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CALCULUS ON SYMPLECTIC MANIFOLDS

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Abstract. On a symplectic manifold, there is a natural elliptic complex replacing the de Rham complex. It can be coupled to a vector bundle with connection and, when the curvature of this connection is constrained to be a multiple of the symplectic form, we find a new complex. In particular, on complex projective space with its Fubini–Study form and connection, we can build a series of differential complexes akin to the Bernstein–Gelfand–Gelfand complexes from parabolic differential geometry.

1. Introduction

Throughout this article M will be a smooth manifold of dimension $2n$ equipped with a symplectic form J_{ab}. Here, we are using Penrose's abstract index notation [15] and non-degeneracy of this 2-form says that there is a skew contravariant 2-form J^{ab} such that $J_{ab}J^{ac} = \delta_b{}^c$ where $\delta_b{}^c$ is the canonical pairing between vectors and co-vectors.

Let $\wedge^k$ denote the bundle of k-forms on M. The homomorphism

$$\wedge^k \to \wedge^{k-2} \ \text{ given by } \ \phi_{abc\cdots d} \mapsto J^{ab}\phi_{abc\cdots d}$$

is surjective for $2 \leq k \leq n$ with non-trivial kernel, corresponding to the irreducible representation

$$\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!- \cdots -\!\overset{0}{\bullet}\!\!-\!\!\underset{\uparrow k^{\text{th}}\text{ node}}{\overset{1}{\bullet}}\!\!-\!\!\overset{0}{\bullet}\!- \cdots -\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \quad \text{of} \quad \mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R})\,.$$

Denoting this bundle by $\wedge^k_\perp$, there is a canonical splitting of the short exact sequence

$$0 \to \wedge^k_\perp \underset{\pi}{\rightleftarrows} \wedge^k \to \wedge^{k-2} \to 0$$

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and an elliptic complex [2, 9, 11, 16, 18]

$$\begin{array}{ccccccccccccccc} 0 & \to & \wedge^0 & \xrightarrow{d} & \wedge^1 & \xrightarrow{d_\perp} & \wedge^2_\perp & \xrightarrow{d_\perp} & \wedge^3_\perp & \xrightarrow{d_\perp} & \cdots & \xrightarrow{d_\perp} & \wedge^n_\perp \\ &&&&&&&&&&&& \big\downarrow d_\perp^2 \\ 0 & \leftarrow & \wedge^0 & \xleftarrow{d_\perp} & \wedge^1 & \xleftarrow{d_\perp} & \wedge^2_\perp & \xleftarrow{d_\perp} & \wedge^3_\perp & \xleftarrow{d_\perp} & \cdots & \xleftarrow{d_\perp} & \wedge^n_\perp \end{array} \tag{1}$$

where

- $d\colon \wedge^0 \to \wedge^1$ is the exterior derivative,
- for $1 \le k < n$, the operator $d_\perp : \wedge^k_\perp \to \wedge^{k+1}_\perp$ is the composition
$$\wedge^k_\perp \hookrightarrow \wedge^k \xrightarrow{d} \wedge^{k+1} \xrightarrow{\pi} \wedge^{k+1}_\perp\,,$$
a first order operator,
- $d_\perp : \wedge^{k+1}_\perp \to \wedge^k_\perp$ are canonically defined first order operators, which may be seen as adjoint to $d_\perp : \wedge^k_\perp \to \wedge^{k+1}_\perp$,
- $d_\perp^2 : \wedge^n_\perp \to \wedge^n_\perp$ is the composition
$$\wedge^n_\perp \xrightarrow{d_\perp} \wedge^{n-1}_\perp \xrightarrow{d_\perp} \wedge^n_\perp\,,$$
a second order operator.

More explicitly, formulæ for these operators may be given as follows. Firstly, it is convenient to choose a *symplectic connection* ∇_a, namely a torsion-free connection such that $\nabla_a J_{bc} = 0$, equivalently $\nabla_a J^{bc} = 0$. As shown in [12], for example, such connections always exist and if ∇_a is one such, then the general symplectic connection is

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b + J^{cd}\Xi_{abc}\phi_d \quad \text{where } \Xi_{abc} = \Xi_{(abc)}\,.$$

Then, for $1 \le k < n$, the operator $d_\perp : \wedge^k_\perp \to \wedge^{k+1}_\perp$ is given by

$$\phi_{def\cdots g} \longmapsto \nabla_{[c}\phi_{def\cdots g]} - \tfrac{k}{2(n+1-k)} J^{ab}(\nabla_a \phi_{b[ef\cdots g})J_{cd]} \tag{2}$$

and $d_\perp : \wedge^{k+1}_\perp \to \wedge^k_\perp$ is given by

$$\psi_{cdef\cdots g} \longmapsto J^{bc}\nabla_b \psi_{cdef\cdots g}\,. \tag{3}$$

Now suppose E is a smooth vector bundle on M and $\nabla\colon E \to \wedge^1 \otimes E$ is a connection. Choosing any torsion-free connection on $\wedge^1$ induces a connection on $\wedge^1 \otimes E$ and, as is well-known, the composition

$$\wedge^1 \otimes E \to \wedge^1 \otimes \wedge^1 \otimes E \to \wedge^2 \otimes E$$

does not depend on this choice. (It is the second in a well-defined sequence of differential operators

$$E \xrightarrow{\nabla} \wedge^1 \otimes E \xrightarrow{\nabla} \wedge^2 \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \wedge^{2n-1} \otimes E \xrightarrow{\nabla} \wedge^{2n} \otimes E \tag{4}$$

known as the *coupled de Rham sequence*.) In particular, we may define a homomorphism $\Theta\colon E \to E$ by

$$J^{ab}\nabla_a\nabla_b\Sigma = \tfrac{1}{2n}\Theta\Sigma \quad \text{for } \Sigma \in \Gamma(E).$$

It is part of the curvature of ∇ and if this is the only curvature, then

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = 2J_{ab}\Theta\Sigma, \tag{5}$$

and we shall say that ∇ is *symplectically flat*. Looking back at (1), it is easy to see that there are coupled operators

$$E \underset{\nabla_\perp}{\overset{\nabla}{\rightleftarrows}} \wedge^1 \otimes E \underset{\nabla_\perp}{\overset{\nabla_\perp}{\rightleftarrows}} \wedge^2_\perp \otimes E \underset{\nabla_\perp}{\overset{\nabla_\perp}{\rightleftarrows}} \cdots \underset{\nabla_\perp}{\overset{\nabla_\perp}{\rightleftarrows}} \wedge^{n-1}_\perp \otimes E \underset{\nabla_\perp}{\overset{\nabla_\perp}{\rightleftarrows}} \wedge^n_\perp \otimes E,$$

explicit formulæ for which are just as in the uncoupled cases (2) and (3). To complete the coupled version of (1) let us use

$$\nabla_\perp^2 - \tfrac{2}{n}\Theta : \wedge^n_\perp \otimes E \longrightarrow \wedge^n_\perp \otimes E \tag{6}$$

for the middle operator. It is evident that

$$E \xrightarrow{\nabla} \wedge^1 \otimes E \xrightarrow{\nabla_\perp} \wedge^2_\perp \otimes E$$

is a complex if and only if ∇ is symplectically flat. The reason for the curvature term in (6) is that this feature propagates as follows.

Theorem 1. *Suppose $E \xrightarrow{\nabla} \wedge^1 \otimes E$ is a symplectically flat connection and define $\Theta\colon E \to E$ by* (5)*. Then the coupled version of* (1)

$$\begin{array}{ccccccccccc} 0 & \to & E & \xrightarrow{\nabla} & \wedge^1 \otimes E & \xrightarrow{\nabla_\perp} & \wedge^2_\perp \otimes E & \xrightarrow{\nabla_\perp} & \cdots & \xrightarrow{\nabla_\perp} & \wedge^n_\perp \otimes E \\ &&&&&&&&&& \big\downarrow \nabla_\perp^2 - \frac{2}{n}\Theta \\ 0 & \leftarrow & E & \xleftarrow{\nabla_\perp} & \wedge^1 \otimes E & \xleftarrow{\nabla_\perp} & \wedge^2_\perp \otimes E & \xleftarrow{\nabla_\perp} & \cdots & \xleftarrow{\nabla_\perp} & \wedge^n_\perp \otimes E \end{array}$$

is a complex. It is locally exact except near the beginning where

$$\ker \nabla : E \to \wedge^1 \otimes E \quad \textit{and} \quad \frac{\ker \nabla_\perp : \wedge^1 \otimes E \to \wedge^2_\perp \otimes E}{\operatorname{im} \nabla : E \to \wedge^1 \otimes E}$$

may be identified with the kernel and cokernel, respectively, of Θ as locally constant sheaves.

More precision and a proof of Theorem 1 will be provided in §2. Our next theorem yields some natural symplectically flat connections.

Theorem 2. *Suppose M is a $2n$-dimensional symplectic manifold with symplectic connection ∇_a. Then there is a natural vector bundle $\mathcal{T}$ on M of rank $2n+2$ equipped with a connection, which is symplectically flat if and only if the curvature $R_{ab}{}^c{}_d$ of ∇_a has the form*

$$R_{ab}{}^c{}_d = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + J_{ad}\mathrm{P}_{be}J^{ce} - J_{bd}\mathrm{P}_{ae}J^{ce} + 2J_{ab}\mathrm{P}_{de}J^{ce}\,, \tag{7}$$

for some symmetric tensor P_{ab}.

In particular, the Fubini–Study connection on complex projective space is symplectic for the standard Kähler form and its curvature is of the form (7) for $\mathrm{P}_{ab} = g_{ab}$, the standard metric. More generally, if the symplectic connection ∇_a arises from a Kähler metric, then we shall see that (7) holds precisely in the case of constant holomorphic sectional curvature.

After proving Theorems 1 and 2, the remainder of this article is concerned with the consequences of Theorem 1 for the vector bundle $\mathcal{T}$ and those bundles, such as $\bigodot^k \mathcal{T}$, induced from it. In particular, these consequences pertain on complex projective space where we shall find a series of elliptic complexes closely following the Bernstein-Gelfand-Gelfand complexes on the sphere S^{2n+1} as a homogeneous space for the Lie group $\mathrm{Sp}(2n+2,\mathbb{R})$.

This article is based on our earlier work [11] but here we focus on the simpler case where we are given a symplectic structure as background. This results in fewer technicalities and in this article we include more detail, especially in constructing the BGG-like complexes in §5. Further indications justifying the shape of our complexes can be found in [3, 4, 5, 6, 7].

2. The Rumin–Seshadri complex

By the *Rumin–Seshadri complex*, we mean the differential complex (1) after [16]. However, the 4-dimensional case is due to R.T. Smith [17] and the general case is also independently due to Tseng and Yau [18]. In this section we shall derive the coupled version of this complex as in Theorem 1, our proof of which includes (1) as a special case. The following lemma is also the key step in [11].

Lemma 1. *Suppose E is a vector bundle on M with symplectically flat connection $\nabla\colon E\to\Lambda^1\otimes E$. Define $\Theta : E\to E$ by (5). Then Θ has constant rank and the bundles* $\ker\Theta$ *and* $\operatorname{coker}\Theta$ *acquire from ∇, flat connections defining locally constant sheaves $\underline{\ker\Theta}$ and $\underline{\operatorname{coker}\Theta}$, respectively. There is an elliptic complex*

$$\begin{tikzcd}
E \arrow[r,"\nabla"] \arrow[ddr] & \Lambda^1\otimes E \arrow[r,"\nabla"] \arrow[ddr] & \Lambda^2\otimes E \arrow[r,"\nabla"] \arrow[ddr] & \Lambda^3\otimes E \arrow[r,"\nabla"] \arrow[ddr] & \Lambda^4\otimes E \\
 & \oplus & \oplus & \oplus & \oplus & \cdots, \\
 & E \arrow[r] \arrow[uur] & \Lambda^1\otimes E \arrow[r] \arrow[uur] & \Lambda^2\otimes E \arrow[r] \arrow[uur] & \Lambda^3\otimes E
\end{tikzcd}$$

where the differentials are given by

$$\Sigma\mapsto\begin{bmatrix}\nabla\Sigma\\ \Theta\Sigma\end{bmatrix}\quad \begin{bmatrix}\phi\\ \eta\end{bmatrix}\mapsto\begin{bmatrix}\nabla\phi-J\otimes\eta\\ \nabla\eta-\Theta\phi\end{bmatrix}\quad \begin{bmatrix}\omega\\ \psi\end{bmatrix}\mapsto\begin{bmatrix}\nabla\omega+J\wedge\psi\\ \nabla\psi+\Theta\omega\end{bmatrix}\quad\cdots.$$

It is locally exact save for the zeroth and first cohomologies, which may be identified with $\underline{\ker\Theta}$ and $\underline{\operatorname{coker}\Theta}$, respectively.

Proof. From (5) the Bianchi identity for ∇ reads

$$0=\nabla_{[a}\big(J_{bc]}\Theta\big)=J_{[ab}\nabla_{c]}\Theta$$

and non-degeneracy of J_{ab} implies that $\nabla_a\Theta=0$. Consequently, the homomorphism Θ has constant rank and the following diagram with exact rows commutes

$$\begin{tikzcd}
0 \arrow[r] & \ker\Theta \arrow[r] & E \arrow[r,"\Theta"] \arrow[d,"\nabla"] & E \arrow[r] \arrow[d,"\nabla"] & \operatorname{coker}\Theta \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^1\otimes\ker\Theta \arrow[r] & \Lambda^1\otimes E \arrow[r,"\Theta"] & \Lambda^1\otimes E \arrow[r] & \Lambda^1\otimes\operatorname{coker}\Theta \arrow[r] & 0
\end{tikzcd}$$

and yields the desired connections on $\ker\Theta$ and $\operatorname{coker}\Theta$, which are easily seen to be flat. Ellipticity of the given complex is readily verified and, by definition, the kernel of its first differential is $\underline{\ker\Theta}$. To identify the higher local cohomology of this complex the key observation is that locally we may choose a 1-form τ such that $d\tau=J$ and, having done this, the connection

$$\Gamma(E)\ni\Sigma\overset{\tilde\nabla}{\longmapsto}\nabla\Sigma-\tau\otimes\Theta\Sigma\in\Gamma(\Lambda^1\otimes E)$$

is flat. The rest of the proof is diagram chasing, using exactness of

$$E \xrightarrow{\tilde{\nabla}} \wedge^1 \otimes E \xrightarrow{\tilde{\nabla}} \wedge^2 \otimes E \xrightarrow{\tilde{\nabla}} \wedge^3 \otimes E \xrightarrow{\tilde{\nabla}} \wedge^4 \otimes E \xrightarrow{\tilde{\nabla}} \cdots .$$

If needed, the details are in [11]. $\square$

Proof of Theorem 1. In [11], the corresponding result [11, Theorem 4] is proved by invoking a spectral sequence. Here, we shall, instead, prove two typical cases 'by hand,' leaving the rest of the proof to the reader.

For our first case, let us suppose $n \geq 3$ and prove local exactness of

$$\wedge^1 \otimes E \xrightarrow{\nabla_\perp} \wedge^2_\perp \otimes E \xrightarrow{\nabla_\perp} \wedge^3_\perp \otimes E .$$

Thus, we are required to show that if ω_{ab} has values in E and

$$\omega_{ab} = \omega_{[ab]} \qquad J^{ab}\omega_{ab} = 0 \qquad \nabla_{[c}\omega_{de]} = \tfrac{1}{n-1} J^{ab}(\nabla_a \omega_{b[c})J_{de]} ,$$

then locally there is $\phi_a \in \Gamma(\wedge^1 \otimes E)$ such that

$$\omega_{cd} = \nabla_{[c}\phi_{d]} - \tfrac{1}{2n} J^{ab}(\nabla_a \phi_b)J_{cd}.$$

If we set $\psi_c \equiv -\frac{1}{n-1} J^{ab}\nabla_a \omega_{bc}$, then $\nabla_{[c}\omega_{de]} + J_{[cd}\psi_{e]} = 0$ so

$$0 = \nabla_{[b}\nabla_c \omega_{de]} + J_{[bc}\nabla_d \psi_{e]} = J_{[bc}\Theta\omega_{de]} + J_{[bc}\nabla_d \psi_{e]}$$

and since $J \wedge \_ \colon \wedge^2 \to \wedge^4$ is injective it follows that

$$\nabla_{[c}\psi_{d]} + \Theta\omega_{cd} = 0 .$$

In other words, we have shown that

$$\begin{aligned} \nabla\omega + J \wedge \psi &= 0 \\ \nabla\psi + \Theta\omega &= 0 \end{aligned}$$

and Lemma 1 locally yields $\phi_a \in \Gamma(\wedge^1 \otimes E)$ and $\eta \in \Gamma(E)$ such that

$$\begin{aligned} \nabla_{[a}\phi_{b]} - J_{ab}\eta &= \omega_{ab} , \\ \nabla_a \eta - \Theta\phi_a &= \psi_a . \end{aligned}$$

In particular,

$$J^{ab}\nabla_a \phi_b - 2n\eta = J^{ab}\big(\nabla_a \phi_b - J_{ab}\eta\big) = J^{ab}\omega_{ab} = 0$$

and, therefore,

$$\nabla_{[c}\phi_{d]} - \tfrac{1}{2n} J^{ab}(\nabla_a \phi_b)J_{cd} = \nabla_{[c}\phi_{d]} - \eta J_{cd} = \omega_{cd} ,$$

as required.

Our second case is more involved. It is to show that

$$\wedge^n_\perp \otimes E \xrightarrow{\nabla^2_\perp - \frac{2}{n}\Theta} \wedge^n_\perp \otimes E \xrightarrow{\nabla_\perp} \wedge^{n-1}_\perp \otimes E \tag{8}$$

is locally exact. As regards $\nabla_\perp : \wedge^n_\perp \otimes E \to \wedge^{n-1}_\perp \otimes E$, notice that

$$J^{bc}\nabla_b \psi_{cdef\cdots g} = \tfrac{n+1}{2} J^{bc}\nabla_{[b}\psi_{cdef\cdots g]}$$

and that if $\phi_{def\cdots g} \in \Gamma(\wedge^k \otimes E)$, then

$$J^{bc}J_{[bc}\phi_{def\cdots g]} = \tfrac{4(n-k)}{(k+1)(k+2)}\phi_{def\cdots g} + \tfrac{k(k-1)}{(k+1)(k+2)} J_{[de}\phi_{f\cdots g]bc}J^{bc} \tag{9}$$

so if $\phi_{def\cdots g} \in \Gamma(\wedge_\perp^{n-1} \otimes E)$, then

$$J^{bc} J_{[bc}\phi_{def\cdots g]} = \tfrac{4}{n(n+1)}\phi_{def\cdots g}\,.$$

Therefore, $\nabla_\perp \psi \in \Gamma(\wedge_\perp^{n-1} \otimes E)$ is characterised by

$$J \wedge \nabla_\perp \psi = \tfrac{2}{n}\nabla\psi \tag{10}$$

as an equation in $\wedge^{n+1} \otimes E$. In particular, in $\wedge^{n+2} \otimes E$ we find

$$J \wedge \nabla\nabla_\perp \psi = \nabla(J \wedge \nabla_\perp \psi) = \tfrac{2}{n}\nabla^2\psi = J \wedge \Theta\psi = 0$$

whence $\nabla\nabla_\perp\psi$ already lies in $\wedge^n \otimes E$ and there is no need to remove the trace as in (2) to form $\nabla_\perp^2\psi$. Therefore, invoking (10) once again, the composition

$$\wedge_\perp^n \otimes E \xrightarrow{\nabla_\perp} \wedge_\perp^{n-1} \otimes E \xrightarrow{\nabla_\perp} \wedge_\perp^n \otimes E \xrightarrow{\nabla_\perp} \wedge_\perp^{n-1} \otimes E$$

is characterised by

$$J \wedge \nabla_\perp^3 \psi = \tfrac{2}{n}\nabla\nabla_\perp^2\psi = \tfrac{2}{n}\nabla^2\nabla_\perp\psi = \tfrac{2}{n}J \wedge \Theta\nabla_\perp\psi = \tfrac{2}{n}J \wedge \nabla_\perp\Theta\psi$$

and, since $J \wedge \_ : \wedge^{n-1} \to \wedge^{n+1}$ is an isomorphism, we conclude that $\nabla_\perp^3\psi = \frac{2}{n}\nabla_\perp\Theta\psi$, equivalently that (8) is a complex.

Before proceeding, let us remark on another consequence of (9), namely that for $\nu_{cdef\cdots g} \in \Gamma(\wedge^n \otimes E)$,

$$J_{[ab}\nu_{cdef\cdots g]} = 0 \iff J^{cd}\nu_{cdef\cdots g} = 0\,. \tag{11}$$

Now to establish local exactness, suppose $\nu \in \Gamma(\wedge_\perp^n \otimes E)$ satisfies $\nabla_\perp\nu = 0$. Equivalently, according to (10) and (11)

$$\nu \in \Gamma(\wedge^n \otimes E) \quad \text{satisfies } \nabla\nu = 0 \text{ and } J \wedge \nu = 0\,.$$

Lemma 1 implies that locally there are

$$\begin{array}{l} \phi \in \Gamma(\wedge^n \otimes E) \\ \eta \in \Gamma(\wedge^{n-1} \otimes E) \end{array} \quad \text{such that} \quad \begin{array}{rcl} \nabla\phi - J \wedge \eta &=& 0 \\ \nabla\eta - \Theta\phi &=& \nu. \end{array}$$

Since

$$0 \to \wedge^{n-2} \xrightarrow{J\wedge\_} \wedge^n \to \wedge_\perp^n \to 0$$

is exact, we can write ϕ uniquely as

$$\phi = \psi + J \wedge \tau\,,$$

where $\psi \in \Gamma(\wedge_\perp^n \otimes E)$ and $\tau \in \Gamma(\wedge^{n-2} \otimes E)$. We conclude that

$$\begin{array}{rcl} \nabla\psi - J \wedge \hat\eta &=& 0 \\ \nabla\hat\eta - \Theta\psi &=& \nu, \end{array} \quad (\text{where } \hat\eta = \eta - \nabla\tau)\,.$$

However, as discussed above, these equations say exactly that

$$\nabla_\perp^2\psi - \tfrac{2}{n}\Theta\psi = \nu\,,$$

and exactness is shown. □

3. Tractor bundles

For the rest of the article we suppose that we are given, not only a manifold M with symplectic form J_{ab}, but also a torsion-free connection ∇_a on the tangent bundle (and hence on all other tensor bundles) such that $\nabla_a J_{bc} = 0$. This is sometimes called a *Fedosov structure* [12] on M. The curvature $R_{ab}{}^c{}_d$ of ∇_a, characterised by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^c = R_{ab}{}^c{}_d X^d\,,$$

satisfies

$$R_{ab}{}^c{}_d = R_{[ab]}{}^c{}_d \qquad R_{[ab}{}^c{}_{d]} = 0 \qquad R_{ab}{}^c{}_d J_{ce} = R_{ab}{}^c{}_e J_{cd}$$

and enjoys the following decomposition into irreducible parts

$$R_{ab}{}^c{}_d = V_{ab}{}^c{}_d + \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + J_{ad}\mathrm{P}_{be}J^{ce} - J_{bd}\mathrm{P}_{ae}J^{ce} + 2J_{ab}\mathrm{P}_{de}J^{ce},$$

for some symmetric P_{ab}, where $V_{ab}{}^a{}_d = 0$ (reflecting the branching

$$\yng(3,1) = \yng(3,1)_\perp \oplus \yng(2)$$

of representations under $\mathrm{GL}(2n,\mathbb{R}) \supset \mathrm{Sp}(2n,\mathbb{R})$). Notice that

$$\mathrm{P}_{bd} = \tfrac{1}{2(n+1)} R_{ab}{}^a{}_d = \tfrac{1}{4(n+1)} J^{ae} R_{ae}{}^c{}_b J_{cd}\,. \tag{12}$$

We define the *standard tractor bundle* to be the rank $2n+2$ vector bundle $\mathcal{T} \equiv \wedge^0 \oplus \wedge^1 \oplus \wedge^0$ with its *tractor connection*

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + J_{ab}\rho + \mathrm{P}_{ab}\sigma \\ \nabla_a\rho - \mathrm{P}_{ab}J^{bc}\mu_c + S_a\sigma \end{bmatrix}, \text{ where } S_a \equiv \tfrac{1}{2n+1} J^{bc}\nabla_c \mathrm{P}_{ab}\,.$$

Readers familiar with conformal differential geometry may recognise the form of this connection as following the tractor connection in that setting [1]. If needs be, we shall write *symplectic tractor connection* to distinguish the connection just defined from any alternatives. We shall need the following curvature identities.

Lemma 2. *Let* $Y_{abc} \equiv \frac{1}{2n+1}\nabla_c V_{ab}{}^c{}_d$. *Then*

$$Y_{abc} = 2\nabla_{[a}\mathrm{P}_{b]c} - 2J_{c[a}S_{b]} + 2J_{ab}S_c \tag{13}$$

and

$$\begin{aligned} J^{ad}\nabla_a Y_{bcd} &= J^{ad}V_{bc}{}^e{}_a \mathrm{P}_{ed} + 4n(J^{ad}\mathrm{P}_{ba}\mathrm{P}_{cd} - \nabla_{[b}S_{c]}) \\ &\quad + 2J_{bc}J^{ad}(\nabla_a S_d - J^{ef}\mathrm{P}_{ae}\mathrm{P}_{df})\,. \end{aligned} \tag{14}$$

Proof. Writing the Bianchi identity $\nabla_{[a}R_{bc]}{}^d{}_e = 0$ in terms of $V_{ab}{}^c{}_d$ and P_{ab} yields

$$\nabla_{[a}V_{bc]}{}^d{}_e = -2\delta_{[b}{}^d \nabla_a \mathrm{P}_{c]e} + 2J^{df}J_{e[b}\nabla_a\mathrm{P}_{c]f} - 2J^{df}J_{[bc}\nabla_{a]}\mathrm{P}_{ef}\,.$$

and contracting over ${}_a{}^d$ gives

$$\begin{aligned} \tfrac{1}{3}\nabla_a V_{bc}{}^a{}_e = \tfrac{4(n-1)}{3}\nabla_{[b}\mathrm{P}_{c]e} &+ \frac{2}{3}\big[\nabla_{[b}\mathrm{P}_{c]e} - (2n+1)J_{e[b}S_{c]}\big] \\ &+ \frac{2}{3}\big[(2n+1)J_{bc}S_e + 2\nabla_{[b}\mathrm{P}_{c]e}\big]\,, \end{aligned}$$

which is easily rearranged as (13). For (14), firstly notice that

$$J^{ad}R_{ab}{}^{e}{}_{d} = J^{ed}R_{ab}{}^{a}{}_{d} = 2(n+1)J^{ed}\mathrm{P}_{bd}$$

and the Bianchi symmetry may be written as $R_{a[b}{}^{e}{}_{c]} = -\frac{1}{2}R_{bc}{}^{e}{}_{a}$. Thus,

$$\begin{aligned} J^{ad}\nabla_a\nabla_b\mathrm{P}_{cd} &= \nabla_b J^{ad}\nabla_a\mathrm{P}_{cd} - J^{ad}R_{ab}{}^{e}{}_{c}\mathrm{P}_{ed} - J^{ad}R_{ab}{}^{e}{}_{d}\mathrm{P}_{ce} \\ &= -(2n+1)\nabla_b S_c - J^{ad}R_{ab}{}^{e}{}_{c}\mathrm{P}_{ed} + 2(n+1)J^{de}\mathrm{P}_{bd}\mathrm{P}_{ce} \end{aligned}$$

and so

$$J^{ad}\nabla_a\nabla_{[b}\mathrm{P}_{c]d} = -(2n+1)\nabla_{[b}S_{c]} + \tfrac{1}{2}J^{ad}R_{bc}{}^{e}{}_{a}\mathrm{P}_{ed} + 2(n+1)J^{de}\mathrm{P}_{bd}\mathrm{P}_{ce}\,.$$

From (13) we see that

$$J^{ad}\nabla_a Y_{bcd} = 2J^{ad}\nabla_a\nabla_{[b}\mathrm{P}_{c]d} + 2\nabla_{[b}S_{c]} + 2J_{bc}J^{ad}\nabla_a S_d\,.$$

Therefore,

$$J^{ad}\nabla_a Y_{bcd} = J^{ad}R_{bc}{}^{e}{}_{a}\mathrm{P}_{ed} - 4n\nabla_{[b}S_{c]} + 4(n+1)J^{de}\mathrm{P}_{bd}\mathrm{P}_{ce} + 2J_{bc}J^{ad}\nabla_a S_d\,.$$

Finally,

$$J^{ad}R_{bc}{}^{e}{}_{a}\mathrm{P}_{ed} = J^{ad}V_{bc}{}^{e}{}_{a}\mathrm{P}_{ed} - 4J^{ad}\mathrm{P}_{ba}\mathrm{P}_{cd} - 2J_{bc}J^{ad}J^{ef}\mathrm{P}_{ae}\mathrm{P}_{df}\,,$$

so

$$\begin{aligned} J^{ad}\nabla_a Y_{bcd} = J^{ad}V_{bc}{}^{e}{}_{a}\mathrm{P}_{ed} &+ 4nJ^{ad}\mathrm{P}_{ba}\mathrm{P}_{cd} - 2J_{bc}J^{ad}J^{ef}\mathrm{P}_{ae}\mathrm{P}_{df} \\ &- 4n\nabla_{[b}S_{c]} + 2J_{bc}J^{ad}\nabla_a S_d\,, \end{aligned}$$

which may be rearranged as (14). □

Proposition 1. *The tractor connection $\mathcal{T} \to \wedge^1 \otimes \mathcal{T}$ preserves the non-degenerate skew form*

$$\left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde\sigma \\ \tilde\mu_c \\ \tilde\rho \end{bmatrix} \right\rangle \equiv \sigma\tilde\rho + J^{bc}\mu_b\tilde\mu_c - \rho\tilde\sigma$$

and its curvature is given by

$$\begin{aligned} (\nabla_a\nabla_a - \nabla_b\nabla_a)\begin{bmatrix} \sigma \\ \mu_d \\ \rho \end{bmatrix} &= \begin{bmatrix} 0 \\ -V_{ab}{}^{c}{}_{d}\mu_c + Y_{abd}\sigma \\ -Y_{abc}J^{cd}\mu_d + \frac{1}{2n}(J^{cd}V_{ab}{}^{e}{}_{c}\mathrm{P}_{de} - J^{cd}\nabla_c Y_{abd})\sigma \end{bmatrix} \\ &\quad + 2J_{ab}\begin{bmatrix} \rho \\ J^{ce}\mathrm{P}_{cd}\mu_e - S_d\sigma \\ S_c J^{cd}\mu_d + \frac{1}{2n}J^{cd}(\nabla_c S_d - J^{ef}\mathrm{P}_{ce}\mathrm{P}_{df})\sigma \end{bmatrix}. \end{aligned}$$

Proof. We expand

$$\left\langle \nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde\sigma \\ \tilde\mu_c \\ \tilde\rho \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \nabla_a \begin{bmatrix} \tilde\sigma \\ \tilde\mu_c \\ \tilde\rho \end{bmatrix} \right\rangle$$

to obtain

$$\begin{aligned}(\nabla_a\sigma-\mu_a)\tilde\rho+\sigma(\nabla\tilde\rho-\mathrm{P}_{ab}J^{bc}\tilde\mu_c+S_a\tilde\sigma)&\\ +J^{bc}(\nabla_a\mu_b+J_{ab}\rho+\mathrm{P}_{ab}\sigma)\tilde\mu_c+J^{bc}\mu_b(\nabla_a\tilde\mu_c+J_{ac}\tilde\rho+\mathrm{P}_{ac}\tilde\sigma)&\\ -(\nabla_a\rho-\mathrm{P}_{ab}J^{bc}\mu_c+S_a\sigma)\tilde\sigma-\rho(\nabla_a\tilde\sigma-\tilde\mu_a)&\end{aligned}$$

in which all terms cancel save for

$$(\nabla_a\sigma)\tilde\rho+\sigma\nabla\tilde\rho+J^{bc}(\nabla_a\mu_b)\tilde\mu_c+J^{bc}\mu_b\nabla_a\tilde\mu_c-(\nabla_a\rho)\tilde\sigma-\rho\nabla_a\tilde\sigma\,,$$

which reduces to

$$\nabla_a\big(\sigma\tilde\rho+J^{bc}\mu_b\tilde\mu_c-\rho\tilde\sigma\big)\,,$$

as required. For the curvature, we readily compute

$$\nabla_{[a}\nabla_{b]}\begin{bmatrix}\sigma\\ \mu_d\\ \rho\end{bmatrix}=\begin{bmatrix}\nabla_{[a}\nabla_{b]}\sigma-J_{ba}\rho\\ \nabla_{[a}\nabla_{b]}\mu_d+J_{d[a}\mathrm{P}_{b]c}J^{ce}\mu_e-\mathrm{P}_{d[a}\mu_{b]}+T_{abd}\sigma\\ \nabla_{[a}\nabla_{b]}\rho-T_{abc}J^{cd}\mu_d+(\nabla_{[a}S_{b]}-J^{cd}\mathrm{P}_{ac}\mathrm{P}_{bd})\sigma\end{bmatrix},$$

where $T_{abc}\equiv\nabla_{[a}\mathrm{P}_{b]c}-J_{c[a}S_{b]}$. Lemma 2, however, states that

$$T_{abc}=\tfrac12Y_{abc}-J_{ab}S_c$$

and

$$\begin{aligned}4n(\nabla_{[a}S_{b]}-J^{cd}\mathrm{P}_{ac}\mathrm{P}_{bd})&=J^{cd}V_{ab}{}^{e}{}_{c}\mathrm{P}_{de}-J^{cd}\nabla_cY_{abd}\\ &\quad+2J_{ab}J^{cd}(\nabla_cS_d-J^{ef}\mathrm{P}_{ce}\mathrm{P}_{df})\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_{[a}\nabla_{b]}\begin{bmatrix}\sigma\\ \mu_d\\ \rho\end{bmatrix}&=\begin{bmatrix}0\\ \nabla_{[a}\nabla_{b]}\mu_d+J_{d[a}\mathrm{P}_{b]c}J^{ce}\mu_e-\mathrm{P}_{d[a}\mu_{b]}+\frac12Y_{abd}\sigma\\ -\frac12Y_{abc}J^{cd}\mu_d+\frac1{4n}(J^{cd}V_{ab}{}^{e}{}_{c}\mathrm{P}_{de}-J^{cd}\nabla_cY_{abd})\sigma\end{bmatrix}\\ &\quad+J_{ab}\begin{bmatrix}\rho\\ -S_d\sigma\\ S_cJ^{cd}\mu_d+\frac1{2n}J^{cd}(\nabla_cS_d-J^{ef}\mathrm{P}_{ce}\mathrm{P}_{df})\sigma\end{bmatrix}.\end{aligned}$$

Finally,

$$R_{ab}{}^{c}{}_{d}\mu_c=V_{ab}{}^{c}{}_{d}\mu_c-2\mathrm{P}_{d[a}\mu_{b]}+2J_{d[a}\mathrm{P}_{b]c}J^{ce}\mu_e+2J_{ab}\mathrm{P}_{de}J^{ce}\mu_c\,,$$

so

$$\nabla_{[a}\nabla_{b]}\mu_d+J_{d[a}\mathrm{P}_{b]c}J^{ce}\mu_e-\mathrm{P}_{d[a}\mu_{b]}=-\tfrac12V_{ab}{}^{c}{}_{d}\mu_c-J_{ab}\mathrm{P}_{de}J^{ce}\mu_c$$

whence

$$\begin{aligned}\nabla_{[a}\nabla_{b]}\begin{bmatrix}\sigma\\ \mu_d\\ \rho\end{bmatrix}&=\begin{bmatrix}0\\ -\frac12V_{ab}{}^{c}{}_{d}\mu_c+\frac12Y_{abd}\sigma\\ -\frac12Y_{abc}J^{cd}\mu_d+\frac1{4n}(J^{cd}V_{ab}{}^{e}{}_{c}\mathrm{P}_{de}-J^{cd}\nabla_cY_{abd})\sigma\end{bmatrix}\\ &\quad+J_{ab}\begin{bmatrix}\rho\\ J^{ce}\mathrm{P}_{cd}\mu_e-S_d\sigma\\ S_cJ^{cd}\mu_d+\frac1{2n}J^{cd}(\nabla_cS_d-J^{ef}\mathrm{P}_{ce}\mathrm{P}_{df})\sigma\end{bmatrix},\end{aligned}$$

as required. □

Corollary 1. *The tractor connection is symplectically flat if and only if the curvature tensor $V_{ab}{}^c{}_d$ vanishes.*

4. Kähler geometry

Kähler manifolds provide a familiar source of symplectic manifolds equipped with a compatible torsion-free connection as in §3. In this case, the connection ∇_a is the Levi-Civita connection of a metric g_{ab} and $J_a{}^b \equiv J_{ac}g^{bc}$ is an almost complex structure on M whose integrability is equivalent to the vanishing of $\nabla_a J_{bc}$. In Kähler geometry, the Riemann curvature tensor decomposes into three irreducible parts:

$$\begin{aligned} R_{ab}{}^c{}_d = {}& U_{ab}{}^c{}_d \\ & + \delta_a{}^c\Xi_{bd} - \delta_b{}^c\Xi_{ad} - g_{ad}\Xi_b{}^c + g_{bd}\Xi_a{}^c \\ & + J_a{}^c\Sigma_{bd} - J_b{}^c\Sigma_{ad} - J_{ad}\Sigma_b{}^c + J_{bd}\Sigma_a{}^c + 2J_{ab}\Sigma^c{}_d + 2J^c{}_d\Sigma_{ab} \\ & + \Lambda(\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} + J_a{}^c J_{bd} - J_b{}^c J_{ad} + 2J_{ab}J^c{}_d)\,, \end{aligned} \tag{15}$$

where indices have been raised using g^{ab} and

– $U_{ab}{}^c{}_d$ is totally trace-free with respect to g^{ab}, $J_a{}^b$, and J^{ab},

– Ξ_{ab} is trace-free symmetric whilst $\Sigma_{ab} \equiv J_a{}^c\Xi_{bc}$ is skew.

Computing the Ricci curvature from this decomposition, we find

$$R_{bd} \equiv R_{ab}{}^a{}_d = 2(n+2)\Xi_{bd} + 2(n+1)\Lambda g_{bd}$$

and therefore from (12) conclude that

$$\mathrm{P}_{ab} = \tfrac{n+2}{n+1}\Xi_{ab} + \Lambda g_{ab}\,.$$

Hence

$$\begin{aligned} J_c{}^a R_{ab}{}^c{}_d &= J_c{}^a V_{ab}{}^c{}_d - J_{bd}\mathrm{P}_a{}^a - 2J_b{}^a\mathrm{P}_{da} \\ &= J_c{}^a V_{ab}{}^c{}_d - 2\frac{n+2}{n+1}\Sigma_{bd} - 2(n+1)\Lambda J_{bd}\,. \end{aligned}$$

On the other hand, from (15) we find

$$J_c{}^a R_{ab}{}^c{}_d = -2(n+2)\Sigma_{bd} - 2(n+1)\Lambda J_{bd}$$

and, comparing these two expressions gives

$$J_c{}^a V_{ab}{}^c{}_d - 2\tfrac{n+2}{n+1}\Sigma_{bd} = -2(n+2)\Sigma_{bd}$$

and we have established the following.

Proposition 2. *Concerning the symplectic curvature decomposition on a Kähler manifold,*

$$J_c{}^a V_{ab}{}^c{}_d = -2\tfrac{n(n+2)}{n+1}\Sigma_{bd}\,.$$

Corollary 2. *The symplectic tractor connection on a Kähler manifold is symplectically flat if and only if the metric has constant holomorphic sectional curvature.*

Proof. According to Corollary 1, we have to interpret the constraint ${V_{ab}}^c{}_d = 0$ in the Kähler case. From (15) it is already clear that ${U_{ab}}^c{}_d = 0$ and Proposition 2 implies that also $\Sigma_{ab} = 0$ so (15) reduces to

$$R_{ab}{}^c{}_d = \Lambda(\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} + J_a{}^c J_{bd} - J_b{}^c J_{ad} + 2J_{ab}J^c{}_d)\,,$$

which is exactly the constancy of holomorphic sectional curvature. □

5. BGG-like complexes on $\mathbb{CP}_n$

Fix a real vector space $\mathfrak{g}_{-1}$ of dimension $2n$, let $\mathfrak{g}_1$ denotes its dual, and fix a non-degenerate 2-form $J_{ab} \in \wedge^2\mathfrak{g}_1$. The $(2n+1)$-dimensional Heisenberg Lie algebra may be realised as

$$\mathfrak{h} = \mathbb{R} \oplus \mathfrak{g}_{-1}\,,$$

where the first summand is the 1-dimensional centre of $\mathfrak{h}$ and the Lie bracket on $\mathfrak{g}_{-1}$ is given by

$$[X, Y] = 2J_{ab}X^aY^b \in \mathbb{R} \hookrightarrow \mathfrak{h}\,.$$

We should admit right away that the reason for this seemingly arcane notation is that we shall soon have occasion to write

$$\mathfrak{sp}(2n+2,\mathbb{R}) = \begin{array}{ccccccccc} \mathfrak{g}_{-2} & \oplus & \mathfrak{g}_{-1} & \oplus & \mathfrak{g}_0 & \oplus & \mathfrak{g}_1 & \oplus & \mathfrak{g}_2 \\ \| & & & & \| & & & & \| \\ \mathbb{R} & & & \mathfrak{sp}(2n,\mathbb{R}) & \oplus\, \mathbb{R} & & & & \mathbb{R} \end{array} \tag{16}$$

(a $|2|$-graded Lie algebra as in [8, §4.2.6]) and, in particular, regard $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{g}_{-1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as a Lie subalgebra of $\mathfrak{sp}(2n+2,\mathbb{R})$. Be that as it may, let us suppose that $\mathbb{V}$ is a finite-dimensional representation of $\mathfrak{h}$. The Lie algebra cohomology $H^r(\mathfrak{h},\mathbb{V})$ may be realised as the cohomology of the Chevalley-Eilenberg complex

$$0 \to \mathbb{V} \to \mathfrak{h}^* \otimes \mathbb{V} \to \cdots \to \wedge^r\mathfrak{h}^* \otimes \mathbb{V} \to \wedge^{r+1}\mathfrak{h}^* \otimes \mathbb{V} \to \cdots \tag{17}$$

as, for example, in [13, Chapter IV]. We shall require, however, the following alternative realisation.

Lemma 3. *There is a complex*

$$\begin{array}{ccccccccccc} 0 & \to & \mathbb{V} & \xrightarrow{\partial} & \mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial_\perp} & \wedge^2_\perp\mathfrak{g}_1 \otimes \mathbb{V} & \xrightarrow{\partial_\perp} & \cdots & \xrightarrow{\partial_\perp} & \wedge^n_\perp\mathfrak{g}_1 \otimes \mathbb{V} \\ & & & & & & & & & & \downarrow \\ 0 & \leftarrow & \mathbb{V} & \xleftarrow{\partial_\perp} & \mathfrak{g}_1 \otimes \mathbb{V} & \xleftarrow{\partial_\perp} & \wedge^2_\perp\mathfrak{g}_1 \otimes \mathbb{V} & \xleftarrow{\partial_\perp} & \cdots & \xleftarrow{\partial_\perp} & \wedge^n_\perp\mathfrak{g}_1 \otimes \mathbb{V} \end{array} \tag{18}$$

whose cohomology realises $H^r(\mathfrak{h},\mathbb{V})$. Here, we are writing

$$\wedge^r_\perp\mathfrak{g}_1 \equiv \{\omega_{abc\cdots d} \in \wedge^r\mathfrak{g}_1 \mid J^{ab}\omega_{abc\cdots d} = 0\},$$

where $J^{ab} \in \wedge^2\mathfrak{g}_{-1}$ is the inverse of $J_{ab} \in \wedge^2\mathfrak{g}_1$ (let's say normalised so that $J_{ab}J^{ac} = \delta_b{}^c$).

Proof. Notice that any representation $\rho : \mathfrak{h} \to \mathrm{End}(\mathbb{V})$ is determined by its restriction to $\mathfrak{g}_{-1} \subset \mathfrak{h}$. Indeed, writing $\partial_a : \mathfrak{g}_{-1} \to \mathrm{End}(\mathbb{V})$ for this restriction, to say that ρ is a representation of $\mathfrak{h}$ is to say that

$$\left.\begin{array}{rcl} (\partial_a\partial_b - \partial_b\partial_a)v & = & 2J_{ab}\theta v \\ (\partial_a\theta - \theta\partial_a)v & = & 0 \end{array}\right\} \quad \forall\, v \in \mathbb{V}\,, \tag{19}$$

where $\theta \in \operatorname{End}(\mathbb{V})$ is $\rho(1)$ for $1 \in \mathbb{R} \subset \mathfrak{h}$.

The splitting $\mathfrak{h}^* = \mathfrak{g}_1 \oplus \mathbb{R}$ allows us to write (17) as

$$\begin{array}{ccccccccc}
\mathbb{V} & \longrightarrow & \mathfrak{h}^*\otimes\mathbb{V} & \longrightarrow & \Lambda^2\mathfrak{h}^*\otimes\mathbb{V} & \longrightarrow & \Lambda^3\mathfrak{h}^*\otimes\mathbb{V} & \longrightarrow & \cdots\\
\| && \| && \| && \| &&\\
\mathbb{V} & \longrightarrow & \mathfrak{g}_1\otimes\mathbb{V} & \longrightarrow & \Lambda^2\mathfrak{g}_1\otimes\mathbb{V} & \longrightarrow & \Lambda^3\mathfrak{g}_1\otimes\mathbb{V} & \longrightarrow & \cdots,\\
& \searrow & \oplus & \nearrow\!\!\!\!\!\!\searrow & \oplus & \nearrow\!\!\!\!\!\!\searrow & \oplus & \nearrow\!\!\!\!\!\!\searrow & \\
& & \mathbb{V} & \longrightarrow & \mathfrak{g}_1\otimes\mathbb{V} & \longrightarrow & \Lambda^2\mathfrak{g}_1\otimes\mathbb{V} & \longrightarrow & \cdots
\end{array} \tag{20}$$

where the differentials are given by

$$v \mapsto \left[\begin{array}{c}\partial_a v\\ \theta v\end{array}\right] \quad \left[\begin{array}{c}\phi_a\\ \eta\end{array}\right] \mapsto \left[\begin{array}{c}\partial_{[a}\phi_{b]} - J_{ab}\eta\\ \partial_a\eta - \theta\phi_a\end{array}\right] \quad \left[\begin{array}{c}\omega_{ab}\\ \psi_a\end{array}\right] \mapsto \left[\begin{array}{c}\partial_{[a}\omega_{bc]} + J_{[ab}\psi_{c]}\\ \partial_{[a}\psi_{b]} + \theta\omega_{ab}\end{array}\right]$$

et cetera. In particular, notice that the homomorphisms

$$\Lambda^{r-1}\mathfrak{g}_1 \ni \psi \longmapsto \pm J \wedge \psi \in \Lambda^{r+1}\mathfrak{g}_1 \tag{21}$$

are

- independent of the representation on $\mathbb{V}$,
- injective for $1 \leq r < n$,
- an isomorphism for $r = n$,
- surjective for $n < r \leq 2n - 1$.

Note that $\Lambda_\perp^{r+1}\mathfrak{g}_1$ is complementary to the image of (21) for $1 \leq r < n$. Also note the isomorphisms

$$\Lambda^{2n+1-r}\mathfrak{g}_1 \xrightarrow{J\wedge J\wedge\cdots\wedge J} \Lambda^{r-1}\mathfrak{g}_1, \quad \text{for } n < r \leq 2n+1\,,$$

under which the kernel of (21) may be identified with

$$\Lambda_\perp^{2n+1-r}\mathfrak{g}_1, \quad \text{for } n < r \leq 2n-1\,.$$

Diagram chasing in (20) (or the spectral sequence of a filtered complex) finishes the proof. □

Remark. Evidently, the equations (19) are algebraic versions of

$$\left.\begin{array}{rcl}(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma &=& 2J_{ab}\Theta\Sigma\\ (\nabla_a\Theta - \Theta\nabla_a)\Sigma &=& 0\end{array}\right\}\quad \forall\,\Sigma \in \Gamma(E)\,,$$

which hold for a symplectically flat connection ∇_a on smooth vector bundle E on M. Also (20) is the evident algebraic counterpart to the differential complex of Lemma 1. It follows that explicit formulæ for the operators $\partial_\perp$ in the complex (18) follow the differential versions (2) and (3) with $\Lambda_\perp^n\mathfrak{g}\otimes\mathbb{V} \to \Lambda_\perp^n\mathfrak{g}\otimes\mathbb{V}$ being given by $\partial_\perp^2 - \frac{2}{n}\theta$.

Let us now consider the tractor connection on $\mathbb{CP}_n$. According to Theorem 2, the remarks following its statement, and the discussions in §3, this is the connection on $\mathcal{T} = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^0$ given by

$$\nabla_a\left[\begin{array}{c}\sigma\\ \mu_b\\ \rho\end{array}\right] = \left[\begin{array}{c}\nabla_a\sigma - \mu_a\\ \nabla_a\mu_b + J_{ab}\rho + g_{ab}\sigma\\ \nabla_a\rho - J_a{}^b\mu_b\end{array}\right] = \left[\begin{array}{c}\nabla_a\sigma\\ \nabla_a\mu_b + g_{ab}\sigma\\ \nabla_a\rho - J_a{}^b\mu_b\end{array}\right] + \left[\begin{array}{c}-\mu_a\\ J_{ab}\rho\\ 0\end{array}\right].$$

The induced operator $\nabla\colon \wedge^1 \otimes \mathcal{T} \to \wedge^2 \otimes \mathcal{T}$ is

$$\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \longmapsto \begin{bmatrix} \nabla_{[a}\sigma_{b]} \\ \nabla_{[a}\mu_{b]c} + g_{c[a}\sigma_{b]} \\ \nabla_{[a}\rho_{b]} - J_{[a}{}^{c}\mu_{b]c} \end{bmatrix} + \begin{bmatrix} \mu_{[ab]} \\ -J_{c[a}\rho_{b]} \\ 0 \end{bmatrix}$$

but Corollary 2 says the tractor connection on $\mathbb{CP}_n$ is symplectically flat so we should contemplate $\nabla_\perp\colon \wedge^1 \otimes \mathcal{T} \to \wedge^2_\perp \otimes \mathcal{T}$ from Theorem 1, viz.

$$\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \longmapsto \begin{bmatrix} \nabla_{[a}\sigma_{b]} - \frac{1}{2n}J^{cd}\nabla_c\sigma_d J_{ab} \\ \cdots \\ \cdots \end{bmatrix} + \begin{bmatrix} \mu_{[ab]} - \frac{1}{2n}J^{cd}\mu_{cd}J_{ab} \\ -J_{c[a}\rho_{b]} - \frac{1}{2n}\rho_c J_{ab} \\ 0 \end{bmatrix}.$$

From these formulæ, let us focus attention on the homomorphisms

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{T} & \to & \wedge^1 \otimes \mathcal{T} & \to & \wedge^2_\perp \otimes \mathcal{T} & \to & \cdots \\ & & \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} & \mapsto & \begin{bmatrix} -\mu_a \\ J_{ab}\rho \\ 0 \end{bmatrix} & & & & \\ & & & & \begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} & \mapsto & \begin{bmatrix} \mu_{[ab]} - \frac{1}{2n}J^{cd}\mu_{cd}J_{ab} \\ -J_{c[a}\rho_{b]} - \frac{1}{2n}\rho_c J_{ab} \\ 0 \end{bmatrix} & & \end{array} \tag{22}$$

It is evident that this is a complex and that its cohomology so far is

$$\wedge^0 \text{ in degree } 0 \quad \text{and} \quad \bigodot^2\wedge^1 \text{ in degree } 1\,.$$

On the other hand, one may check that the defining representation of the Lie algebra $\mathfrak{sp}(2n+2,\mathbb{R})$ on $\mathbb{R}^{2n+2} = \mathbb{R} \oplus \mathbb{R}^{2n} \oplus \mathbb{R}$ restricts via (16) to a representation of the Heisenberg Lie algebra $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{g}_{-1}$, given explicitly by

$$\begin{array}{ccccccc} \mathbb{R}^{2n+2} & \xrightarrow{\theta} & \mathbb{R}^{2n+2} & \text{and} & \mathbb{R}^{2n+2} & \xrightarrow{\partial_a} & \mathfrak{g}_1 \otimes \mathbb{R}^{2n+2} \\ \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} & \longmapsto & \begin{bmatrix} \rho \\ 0 \\ 0 \end{bmatrix} & & \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} & \longmapsto & \begin{bmatrix} -\mu_a \\ J_{ab}\rho \\ 0 \end{bmatrix} \end{array}$$

(noticing that equations (19) hold, as they must). We may also find θ as part of the curvature of the tractor connection on $\mathbb{CP}_n$. Specifically, the formula from Proposition 1 reduces to

$$(\nabla_a\nabla_a - \nabla_b\nabla_a)\begin{bmatrix} \sigma \\ \mu_d \\ \rho \end{bmatrix} = 2J_{ab}\begin{bmatrix} \rho \\ J_d{}^e\mu_e \\ -\sigma \end{bmatrix} \tag{23}$$

and we find θ as the top component of $\Theta\colon \mathcal{T} \to \mathcal{T}$ where Θ is defined by (5). If we now consider the entire complex from Theorem 1, with filtration induced by

$$\begin{array}{ccccccc} \wedge^0 & \subset & \wedge^1 \oplus \wedge^0 & \subset & \wedge^0 \oplus \wedge^1 \oplus \wedge^0 & = & \mathcal{T} \\ \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} & & \begin{bmatrix} 0 \\ \mu_b \\ \rho \end{bmatrix} & & \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} & & \end{array}$$

of $\mathcal{T}$, then the associated spectral sequence (or corresponding diagram chasing) yields (22) continuing as in (18) including the middle operator $\nabla_\perp^2 - \frac{2}{n}\theta\colon \wedge_\perp^n \to \wedge_\perp^n$. The same reasoning pertains for any Fedosov structure with $V_{ab}{}^c{}_d = 0$ as in Corollary 1. Evidently, this sequence of vector bundle homomorphisms is induced by the complex (18) and, together with Lemma 3, the spectral sequence of a filtered complex (or the appropriate diagram chasing) immediately yields the following.

Theorem 3. *Suppose ∇_a is a torsion-free connection on a symplectic manifold (M, J_{ab}), such that $\nabla_a J_{bc} = 0$ and so that the corresponding curvature tensor $V_{ab}{}^c{}_d$ vanishes. Fix a finite-dimensional representation $\mathbb{E}$ of $\mathrm{Sp}(2n+2,\mathbb{R})$ and let E denote the associated 'tractor bundle' induced from the standard tractor bundle and the representation $\mathbb{E}$ (so that the standard representation of $\mathrm{Sp}(2n+2,\mathbb{R})$ on $\mathbb{R}^{2n+2}$ yields the standard tractor bundle). In accordance with Corollary 1, the induced 'tractor connection' $\nabla : E \to \wedge^1 \otimes E$ is symplectically flat and we may define $\Theta : E \to E$ by (5). Having done this, there are complexes of differential operators*

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathfrak{h}, E) & \to & H^1(\mathfrak{h}, E) & \to & H^2(\mathfrak{h}, E) & \to \cdots \to & H^n(\mathfrak{h}, E) \\ & & & & & & & \downarrow \\ 0 \leftarrow & H^{2n+1}(\mathfrak{h}, E) & \leftarrow & H^{2n}(\mathfrak{h}, E) & \leftarrow & H^{2n-1}(\mathfrak{h}, E) & \leftarrow \cdots \leftarrow & H^{n+1}(\mathfrak{h}, E) \end{array}$$

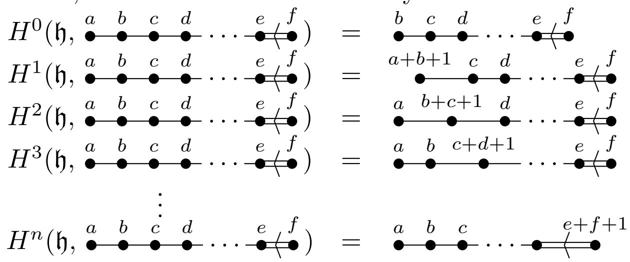
where $H^r(\mathfrak{h}, E)$ denotes the tensor bundle on M that is induced by the cohomology $H^r(\mathfrak{h}, \mathbb{E})$ as a representation of $\mathrm{Sp}(2n,\mathbb{R})$. This complex is locally exact except near the beginning where

$$\ker : H^0(\mathfrak{h}, E) \to H^1(\mathfrak{h}, E) \quad \text{and} \quad \frac{\ker : H^1(\mathfrak{h}, E) \to H^2(\mathfrak{h}, E)}{\mathrm{im} : H^0(\mathfrak{h}, E) \to H^1(\mathfrak{h}, E)}$$

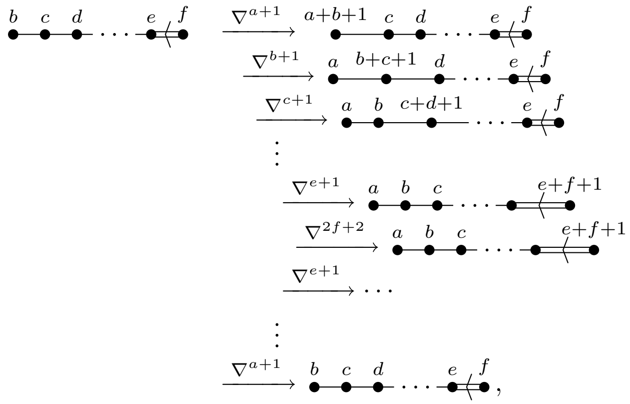
may be identified with the locally constant sheaves $\underline{\ker\Theta}$ and $\underline{\mathrm{coker}\,\Theta}$, respectively. In particular, for $\mathbb{CP}_n$ with its Fubini–Study connection, these sheaves vanish and the complex is locally exact everywhere.

Proof. It remains only to observe that for the Fubini–Study connection we see from (23) that $\Theta : \mathcal{T} \to \mathcal{T}$ is an isomorphism. $\square$

The main point about Theorem 3, however, is that if the representation $\mathbb{E}$ of $\mathrm{Sp}(2n+2,\mathbb{R})$ is irreducible, then the representations $H^r(\mathfrak{h}, \mathbb{E})$ of $\mathrm{Sp}(2n,\mathbb{R})$ are also irreducible and are computed by a theorem due to Kostant [14]. Specifically, if we denote the irreducible representations of $\mathrm{Sp}(2n+2,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$ by writing the highest weight as a linear combination of fundamental weights and recording the coefficients over the corresponding nodes of the Dynkin diagrams for C_{n+1} and C_n, as is often done, then Kostant's Theorem says that



and for $r \geq n+1$, there are isomorphisms $H^r(\mathfrak{h}, \mathbb{E}) = H^{2n+1-r}(\mathfrak{h}, \mathbb{E})$. Using the same notation for the bundles $H^r(\mathfrak{h}, E)$, the complexes of Theorem 3 become



for arbitrary non-negative integers $a, b, c, d, \cdots, e, f$. When all these integers are zero, this is the Rumin–Seshadri complex. Just the first three terms in this complex, in the special case when only a is non-zero, are already essential in [10]. For example, if $a = 1$, then the first two differential operators are

$$\sigma \mapsto \nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma \quad \text{and} \quad \phi_{bc} \mapsto \big(\nabla_a \phi_{bc} - \nabla_b \phi_{ac}\big)_\perp$$

where ϕ_{bc} is symmetric and $(\ \ )_\perp$ means to take the trace-free part with respect to J_{ab}. From the curvature decomposition and Bianchi identity we find that their composition is

$$\sigma \longmapsto V_{ab}{}^d{}_c \nabla_d \sigma + Y_{abc}\sigma\,,$$

which vanishes in case $V_{ab}{}^c{}_d = 0$. In case Θ is invertible, as for the Fubini–Study connection, we conclude that this sequence of differential operators is locally exact.

References

[1] Bailey, T.N., Eastwood, M.G., Gover, A.R., *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. **24** (1994), 1191–1217.

[2] Bryant, R.L., Eastwood, M.G., Gover, A.R., Neusser, K., *Some differential complexes within and beyond parabolic geometry*, `arXiv:1112.2142`.

[3] Cahen, M., Schwachofer, L. J., *Special symplectic connections*, J. Differential Geom. **83** (2009), 229–271.

[4] Čap, A., Salač, T., *Parabolic conformally symplectic structures I; definition and distinguished connections*, Forum Math., to appear, `arXiv:1605.01161`.

[5] Čap, A., Salač, T., *Parabolic conformally symplectic structures II; parabolic contactization*, Ann. Mat. Pura Appl., to appear, `arXiv:1605.01897`.

[6] Čap, A., Salač, T., *Parabolic conformally symplectic structures III; invariant differential operators and complexes*, `arXiv:1701.01306`.

[7] Čap, A., Salač, T., *Pushing down the Rumin complex to conformally symplectic quotients*, Differential Geom. Appl. **35** (2014), 255–265, `arXiv:1312.2712`.

[8] Čap, A., Slovák, J., *Parabolic Geometries I: Background and General Theory*, Math. Surveys Monogr. **154** (209).

[9] Eastwood, M.G., *Extensions of the coeffective complex*, Illinois J. Math. **57** (2013), 373–381.

[10] Eastwood, M.G., Goldschmidt, H., *Zero-energy fields on complex projective space*, J. Differential Geom. **94** (2013), 129–157.

[11] Eastwood, M.G., Slovák, J., *Conformally Fedosov manifolds*, `arXiv:1210.5597`.

[12] Gelfand, I.M., Retakh, V.S., Shubin, M.A., *Fedosov manifolds*, Adv. Math. **136** (1998), 104–140.

[13] Knapp, A.W., *Lie Groups, Lie Algebras, and Cohomology*, Princeton University Press, 1988.

[14] Kostant, B., *Lie algebra cohomology and the generalized Borel–Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387.

[15] Penrose, R., Rindler, W., *Spinors and Space-time*, vol. 1, Cambridge University Press, 1984.

[16] Seshadri, N., Private communication, September 2007.

[17] Smith, R.T., *Examples of elliptic complexes*, Bull. Amer. Math. Soc. (N.S.) **82** (1976), 294–299.

[18] Tseng, L.-S., Yau, S.-T., *Cohomology and Hodge theory on symplectic manifolds: I and II*, J. Differential Geom. **91** (2012), 383–416, 417–443.

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