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PROPERADS AND HOMOLOGICAL DIFFERENTIAL OPERATORS RELATED TO SURFACES

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Abstract. We give a biased definition of a properad and an explicit example of a closed Frobenius properad. We recall the construction of the cobar complex and algebra over it. We give an equivalent description of the algebra in terms of Barannikov's theory which is parallel to Barannikov's theory of modular operads. We show that the algebra structure can be encoded as homological differential operator. Example of open Frobenius properad is mentioned along its specific properties.

1. Introduction

Operads are objects that model operations with several inputs and one output. As such, they capture the composition of operations and the permutation of variables. Such a structure can be considered in the context of graphs, strictly speaking in the context of directed trees. The composition of an m-ary operation and an n-ary operation is in this context given by grafting one directed rooted tree with m leaves (external incoming edges) into another directed rooted tree with n leaves. This produces a new directed tree with $(m + n - 1)$ leaves. The permutation of variables then corresponds to the relabeling of leaves.

This structure could be generalized in two possible ways. The first way is by using undirected graphs with several inputs and leads to the notions of cyclic and modular operads. The second possibility is by using connected directed acyclic graphs[1] with several inputs and several outputs and leads to the notion of properads. Both these generalization include examples, which can be

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[1]See Definition 6

interpreted in terms of 2-dimensional surfaces with boundaries and punctures. The punctures could be positioned in the interior or on the boundaries or both, i.e. describing closed, open or open-closed strings. The aim of this article is to continue in the work in [2] by Doubek, Jurčo, and Münster but this time for properads.

Properads were first introduced in [10] by Vallette as connected parts of PROPs. In [10] he gives both an unbiased as well as a biased definition. The biased definition which is used in this article is at closest to the one in [5]. The properad, in this case, is indexed by two finite sets. The main example discussed in this article is the closed Frobenius properad serving as the description of closed strings.

There is a fundamental difference between modular operads and properads. In the case of modular operads, the structure operations are iterations of those corresponding to the contraction of one edge of the underlying pasting scheme[2]. But the same is not true for properads. In the case of properads, one needs to contract all edges connecting two vertices at once.

The construction of the cobar complex is a useful general construction. It is defined as a functor from the category of co(pr)operads to the category of augmented differential graded (pr)operads. The version for modular operads is also known as the Feynman transform[3]. To avoid the new definitions of coproperads, cocomposition maps, coproperad morphism etc., we consider the construction of the cobar complex in the same manner as in [6] by Markl, Schnider, and Stasheff. In short, the cobar complex of a properad is the free properad over its suspended linear dual equipped with the differential induced by the duals of the structure operations.

For some (pr)operads, namely for Koszul operads, the cobar complex can be used to construct a minimal model of this operad. But we will not discuss this aspect. Instead, we focus on the result of Barannikov in [1]. In [1], he showed how an algebra over the cobar complex of a modular operad can equivalently be described as a solution of a master equation for certain generalized BV algebra. This article gives an analogous description for properads.

The paper is organized as follows. Sect. 2 introduces properads with some additional conditions and presents the main example used in the paper, the closed (commutative) Frobenius properad. In Sect. 3 we recall the cobar construction of a properad. Sect. 4 contains the definition of the endomorphism properad. Further, we give an explicit description of algebras over properads and algebras over the cobar complex of the closed Frobenius properad. Finally,

[2]In [2] called operadic compositions and self-contractions.

[3]The Feynman transform produces, out of a modular operad, a twisted modular operad.

in Sect. 5, we give an analog of Barannikov's theory for algebras over the cobar complex and interpret the composition of this algebra as composition of differential operators.

2. Properads

Denote by Cor the category of finite sets and their isomorphisms (corollas).

Definition 1. Denote by $\mathsf{DCor} := \mathsf{Cor} \times \mathsf{Cor}$ the category of directed corollas: the objects are pairs (C, D) with C and D finite sets which are called the outputs and inputs. A morphism $(\rho, \sigma)\colon (C, D) \to (C', D')$ is a pair of bijections $\rho\colon C \xrightarrow{\sim} C'$, $\sigma\colon D \xrightarrow{\sim} D'$.

Definition 2. A properad $\mathcal{P}$ consists of a collection

$$\{\mathcal{P}(C, D) \mid (C, D) \in \mathsf{DCor}\}$$

of dg vector spaces and two collections of degree 0 morphisms of dg vector spaces

$$\{\mathcal{P}(\rho, \sigma)\colon \mathcal{P}(C, D) \to \mathcal{P}(C', D') \mid (\rho, \sigma) : (C, D) \to (C', D')\}$$

$$\left\{ {}_B\overset{\eta}{\circ}_A\colon \mathcal{P}(C_1, D_1 \sqcup B) \otimes \mathcal{P}(C_2 \sqcup A, D_2) \to \mathcal{P}(C_1 \sqcup C_2, D_1 \sqcup D_2) \mid \eta\colon B \xrightarrow{\sim} A \right\}.$$

Where A, B are arbitrary isomorphic finite nonempty sets. These data are required to satisfy the following axioms:

(1) $\mathcal{P}((1_C, 1_D)) = 1_{\mathcal{P}(C,D)}, \quad \mathcal{P}((\rho\rho', \sigma'\sigma)) = \mathcal{P}((\rho, \sigma))\ \mathcal{P}((\rho', \sigma'))$

(2) $(\mathcal{P}((\rho_1 \sqcup \rho_2|_{C_2}, \sigma_1|_{D_1} \sqcup \sigma_2))\ {}_B\overset{\eta}{\circ}_A = {}_{\sigma_1(B)}\overset{\rho_2\eta\sigma_1^{-1}}{\circ}_{\rho_2(A)}\ (\mathcal{P}((\rho_1, \sigma_1)) \otimes \mathcal{P}((\rho_2, \sigma_2))$

(3) ${}_{B_2 \sqcup B_3}\overset{\epsilon}{\circ}_{A_2 \sqcup A_3}\ (\ {}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1} \otimes 1) = {}_{B_1 \sqcup B_3}\overset{\eta}{\circ}_{A_1 \sqcup A_3}\ (1 \otimes\ {}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2})$ where $\tilde{\eta}, \tilde{\epsilon}$ are restrictions of η, ϵ to the pairs of nonempty sets A_1, B_1 and A_2, B_2, respectively.
For A_1, B_1 empty sets
${}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2}\ (\ {}_{B_3}\overset{\eta}{\circ}_{A_3} \otimes 1) = {}_{B_3}\overset{\eta}{\circ}_{A_3}\ (1 \otimes\ {}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2})$.
For A_2, B_2 empty sets
${}_{B_3}\overset{\epsilon}{\circ}_{A_3}\ (\ {}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1} \otimes 1) = {}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1}\ (1 \otimes\ {}_{B_3}\overset{\epsilon}{\circ}_{A_3})$.

whenever the expressions make sense.

By $\mathsf{Pr}_{\mathsf{DCor}}$ we will denote the category of properads with the obvious morphisms.

Remark 3. If we only consider Axiom 1., the resulting structure is called a Σ-bimodule. Obviously, by forgetting the composition maps, a properad gives rise to its underlying Σ-module.

All these notions are equivalent to their usual counterparts in [10]. For example, Axiom 1. stands for the left and right Σ-actions on C, D respectively, 2. expresses the equivariance and 3. expresses the associativity of the structure maps.

In this paper, we consider only properads such that the dg vector spaces $\mathcal{P}(C,D)$ have an additional $\mathbb{N}_0$ grading by a degree which will be denoted by G. The differential and both Σ-actions are assumed to preserve the degree G-components $\mathcal{P}(C,D,G)$. For operations ${}_B\overset{\eta}{\circ}_A$, we assume that they map the components with respective degrees G_1 and G_2 into the component of degree $G(G_1,G_2,A,B,\eta)$ which is determined, in general, by the degrees G_1, G_2 sets A, B and their identification η.

In our main example, i.e., in the example of the closed Frobenius properad, the result of such composition is in component $G_1+G_2+|A|-1$. This is not necesarily true in general. Sometimes, on relevant places, we will comment on the general case.

Also, let us introduce $\chi := 2G+|C|+|D|-2$. Correspondingly, we will use the notation $\mathcal{P}(C,D,\chi)$ for $\mathcal{P}(C,D,G)$ with $2G=\chi-|C|-|D|+2\geq 0$. Having in mind the example of the closed Frobenius properad, we refer to χ as the "Euler characteristic".

We will assume the stability condition $\chi>0$, unless explicitly mentioned otherwise. In particular, this means that for $G=0$, $|C|+|D|\geq 3$ and for $G=1$, $|C|+|D|\geq 1$. For $G>1$, there is no restriction on the number of inputs and outputs.

Here we should mention that we use slightly different conventions than in [10], where it is assumed that the sets C and D are always non-empty, i.e., there is always at least one input and one output. Also, in [10], one input and one output are allowed for $G=0$. We will comment on this further when describing the cobar complex and algebras over it.

Example 4 (The (closed) Frobenius properad $\mathcal{F}$)**.** For each $(C,D)\in\mathsf{DCor}$ and $\chi>0$, put $\mathcal{F}(C,D,\chi)=\Bbbk$, i.e., the linear span on one generator $p_{C,D,\chi}$ in degree zero. The Frobenius properad has the trivial differential and the trivial Σ-bimodule structure. The operations ${}_B\overset{\eta}{\circ}_A$ do not depend on sets A,B and η,

$${}_B\overset{\eta}{\circ}_A\colon p_{C_1,D_1\sqcup B,\chi_1}\otimes p_{C_2\sqcup A,D_2,\chi_2}\mapsto p_{C_1\sqcup C_2,D_1\sqcup D_2,\chi_1+\chi_2}\,.$$

Geometrically, this properad consists of homeomorphism classes of 2-dimensional compact oriented surfaces with two kinds of labeled boundary components, the inputs and outputs. Here, $G=g$, the geometric genus of the surface. Under the operation ${}_B\overset{\eta}{\circ}_A$, we have $g=g_1+g_2+|A|-1$. Hence, the Euler characteristic χ is indeed additive as indicated in the formula above. Bijections act by relabeling the inputs and outputs independently. The operation ${}_B\overset{\eta}{\circ}_A$ for a non-trivial pair of sets (A, B) consists of gluing surfaces along the inputs in B and outputs in A identified according to η.

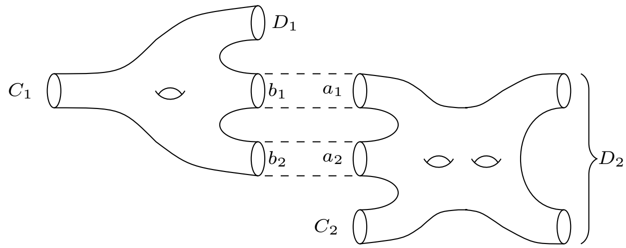

FIG. 1: ${}_B\overset{\eta}{\circ}_A$, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $\eta(b_1) = a_1, \eta(b_2) = a_2$

Remark 5. The closed Frobenius properad can actually be seen as an oriented 2-dimensional Riemann surface with punctures in the interior. The operation ${}_B\overset{\eta}{\circ}_A$ "glue" together these punctures according to the orientation of the surface. Another example with similar geometrical interpretation is the open Frobenius properad, where the punctures are within the boundaries of the surface. The boundaries can be permuted freely among themselves and the punctures on each boundary can also be cyclically permuted. The result of ${}_B\overset{\eta}{\circ}_A$ is obtained by (orientation preserving) gluing of two surfaces.

In this case, the Euler characteristic, contrary to the closed Frobenius properad, is not additive anymore. Concerning the genus of the resulting surface, it is given by a sum of genera of the original surfaces and the number of distinct pairs of boundaries which were "glued together". Therefore, the χ of the resulting surface is given by a a more complicated expression for $\chi(\chi_1, \chi_2, A, B, \eta)$[4].

3. Cobar complex

In this section we will introduce the cobar complex. The cobar complex of a properad $\mathcal{P}$ is a properad denoted by $C\mathcal{P}$. It is the free properad generated by the suspended dual of $\mathcal{P}$, with the differential induced by the duals of structure

[4]which is determined by the degrees χ_1, χ_2 sets A, B and their identification η

maps. Roughly speaking, $C\mathcal{P}$ is spanned by directed acyclic graphs, i.e. graphs with no directed circuits, and their vertices are decorated with elements of $\mathcal{P}^{\#}$.

To avoid problems with duals, we assume that the dg vector space $\mathcal{P}(C, D, G)$ is finite dimensional for any triple (C, D, G) whenever $C\mathcal{P}$ appears.

Definition 6. A graph consists of vertices and half-edges. Exactly one end of every half-edge is attached to a vertex. The other end is either unattached (such an half-edge is called a leg) or attached to the end of another half-edge (in that case, these two half-edges form an edge). Every end is attached to at most one vertex/end. The half-edge structure for vertex G_1 of the graph G is indicated on the following Fig. 2 on the left.

Definition 7. In a directed graph, every half-edge has assigned an orientation such that two half-edges composing one edge have the same orientation. The half-edges attached to each vertex are partitioned into incoming and outgoing half-edges.

A directed circuit in such graph is a set of edges such that we can go along them following their orientation and get to the point where we started.

We require that every vertex V_i a nonnegative integer G_i is assigned. We define

$$G := \dim_{\mathbb{Q}} H_1(\mathsf{G}, \mathbb{Q}) + \sum_i G_i$$

to be the genus of the graph. The stable graphs then fulfill the condition

$$\chi_i = 2(G_i - 1) + |C_i| + |D_i| > 0\,,$$

for every vertex V_i, where $|C_i|$ and $|D_i|$ denotes the number of outgoing resp. incoming half-edges attached to V_i.

Consider a finite directed graph G with no directed circuits and with integers G_i assigned to each vertex as is indicated on the picture on the right.

Finally, we require that the incoming legs of G are in bijection with the set D and outgoing legs with C.[5] The graph G is "decorated" by an element

$$(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n)\,, \tag{1}$$

where $V_1, \ldots V_n$ are all vertices of G, $\uparrow V_i$'s are formal elements of degree $+1$, $\wedge$ stands for the graded symmetric tensor product and $P_i \in \mathcal{P}(C_i, D_i, G_i)^{\#}$,

[5] In [4], it is shown that the number of isomorphism classes of (ordinary) stable graphs with legs labeled by the set $[n]$ and with the fixed genus G is finite. The additional conditions on graphs, i.e., being directed with no directed circuits, will obviously not change this.

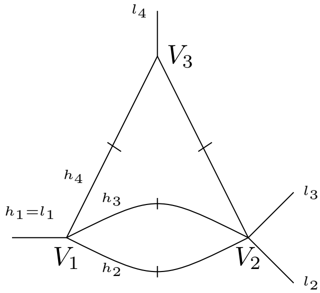

FIG. 2: Half-edge structure of the graph G and the directed graph G with integers G_i assigned to its vertices.

for every vertex V_i. Then the isomorphism class of G together with decoration 1 is an actual element of $C\mathcal{P}(C,D,G)$.

The operation $({}_B\overset{\eta}{\circ}{}_A)_{C\mathcal{P}}$ is defined by grafting of graphs, attaching together $|A|$ pairs of incoming and outgoing legs with the suitable orientation so that no directed circuits are formed.

The differential $\partial_{C\mathcal{P}}$ on $C\mathcal{P}$ is the sum of the differential $d_{P\#}$ and of the differential given by the dual of $({}_B\overset{\eta}{\circ}{}_A)$ which adds one vertex V, $|A|$ edges attached to it and modifies the decoration of G. The differential for the closed Frobenius properad on one vertex is given as

$$\partial_{C\mathcal{P}} = d_{P\#}\otimes 1 + \sum_{\substack{C_1\sqcup C_2=C\\ D_1\sqcup D_2=D\\ 1\leq |A|\leq G+1\\ G_1+G_2+|A|-1=G\\ \eta}} \frac{1}{|A|!}\left({}^{(C_1,D_1\sqcup B,G_1)}{}_B\overset{\eta}{\circ}{}_A{}^{(C_2\sqcup A,D_2,G_2)}\right)^{\#}_{P}\otimes(\uparrow V\wedge\cdot), \tag{2}$$

where

$$\begin{aligned}\left({}^{(C_1,D_1\sqcup B,G_1)}{}_B\overset{\eta}{\circ}{}_A{}^{(C_2\sqcup A,D_2,G_2)}\right)^{\#}_{P} &: P(C,D,G)^{\#}\\ &\to P(C_1,D_1\sqcup B,G_1)^{\#}\quad\otimes\quad P(C_2\sqcup A,D_2,G_2)^{\#},\end{aligned} \tag{3}$$

for stable vertices $(C_1,D_1\sqcup B,G_1)$ and $(C_2\sqcup A,D_2,G_2)$. For a general stable graph, the differential extends by the Leibniz rule.

Remark 8. As we have seen in Remark 5, not all properads have an additive Euler characteristic. Hence in general the sum in the differential (2) is over all $\chi=\chi(\chi_1,\chi_2,A,B,\eta)$ with χ, χ_1, χ_2 corresponding to G, G_1, G_2, respectively.

Nevertheless, the right-hand side still does not contain infinitely many terms as it may appear on the first view. This is due to condition $1\leq |A|\leq G+1$ (in general, the number $|A|$ of new edges is bounded by the genus of the original graph) and the condition of stability (every vertex V_i of the graph has $\chi_i > 0$). These conditions together restrict the number of terms on the right-hand side.

4. The endomorphism properad and algebras over a properad

Definition 9. For any set C, $|C| = n$, we define the unordered product $\bigodot_{c\in C} V_c$ of the collection of vector spaces $\{V_c\}_{c\in C}$ as the vector space of equivalence classes of usual tensor products

$$v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \in V_{\omega(1)} \otimes \cdots \otimes V_{\omega(n)},\ \omega\colon [n] \xrightarrow{\cong} C\,,$$

modulo the identifications

$$v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \sim \epsilon(\sigma)\ v_{\omega\sigma(1)} \otimes \cdots \otimes v_{\omega\sigma(n)},\ \sigma \in \Sigma_n\,,$$

where $\epsilon(\sigma)$ is the Koszul sign of the permutation σ.

It is possible to show by a direct computation, that there is a natural map

$$\overline{\sigma} : \bigodot_{c\in C} V_c \to \bigodot_{d\in D} V_d$$

of unordered products given by the assignment

$$\bigodot_{c\in C} V_c \ni \left[v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)}\right] \longmapsto \left[w_{\sigma\omega(1)} \otimes \cdots \otimes w_{\sigma\omega(n)}\right] \in \bigodot_{d\in D} V_d\,, \tag{4}$$

where $\sigma\colon C \to D$ is an isomorphism of finite sets and $\{V_c\}_{c\in C}$ and $\{W_d\}_{d\in D}$ are collections of graded vector spaces, such that $V_c = W_d = V$ for all $c \in C$, $d \in D$ with $w_{\sigma\omega(i)} := v_{\omega(i)} \in V_{\sigma\omega(i)}$, $1 \le i \le n$.

One also has a canonical isomorphism

$$\bigodot_{c'\in C'} V_{c'} \otimes \bigodot_{c''\in C''} V_{c''} \cong \bigodot_{c\in C'\sqcup C''} V_c$$

for two disjoint sets C', C''. By iterating this isomorphism we obtain a canonical isomorphism

$$\bigodot_{c\in C} V_c \cong V_{c_1} \otimes \cdots \otimes V_{c_n}$$

This allows us to define the endomorphism properad $\mathcal{E}_V$.

Definition 10. For $(C, D) \in \mathsf{DCor}$, $\chi > 0$ define

$$\mathcal{E}_V(C, D, \chi) := \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V)\,.$$

Let $\bar{f} \in \mathrm{Hom}_{\Bbbk}(\bigotimes_D V, \bigotimes_C)$ correspond to $f \in \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V)$, under the above isomorphism. Then the differential on $\mathcal{E}_V$ is given, by abuse of notation, by

$$(5) \quad d(\bar{f}) = \sum_{i=0}^{m-1} (1^{\otimes i} \otimes d \otimes 1^{\otimes m-i-1})\bar{f} - (-1)^{|\bar{f}|} \sum_{i=0}^{n-1} \bar{f}(1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1})$$

Given a morphism $(\rho, \sigma)\colon (C, D) \to (C', D')$ in DCor, define

$$\mathcal{E}_V(\rho, \sigma)\colon \mathcal{E}_V(C, D, \chi) \to \mathcal{E}_V(C', D', \chi)$$
$$f \mapsto \overline{\rho}\, f\, \overline{\sigma}\,,$$

for $f \in \mathrm{Hom}_{\Bbbk}(\bigodot_D V, \bigodot_C V) \in \mathcal{E}_V(C, D, \chi)$ and $\overline{\rho}$, $\overline{\sigma}$ as in (4).

For $f \in \mathcal{E}_V\big(C_2 \sqcup A, D_2, \chi_2\big)$ and $g \in \mathcal{E}_V\big(C_1, D_1 \sqcup B, \chi_1\big)$ let

$$g \;{}_B\overset{\eta}{\circ}_A\; f \in \mathcal{E}_V\big(C_1 \sqcup C_2, D_1, \sqcup D_2, \chi\big)\,.$$

Then the collection $\mathcal{E}_V = \{\mathcal{E}_V(C, D, \chi) | (C, D) \in \mathsf{DCor}, \chi > 0\}$ with the above operations is called the endomorphism properad.

Definition 11. Let $\mathcal{P}$ be a properad. An algebra over $\mathcal{P}$ on a dg vector space V is a properad morphism

$$\alpha\colon \mathcal{P} \to \mathcal{E}_V\,,$$

i.e. it is a collection of dg vector space morphisms

$$\{\alpha(C, D, \chi)\colon \mathcal{P}(C, D, \chi) \to \mathcal{E}_V(C, D, \chi) \mid (C, D) \in \mathsf{DCor}, \chi > 0\}$$

such that

(1) $\alpha \circ \mathcal{P}(\rho, \sigma) = \mathcal{E}_V(\rho, \sigma) \circ \alpha$ for any morphism (ρ, σ) in DCor

(2) $\alpha \circ (\,{}_B\overset{\eta}{\circ}_A)_{\mathcal{P}} = (\,{}_B\overset{\eta}{\circ}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha)$

(we drop the notation (C, D, χ) at $\alpha(C, D, \chi)$, for brevity)

Remark 12. Note that the above formula 2. is compatible with any composition map for the degree G, or equivalently for the Euler characteristic χ. This is because, for fixed sets C, D, the vector spaces $\mathcal{E}_V(C, D, \chi)$ are independent of the actual value of χ. So we always can choose the composition law for χ in the endomorphism properad $\mathcal{E}_V$ so that it respects the one for $\mathcal{P}$.

The following theorem is essentially the only thing we need from the theory of the cobar transform. Compare to Feynman transform for modular operads [1].

In order to describe an algebra over the cobar complex, it is enough to consider graphs with one vertex.

Theorem 13. *An algebra over the cobar complex $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection of degree 1 linear maps*

$$\left\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi)^{\#} \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathsf{DCor}, \chi > 0\right\},$$

(no compatibility with differential on $\mathcal{P}(C,D,\chi)^{\#}$!) such that

$$\mathcal{E}_V(\rho,\sigma) \circ \alpha(C,D,\chi) = \alpha(C',D',\chi) \circ \mathcal{P}(\rho^{-1},\sigma^{-1})^{\#}$$

for any pair of bijections $(\rho,\sigma) : (C,D) \xrightarrow{\sim} (C',D')$ and

$$\begin{aligned} d \circ \alpha(C,D,\chi) &= \alpha(C,D,\chi) \circ d_{\mathcal{P}^{\#}} \\ &\quad + \sum_{\substack{C_1 \sqcup C_2 = C \\ D_1 \sqcup D_2 = D \\ \chi = \chi(\chi_1,\chi_2,A,B,\eta) \\ \chi_1,\chi_2 > 0}} \frac{1}{|A|!} ({}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V} \circ \big(\alpha(C_1, D_1 \sqcup B, \chi_1) \\ &\quad \otimes \alpha(C_2 \sqcup A, D_2, \chi_2)\big) \circ ({}_B\overset{\eta}{\circ}{}_A)^{\#}_{\mathcal{P}}, \end{aligned} \tag{6}$$

where $({}_B\overset{\eta}{\circ}{}_A)^{\#}_{\mathcal{P}}$ is a shorthand notation for $({}^{(C_1,D_1\sqcup B,\chi_1)}{}_B\overset{\eta}{\circ}{}_A{}^{(C_2\sqcup A,D_2,\chi_2)})^{\#}_{P}$ from (3)

$$({}_B\overset{\eta}{\circ}{}_A)^{\#}_{\mathcal{P}} : \mathcal{P}(C,D,\chi)^{\#} \to \mathcal{P}(C_1, D_1 \sqcup B, \chi_1)^{\#} \otimes \mathcal{P}(C_2, D_2 \sqcup A, \chi_2)^{\#}.$$

Remark 14. For the closed Frobenius properad, the conditions on χ_1, χ_2, A, B could simply be formulated as a summation over A and G_1, G_2 s.t. $1 \le A \le G+1$, $G_1 + G_2 + |A| - 1 = G$.

5. Barannikov's type theory and homological differential operators

In Theorem 1 of [1], Barannikov observed that an algebra over the Feynman transform of modular operad $\mathcal{P}$ is equivalently described as a solution of a certain master equation in an algebra succinctly defined in terms of $\mathcal{P}$, cf. also Theorem 20 in [2]. In this section, we formulate the corresponding theorem for properads in our formalism and then adapt it to our applications.

Let us for simplicity assume $C = [m]$, $D = [n]$. We can show that there is an isomorphism sending a collection of α's from Theorem 13 to $\Sigma_m \times \Sigma_n$-invariant[6] in the space $(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi))$ given by

$$\mathrm{Hom}_{\Sigma_m\times\Sigma_n}(\mathcal{P}([m],[n],\chi)^{\#}, \mathcal{E}_V([m],[n],\chi)) \xrightarrow{\cong} {}^{\Sigma_m}(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi))^{\Sigma_n}$$

[6] In the following, the left upper index Σ_m denotes the invariants under the left action of Σ_m. And similarly for right upper index Σ_n.

$$\alpha \mapsto \sum_i p_i \otimes \alpha(p_i^{\#})$$

where $\{p_i\}$ is a $\Bbbk$-basis of $\mathcal{P}([m],[n],\chi)$ and $\{p_i^{\#}\}$ is its dual basis.

For a properad $\mathcal{P}$, define

$$P(m,n,\chi) := {}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n}$$

The differential transferred by the isomorphism is

$$d = d_{\mathcal{P}} \otimes 1_{\mathcal{E}_V} - 1_{\mathcal{P}} \otimes d_{\mathcal{E}_V}$$

and the composition $\circ$ is without lost of generality described as follows: Assume $g \in P(m_1, n_1 + |N|, \chi_1)$, $h \in P(m_2 + |M|, n_2, \chi_2)$ and $|N| = |M|$, then the $(m = m_1 + m_2, n = n_1 + n_2, \chi = \chi(\chi_1, \chi_2, A, B, \eta))$ component of the composition $g \circ h$ is given by

$$\sum (\mathcal{P}(\rho,\sigma) \otimes \mathcal{E}_V(\rho,\sigma))\Big(({}_N\overset{\xi}{\circ}{}_M)_{\mathcal{P}} \otimes ({}_N\overset{\xi}{\circ}{}_M)_{\mathcal{E}_V}\Big)\sigma_{23}(g \otimes h)\,, \tag{7}$$

with the sum running over all (m_1, m_2)-shuffles ρ and (n_2, n_1)-shuffles σ and σ_{23} is the flip exchanging the two middle factors. This allows us to reformulate Theorem 13:

Theorem 15. *An algebra over the cobar complex $C\mathcal{P}$ on a dg vector space V is equivalently given by a degree 1 element*

$$L \in P := \prod_{\substack{m,n\\ \chi>0}} {}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n}$$

satisfying the master equation

$$d(L) + L \circ L = 0\,. \tag{8}$$

The set of invariants is isomorphic to coinvariants with respect to diagonal $\Sigma_m \times \Sigma_n$-action[7]. Put $f_{p_i} := \bar{\alpha}(p_i^{\#})\colon V^{\otimes n} \to V^{\otimes m}$. Also, pick a homogeneous basis $\{a_i\}$ of V and denote $f^J_{p_i I}$ the respective coordinates of f_{p_i}, where $I := (i_1, \ldots, i_n)$ and $J := (j_1, \ldots, j_m)$ are multi-indices in $[\dim V]^{\times n}$ and in $[\dim V]^{\times m}$, respectively.

Hence, on components we have an isomorphism

$${}^{\Sigma_m}\left(\mathcal{P}([m],[n]\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n} \cong \mathcal{P}([m],[n],\chi)\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} V^{\otimes m} \otimes ((V^{\#})^{\otimes n})$$

$$\sum_i p_i \otimes \alpha(p_i^{\#}) \mapsto \frac{1}{m!n!}\sum_{i,I,J} f^J_{p_i I}(p_i\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (a_J \otimes \phi^I))\,. \tag{9}$$

[7]The argument is the same as in [7] after Proposition 6. just applied to both component of the tensor product.

Put $\tilde{P} := \prod_{m,n,\chi} \big(\mathcal{P}([m],[n],\chi)\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (V^{\otimes m} \otimes (V^{\#})^{\otimes n}\big)$, then we have also an isomorphism $P \cong \tilde{P}$ with the transferred differential

$$\tilde{d}\left(p\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (a_J \otimes \phi^I)\right) \\ = d_{\mathcal{P}}(p)\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (a_J \otimes \phi^I) - (-1)^{|p|} p\,{}_{\Sigma_m}\!\otimes_{\Sigma_n} d_{\mathcal{E}_V}(a_J \otimes \phi^I)$$

and the composition

$$\left(p_1\,{}_{\Sigma_{m_1}}\!\otimes_{\Sigma_{n_1}} (a_{J_1} \otimes \phi^{I_1})\right) \tilde{\circ} \left(p_2\,{}_{\Sigma_{m_2}}\!\otimes_{\Sigma_{n_2}} (a_{J_2} \otimes \phi^{I_2})\right) = \sum_{M,N,\xi} \left(({}_N\overset{\xi}{\circ}{}_M)_{\mathcal{P}}(p_1 \otimes p_2)\right)$$

$$_{\Sigma_{m_1+m_2-|M|}} \otimes_{\Sigma_{n_1+n_2-|M|}} \left(({}_N\overset{\xi}{\circ}{}_M)_{\mathcal{E}_V} (a_{J_1} \otimes \phi^{I_1}) \otimes (a_{J_2} \otimes \phi^{I_2})\right) \tag{10}$$

Finally, it can be useful to have the following interpretation of the operation $\tilde{\circ}$. Here we shall assume that our corollas have always at least one input and one output, i.e. we assume $\mathcal{P}(C, D, \chi)$ to be nontrivial only if both C and D are non-empty and $m + n > 2$, for $G = 0$. In this case, we introduce, similarly to [2], positional derivations

$$\frac{\partial^{(k)}}{\partial a_j} (a_{i_1} \otimes \ldots \otimes a_{i_m}) \\ = (-1)^{|a_j|(|a_{i_1}|+\ldots|a_{i_{k-1}}|)}\, \delta_j^{i_k}\, (a_{i_1} \otimes \ldots \otimes \widehat{a_{i_k}} \otimes \ldots \otimes a_{i_m}) \tag{11}$$

where $\delta_j^{i_k}$ is Kronecker delta. For sets $J = \{j_1, \ldots j_{|N|}\}$ and $K = \{k_1, \ldots k_{|N|}\}$

$$\frac{\partial^{(K)}}{\partial a_J} = \frac{\partial^{(k_1)}}{\partial a_{j_1}} \cdots \frac{\partial^{(k_{|N|})}}{\partial a_{j_{|N|}}}\,.$$

Although the formula defining the positional derivative might seem obscure at the first sight, its usefulness will be obvious from the forthcoming formula (12). The meaning of the positional derivative $\dfrac{\partial^{(k)}}{\partial a_j}$ is simple. Applied to a tensor product like $a_{i_1} \otimes \ldots \otimes a_{i_m}$ it is zero unless there is a tensor factor a_j at the k-th position, in which case it cancels this factor and produces the relevant Koszul sign. We have introduced it because, in contrary to the left derivative familiar from the supersymmetry literature, here we do not have a rule how to commute the tensor factor a_j to the left. The “inputs" from $(V^{\#})^{\otimes n_1}$ in equation (10) can then be interpreted as the partial derivations acting on the “outputs" from $V^{\otimes m_2}$, and hence we can interpret elements of $\tilde{P} = \prod_{m,n,\chi} \tilde{P}(m,n,\chi)$ as differential operators acting on $\tilde{P}_+ := \prod_k \tilde{P}(k,0,\chi)$

as

$$p_1\,{}_{\Sigma_{m_1}}\otimes_{\Sigma_{n_1}}(a_{J_1}\otimes\phi^{I_1})\colon p_2\,{}_{\Sigma_{m_2}}\otimes a_{J_2}\mapsto \pm\sum_{M,N,\xi}\frac{\partial^{\xi(N)}}{\partial a_N}(a_M)\,({}_N\overset{\xi}{\circ}{}_M)_{\mathcal{P}}(p_1\otimes p_2)\,{}_{\Sigma_{m_1+m_2-|M|}}\otimes_{\Sigma_{n_1-|M|}} a_{J_1}a_{J_2-M}\,, \tag{12}$$

where the sign $\pm$ is given as in (11). Hence, in the master equation $\tilde{d}\tilde{L}+\tilde{L}\tilde{\circ}\tilde{L}=0$ where $\tilde{L}=Y(L)$ with Y being the iso (9), the operation $\tilde{\circ}$ becomes the composition of differential operators. For this, recall that $\tilde{L}$ is of degree 1 so we can write $\tilde{L}\tilde{\circ}\tilde{L}=\frac{1}{2}[\tilde{L}\overset{\circ}{,}\tilde{L}]$ as the graded commutator.

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