of the Fermi transport of spinors along a timelike line [5]. Actually Y is related to the electromagnetic potential and, in pure electrodynamics, can be just interpreted as such.

Adding further internal degrees of freedom means considering new vector bundles, say $F \rightarrowtail M$, whose fibers are not soldered to spacetime geometry, and taking tensor products such as $U \otimes F$. In general, in such enlarged setting, one has no well-defined notion of Lie derivatives of matter fields and gauge fields with respect to vector fields on the base manifold M. On the other hand, the notion of Lie derivative with respect to a vector field on the total manifold is well-defined, and an important tool in Lagrangian field theory — with particular regard to symmetries.

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