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BOUNDS FOR THE CHARACTERISTIC RANK AND CUP-LENGTH OF ORIENTED GRASSMANN MANIFOLDS

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Abstract. We estimate the characteristic rank of the canonical k–plane bundle over the oriented Grassmann manifold $\widetilde{G}_{n,k}$. We then use it to compute uniform upper bounds for the $\mathbb{Z}_2$–cup-length of $\widetilde{G}_{n,k}$ for n belonging to certain intervals.

1. Introduction and preliminaries

Let us denote $G_{n,k}$ the Grassmann manifold of k–dimensional vector subspaces in $\mathbb{R}^n$, i.e. the space $O(n)/(O(k)\times O(n-k))$. Next, denote $\widetilde{G}_{n,k}$ the *oriented* Grassmann manifold of *oriented* k–dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k)\times SO(n-k))$. We may suppose that $k\le n-k$ for both of them.

For a topological space X we can define its $\mathbb{Z}_2$-cup-length $\mathrm{cup}_{\mathbb{Z}_2}(X)$ as the greatest number r such that there exist $x_1,\dots,x_r\in\widetilde{H}^*(X;\mathbb{Z}_2)$ with cup-product $x_1\cdots x_r\neq 0$. For a path connected space X, the condition is equivalent to the existence of cohomology classes $x_1,\dots,x_r\in H^*(X;\mathbb{Z}_2)$ in positive dimensions such that $x_1\cdots x_r\neq 0$.

In this paper we will be considering only cohomology with $\mathbb{Z}_2$ coefficients, thus we will abbreviate $H^j(X;\mathbb{Z}_2)$ to $H^j(X)$ and $\mathrm{cup}_{\mathbb{Z}_2}(X)$ to $\mathrm{cup}(X)$ henceforth.

The cohomology ring of the Grassmann manifold $G_{n,k}$ is (see [1])

$$H^*(G_{n,k})=\mathbb{Z}_2[w_1,w_2,\dots,w_k]/I_{n,k}\,, \tag{1.1}$$

where $\dim(w_i)=i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1},\bar{w}_{n-k+2},\dots,\bar{w}_n$, where each $\bar{w}_i$ denotes the i–dimensional component of the formal power series

$$1+(w_1+w_2+\dots+w_k)+(w_1+w_2+\dots+w_k)^2+(w_1+w_2+\dots+w_k)^3+\cdots.$$

Each indeterminate w_i is a representative of the ith Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical k-plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

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On the other hand, the cohomology ring of the oriented Grassmann manifold $\widetilde{G}_{n,k}$ is fully known for spheres $\widetilde{G}_{n,1} \cong S^{n-1}$, complex quadrics $\widetilde{G}_{n,2}$ and some other cases (see e.g. [6]), but there is no general formula similar to (1.1).

There has been some work done to compute cup-length of some families of oriented Grassmann manifolds $\widetilde{G}_{n,3}$. In particular, the paper by Fukaya [2] (where a slightly different notation for Grassmann manifolds is used; $\widetilde{G}_{n,3}$ corresponds to $\widetilde{G}_{n+3,3}$ in this paper) contains the proof, that $\operatorname{cup}(\widetilde{G}_{2^t-1,3}) = 2^t - 3$ for $t \geq 3$, and the following interesting conjecture [2, Conjecture 1.2] (adjusted to our notation and replacing the letter k with a to avoid ambiguity)

$$\operatorname{cup}(\widetilde{G}_{n,3}) = \begin{cases} 2^t - 3, & \textit{when } 2^t - 1 \leq n \leq 2^t + 2^{t-1} - 3\,, \\ 2^t - 3 + a, & \textit{when } n = 2^t + 2^{t-1} - 2 + a, 0 \leq a \leq 2\,, \\ 2^t + 2^{t-1} + \cdots & \textit{when } n = 2^t + 2^{t-1} + \cdots + 2^j + 1 + a, \\ +2^{j+1} + 2^{j-1} + a, & 0 \leq a \leq 2^{j-1} - 1. \end{cases}$$

The value $\operatorname{cup}(\widetilde{G}_{2^t-1,3}) = 2^t - 3$ has been obtained independently by Korbaš [4] employing an approach using the notion of characteristic rank. Making use of refined version of this idea, some other parts of the conjecture have been proved in papers [10], [11]. Namely, the cases corresponding to n in the interval $2^t - 1 \leq n < 2^t - 1 + \frac{2^t}{3}$ for $t \geq 3$ and $n = 2^t + 2^{t-1} + a$ for $a = 1, 2$ and $t \geq 3$.

The characteristic rank of a manifold was introduced by Korbaš [4] and later generalized by Naolekar and Thakur [9] to the characteristic rank of a vector bundle.

Definition 1.1. Let X be a connected, finite CW-complex and ξ a real vector bundle over X. The *characteristic rank* of the vector bundle ξ, $\operatorname{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

Following theorem illustrates how characteristic rank can be used to obtain upper bounds for the cup-length of a manifold.

Theorem 1.2 ([9, Theorem 1.2]). *Let X be a connected closed smooth d-manifold and let ξ be a vector bundle over X, such that there exists j, $j \leq \operatorname{charrank}(\xi)$, such that every monomial $w_{i_1}(\xi) \dots w_{i_r}(\xi)$ for $0 \leq i_t \leq j$ of degree d is zero. Then*

$$\operatorname{cup}(X) \leq 1 + \frac{d - j - 1}{r_X}\,,$$

where r_X is the smallest positive integer, such that $\widetilde{H}^{r_X}(X; \mathbb{Z}_2) \neq 0$.

The main result of this paper is the following theorem, listed as Theorem 4.1, providing a lower bound for the characteristic rank of the canonical k–plane bundle $\widetilde{\gamma}_{n,k}$ over $\widetilde{G}_{n,k}$.

Theorem A. *Let $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$. For any $x \geq 0$ and n such that*

$$(k-1) \cdot 2^{t-1} + \frac{k-3}{k-1} \cdot 2^{t-1} + \frac{x}{k-1} - 1 < n \leq k \cdot 2^{t-1} - 1 - x$$

we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \geq n-k+x\,.$$

This result, combined with Theorem 1.2, leads to an upper bound for the cup-length of the oriented Grassmann manifold $\widetilde{G}_{n,k}$.

Denoting $x_{k,t}$ the smallest positive integer such that $2^t - x_{k,t}$ is divisible by $k-1$ and

$$n_{k,t} = k \cdot 2^{t-1} - \frac{2^t - x_{k,t}}{k-1} - 1,$$

we obtain the associated result on cup-length, listed as Theorem 4.3.

Theorem B. *For any $k \geq 3$, $t \geq \max\{3, \log_2(k-1)\}$ and $a \geq 0$, such that a satisfies $ka + x_{k,t} - 1 \leq \frac{2^t - x_{k,t}}{k-1}$ we have*

$$\operatorname{cup}(\widetilde{G}_{n_{k,t}+a,k}) \leq 1 + \frac{k(n_{k,t}-k) - (n_{k,t}-k+x_{k,t})}{2}\,.$$

The interesting feature of this upper bound is that it is uniform. For each k and t satisfying the conditions there is a certain interval starting at $n_{k,t}$, such that for any $n = n_{k,t} + a$ in this interval the cup-length of $\widetilde{G}_{n,k}$ is bounded by the right-hand side, which does not depend on a. As we will discuss in Section 4, if this upper bound for some $n_{k,t}$ happens to be the exact value of the cup-length of $\widetilde{G}_{n_{k,t},k}$, due to uniformity it becomes the exact value of the cup-length of all $\widetilde{G}_{n_{k,t}+a,k}$ in the interval.

The paper is organized as follows.

In Section 2 of this paper a review of the method to obtain lower bounds for the characteristic rank of $\widetilde{\gamma}_{n,k}$ is presented. It concludes with Proposition 2.1, which is the foundation for the rest of the paper.

Section 3 consists of additional observations and further analysis. It also contains all technical lemmas.

Finally, proofs of the results are the content of Section 4.

2. Estimating the characteristic rank of $\widetilde{\gamma}_{n,k}$

For our purposes, the cohomology of the oriented Grassmann manifold $\widetilde{G}_{n,k}$ is best described through its relation to the cohomology of the (unoriented) Grassmann manifold $G_{n,k}$; the following approach is the same as utilized in papers [5, 6, 7, 11].

There is a covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n,k) \neq (2,1)$. To this 2-fold covering, there is an associated line bundle ξ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence ([8, Corollary 12.3])

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} \tag{2.1}$$

where $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel–Whitney class $w_1(\xi) = w_1(\gamma_{n,k})$.

Since the pullback $p^*\gamma_{n,k}$ is isomorphic to $\widetilde{\gamma}_{n,k}$, the covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$ induces the ring homomorphism $p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$, which maps each Stiefel–Whitney class $w_i(\gamma_{n,k})$ to $w_i(\widetilde{\gamma}_{n,k})$.

Consequently, the image $\operatorname{Im}(p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$–vector space $H^j(\widetilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel–Whitney characteristic classes of $\widetilde{\gamma}_{n,k}$. We shall call it *the characteristic subspace* and denote it $C(j;n,k)$. Moreover (see [12]), the image $\operatorname{Im}(p^*)$ of the ring homomorphism $p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$ is a *self-annihilating* subspace of $H^*(\widetilde{G}_{n,k})$ (that is, for any $x \in C(j;n,k)$ and $y \in C(j';n,k)$ we have $xy = 0$ if $j + j' = k(n-k) = \dim(\widetilde{G}_{n,k})$).

This implies that the characteristic rank of $\widetilde{\gamma}_{n,k}$ is equal to the greatest integer q, such that the homomorphism $p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k})$ is surjective for all j, $0 \le j \le q$, or equivalently, by (2.1), that the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective for all j, $0 \le j \le q$.

Hence, in order to compute the characteristic rank of $\widetilde{\gamma}_{n,k}$, it is necessary to study the kernel of $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$. The following is a brief summary of the approach employed in the work of Korbaš and Rusin [7].

For the $\mathbb{Z}_2$-vector space $H^j(G_{n,k})$ the set

$$\big\{w_1(\gamma_{n,k})^{a_1} w_2(\gamma_{n,k})^{a_2} \dots w_k(\gamma_{n,k})^{a_k} : \sum_{i=1}^{k} i a_i = j, \sum_{i=1}^{k} a_i \le n-k\big\}$$

is an additive basis ([3]). We will call it *the standard basis* for $H^j(G_{n,k})$. An element of the standard basis for $H^j(G_{n,k})$ is called *regular*, if its image under the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is an element of the standard basis for $H^{j+1}(G_{n,k})$. Otherwise it is called *singular*.

We can compute the number of singular elements of the standard basis for $H^j(G_{n,k})$, which is an upper bound for the dimension of the kernel of the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$. For any $j \le n-k-1$, all the elements in the standard basis for $H^j(G_{n,k})$ are regular ([7, Proposition 2.1]), thus $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective, hence we recover the known inequality ([5, (2.5)])

$$\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \ge n-k-1\,.$$

Further analysis shows that there is a better sufficient condition for the injectivity of the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$, which will provide sharper lower bounds for the characteristic rank of $\widetilde{\gamma}_{n,k}$. First, let us take a closer look at the description of the cohomology ring $H^*(G_{n,k})$ (see (1.1)). The ideal $I_{n,k}$ is generated by the polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \dots, \bar{w}_n$. Thus cohomology classes corresponding to these polynomials are zero classes. Hence by denoting g_i the reduction of the polynomial $\bar{w}_i$ modulo w_1, the polynomials $g_{n-k+1}, g_{n-k+2}, \dots, g_n \in \mathbb{Z}_2[w_2, \dots, w_k]$ become representatives of $w_1(\gamma_{n,k})$-multiples of some cohomology classes in $H^*(G_{n,k})$. Let us denote $g_i(\gamma_{n,k})$ the cohomology class corresponding to the polynomial g_i. Then for $i \in \{n-k+1, n-k+2, \dots, n\}$ the class $g_i(\gamma_{n,k})$ lies in the image of $w_1\colon H^{i-1}(G_{n,k}) \longrightarrow H^i(G_{n,k})$.

By estimating the dimension of the image of this homomorphism, we obtain upper bound for the dimension of its kernel.

Proposition 2.1 ([7, Proposition 2.4. 3)]). *For a non-negative integer x, we associate with $H^{n-k+x+1}(G_{n,k})$ $(2 \le k \le n-k)$ the set*

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}.$$

If $x \le n-k-1$ and the set $N_x(G_{n,k})$ is linearly independent, then

$$w_1 \colon H^{n-k+x}(G_{n,k}) \longrightarrow H^{n-k+x+1}(G_{n,k})$$

is a monomorphism.

3. The polynomials g_i

The polynomial g_i associated with $G_{n,k}$ is a representative of the cohomology class $g_i(\gamma_{n,k})$ in $H^i(G_{n,k})$. We can also consider arbitrary polynomial $f_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ as a representative of some cohomology class.

Definition 3.1. If f_i is of the form

$$f_i = \sum_{2a_{j,2}+\cdots+ka_{j,k}=i} c_j w_2^{a_{j,2}} \cdots w_k^{a_{j,k}} \ ; \ c_j \in \mathbb{Z}_2,$$

we denote the corresponding cohomology class

$$f_i(\gamma_{n,k}) = \sum_{2a_{j,2}+\cdots+ka_{j,k}=i} c_j w_2(\gamma_{n,k})^{a_{j,2}} \cdots w_k(\gamma_{n,k})^{a_{j,k}} \in H^i(G_{n,k})$$

and we will say that the polynomial f_i *lies in the dimension i.*

Note. The zero polynomial lies in the dimension i for any i.

Later, we will consider sums of such polynomials.

Definition 3.2. Let $i \in \mathbb{N}$ and $f \in \mathbb{Z}_2[w_2, \ldots, w_k]$ be a polynomial that is a sum of polynomials each of which lies in dimension $j \le i$. We will say that such a polynomial f is *contained within dimension i.*

We already know that for every $i \ge 1$ the polynomial g_i lies in the dimension i. Now, of course, if we fix some k, the corresponding cohomology class $g_i(\gamma_{n,k}) \in H^i(G_{n,k})$ depends on the n, but the polynomial itself does not. We will have this in mind while we further explore properties of the polynomials g_i.

For $i - k \ge 1$ there is a recurrence formula implied by [5, (2.4)]

$$g_i = \sum_{j=2}^{k} w_j g_{i-j} = w_2 g_{i-2} + w_3 g_{i-3} + \cdots + w_k g_{i-k}. \tag{3.1}$$

By applying the formula twice we obtain

$$g_i = \sum_{j=2}^{k} w_j g_{i-j} = \sum_{j=2}^{k} w_j \sum_{l=2}^{k} w_l g_{i-j-l} = \sum_{j,l=2}^{k} w_j w_l g_{i-j-l} = \sum_{j=2}^{k} w_j^2 g_{i-2j},$$

and it is apparent that induction leads to the generalized formula

$$g_i = \sum_{j=2}^{k} w_j^{2^s} g_{i-j\cdot 2^s}\,, \tag{3.2}$$

valid for all s such that $i - k \cdot 2^s \geq 1$.

For convenience we introduce the following formalism expanding the definition of polynomials g_i associated with $G_{n,k}$.

Definition 3.3. For negative integers i we define formal Laurent series g_i associated with $G_{n,k}$ in indeterminates $w_2, w_3, \ldots, w_k$ recursively by the relation $g_i = w_2 g_{i-2} + w_3 g_{i-3} + \cdots + w_k g_{i-k}$.

This allows us to use (3.1) without restrictions on the integer i. And so, in the system $\{g_i\}_{i\in\mathbb{Z}}$ of polynomials (for $i \geq 1$) and formal Laurent series (for $i \leq 0$) the relation

$$g_i = \sum_{j=2}^{k} w_j^{2^s} g_{i-j\cdot 2^s} \tag{3.3}$$

is satisfied for all $i \in \mathbb{Z}$ and all $s \geq 0$.

These formal Laurent series do not represent elements in $H^*(G_{n,k})$, but they can be used to derive information about the polynomials g_i, which do.

For example, the first few corresponding formal Laurent series g_i associated with $G_{n,3}$ are as follows

$$\begin{aligned}
g_{-1} &= 0\\
g_{-2} &= 0\\
g_{-3} &= w_3^{-1}\\
g_{-4} &= w_2 w_3^{-2}\\
g_{-5} &= w_2^2 w_3^{-3}\\
g_{-6} &= w_2^3 w_3^{-4} + w_3^{-2}
\end{aligned}$$

We wish to gather more information about these formal Laurent series. For any $k \geq 2$ we obtain a corresponding sequence $\{g_i\}_{i\in\mathbb{Z}}$ of formal Laurent series. Until now, it was sufficient to consider each sequence $\{g_i\}_{i\in\mathbb{Z}}$ separately, but to study their properties, we will make use of some interplay between them.

Lemma 3.4. *Let us denote $g_{i,k}$ the polynomials associated with $G_{n,k}$. For all $k \geq 2$ we have the following relations :*

i) *We have $g_{0,k} = 1$.*

ii) *Reduction of the polynomial $g_{i,k}$ modulo w_k is exactly $g_{i,k-1}$.*

iii) *For $0 \leq i < k$ we have $g_{i,k} = g_{i,k-1}$*

iv) *We have*

$$\begin{aligned} g_{-1,k} &= 0 \\ g_{-2,k} &= 0 \\ &\vdots \\ g_{-(k-1),k} &= 0 \\ g_{-k,k} &= w_k^{-1} \end{aligned}$$

Proof. The first two parts are immediate consequences of the definition of the polynomials $g_{i,k}$.

Part iii) is directly implied by the second part, since $g_{i,k}$ lies in dimension $i < k$ and thus it cannot contain any terms divisible by w_k.

For part iv), it is easy to check that for $k = 2$ we indeed have $g_{-1,2} = 0$ and $g_{-2,2} = w_2^{-1}$. Now we proceed by induction on k.

Suppose that for some $k \geq 3$ we have

$$g_{-i,k-1} = 0 \quad \text{for} \quad i = 1, 2, \ldots, k-2 \tag{3.4}$$

$$g_{-(k-1),k-1} = w_{k-1}^{-1} \, . \tag{3.5}$$

First, we will prove, by induction on i, that $g_{-i,k} = 0$ for $i = 1, \ldots, k-1$. We have, by (3.3),

$$\begin{aligned} g_{k-1,k} &= \sum_{j=2}^{k} w_j g_{k-j-1,k} \, , \\ g_{k-1,k} &= \sum_{j=2}^{k-1} (w_j g_{k-j-1,k}) + w_k g_{-1,k} \, , \\ w_k g_{-1,k} &= g_{k-1,k} + \sum_{j=2}^{k-1} (w_j g_{k-j-1,k}) \, , \end{aligned}$$

and by part iii), also

$$w_k g_{-1,k} = g_{k-1,k-1} + \sum_{j=2}^{k-1} (w_j g_{k-j-1,k-1}) \, ,$$

but, by the recurrence relation for polynomials $g_{i,k-1}$, the RHS is zero. Hence $g_{-1,k} = 0$.

Now, suppose $g_{-1,k} = \cdots = g_{-(i-1),k} = 0$ and $i \le k-1$. By the recurrence formula for $g_{k-i,k}$ we have

$$g_{k-i,k} = \sum_{j=2}^{k} w_j g_{k-i-j,k} \,,$$

$$g_{k-i,k} = \sum_{j=2}^{k-1} (w_j g_{k-i-j,k}) + w_k g_{-i,k} \,,$$

$$w_k g_{-i,k} = g_{k-i,k} + \sum_{j=2}^{k-1} w_j g_{k-i-j,k} \,,$$

$$w_k g_{-i,k} = g_{k-i,k} + \sum_{j=2}^{k-i} w_j g_{k-i-j,k} + \sum_{j=k-i+1}^{k-1} w_j g_{k-i-j,k} \,.$$

The last sum is zero by the induction hypothesis and thus by part iii) we have

$$w_k g_{-i,k} = g_{k-i,k-1} + \sum_{j=2}^{k-i} w_j g_{k-j-i,k-1} \,.$$

By substituting the recurrence relation for the polynomial $g_{k-i,k-1}$, we obtain

$$w_k g_{-i,k} = \sum_{j=k-i+1}^{k-1} w_j g_{k-j-i,k-1} \,,$$

$$w_k g_{-i,k} = w_{k-i+1} g_{-1,k-1} + \cdots + w_{k-1} g_{-(i-1),k-1} \,.$$

Since $i \le k-1$ and therefore $i-1 \le k-2$, by (3.4) the RHS is zero and $g_{-i,k} = 0$ as well.

Finally, we have

$$1 = g_{0,k} \,,$$

$$1 = \sum_{j=2}^{k} w_j g_{-j,k} \,,$$

$$1 = \sum_{j=2}^{k-1} (w_j g_{-j,k}) + w_k g_{-k,k} \,.$$

And since $g_{-j,k} = 0$ for $j = 2, \dots, k-1$, this yields $g_{-k,k} = w_k^{-1}$. □

From now on fix some k and again write simply g_i instead of $g_{i,k}$. Examining the occurrence of powers of w_k in the formal Laurent series g_i leads to discovering a nice pattern in the form of polynomials g_i for certain values of i.

Lemma 3.5. *For $i \ge 1$ the formal Laurent series g_{-i-k+1} is of the form*

$$g_{-i-k+1} = w_k^{-i} h_i \,,$$

where $h_i \in \mathbb{Z}_2[w_2, w_3, \dots, w_k]$ is a polynomial of the form

$$h_i = w_{k-1}^{i-1} + \text{ terms divisible by } w_k \,, \tag{3.6}$$

and h_i lies in the dimension $(k-1)(i-1)$.

Proof. We have $g_{-1} = g_{-2} = \cdots = g_{-(k-1)} = 0$ and $g_{-k} = w_k^{-1}$. For all $i \geq 2-k$ let us define formal Laurent series $h_i = w_k^i g_{-i-k+1}$.

We have that $h_{2-k} = h_{3-k} = \cdots = h_0 = 0$ and $h_1 = 1$. By the recurrence formula (3.3), we have for all $i \geq 2$

$$\begin{aligned}
g_{-i+1} &= \sum_{j=2}^{k} w_j g_{-i-j+1} \\
w_k^{i-1} g_{-i+1} &= \sum_{j=2}^{k} w_j w_k^{i-1} g_{-i-j+1} \\
w_k^{k-1} w_k^{i-k} g_{-i+1} &= \sum_{j=2}^{k} w_j w_k^{k-j-1} w_k^{i+(j-k)} g_{-i-j+1} \\
w_k^{k-1} h_{i-k} &= \sum_{j=2}^{k} w_j w_k^{k-j-1} h_{i+(j-k)} \\
w_k^{k-1} h_{i-k} &= h_i + w_{k-1} h_{i-1} + \sum_{j=2}^{k-2} w_j w_k^{k-j-1} h_{i+(j-k)} \\
h_i &= w_{k-1} h_{i-1} + \sum_{j=2}^{k-2} \left(w_j w_k^{k-j-1} h_{i+(j-k)} \right) + w_k^{k-1} h_{i-k}
\end{aligned}$$

and the statement follows from the obvious induction. □

Lemma 3.6. *For $t \geq 1$ and $1 \leq a \leq 2^{t-1}$ we have*

$$\begin{aligned}
g_{(k-1)\cdot 2^{t-1}-k+a} = \sum_{j=2}^{k-1} &\left(w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a} \right) + w_{k-1}^{2^{t-1}-a} w_k^{a-1} + \\
&+ \text{terms divisible by } \ w_k^a .
\end{aligned}$$

Proof. By (3.3), we have

$$\begin{aligned}
g_{(k-1)\cdot 2^{t-1}-k+a} &= \sum_{j=2}^{k} w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a}, \\
&= \sum_{j=2}^{k-1} \left(w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a} \right) + w_k^{2^{t-1}} g_{-2^{t-1}-k+a}, \\
&= \sum_{j=2}^{k-1} \left(w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a} \right) + w_k^{a-1} h_{2^{t-1}+1-a}
\end{aligned}$$

and applying Lemma 3.5 gives the desired result. □

In order to utilize Proposition 2.1 we need to consider linear combinations of elements of

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}\,.$$

Any such linear combination is a polynomial

$$f_x g_{n-k+1} + f_{x-1} g_{n-k+2} + \cdots + f_{x-k+1} g_n\,,$$

where $f_x, f_{x-1}, \ldots, f_{x-k+1} \in \mathbb{Z}_2[w_2, w_3, \ldots, w_k]$ are some polynomials in dimensions $x, x-1, \ldots, x-k+1$ respectively.

We can think of this in the following way. Let us define an element g in the ring of formal power series $\mathbb{Z}_2[[w_2, w_3, \ldots, w_k]]$ by

$$g = \sum_{i=0}^{\infty} g_i = g_0 + g_1 + g_2 \cdots\,.$$

If we denote $f = f_x + f_{x-1} + \cdots + f_{x-k+1}$, then in the formal series $f \cdot g$ the sum of terms lying in the dimension $n - k + 1 + x$ is exactly

$$f_x g_{n-k+1} + f_{x-1} g_{n-k+2} + \cdots + f_{x-k+1} g_n\,.$$

Hence from the polynomial f we can recover the original polynomial $f_x g_{n-k+1} + f_{x-1} g_{n-k+2} + \cdots + f_{x-k+1} g_n$ just by remembering in which dimension it lies.

Definition 3.7. For any polynomial $f \in \mathbb{Z}_2[w_2, w_3, \ldots, w_k]$ let us denote $[f \cdot g]_i$ the sum of all terms of the formal power series $f \cdot g$ lying in the dimension i.

Lemma 3.8. *For any i, k, $a \in \mathbb{N}, j \leq k$ and f, $f' \in \mathbb{Z}_2[w_2, w_3, \ldots, w_k]$ we have equalities*

$$[(f + f') \cdot g]_i = [f \cdot g]_i + [f' \cdot g]_i, \tag{3.7}$$

$$w_j^a [f \cdot g]_i = [w_j^a f \cdot g]_{i+ja}\,. \tag{3.8}$$

Whenever the polynomial f is divisible by $1 + w_2 + \cdots + w_k$ and contained in dimension i, we have

$$[f \cdot g]_i = 0\,. \tag{3.9}$$

Proof. The first two equalities are obvious from the definition. To prove the last statement suppose that

$$f = (1 + w_2 + \cdots + w_k)(f_0 + \cdots + f_{i-k})$$

for some polynomials in dimensions $0, \ldots, i-k$ respectively.

For the sake of brevity, let us denote $w_1 = 0$ and $w_0 = 1$ as elements of $\mathbb{Z}_2[w_2, w_3, \ldots, w_k]$ until the rest of the proof. This allows us to write

$$f = \sum_{j=0}^{k} \sum_{x=0}^{i-k} w_j f_x\,,$$

which, by (3.7), implies

$$[f \cdot g]_i = \sum_{j=0}^{k} \sum_{x=0}^{i-k} [w_j f_x \cdot g]_i \,.$$

Each f_x is a sum of monomials in the same dimension x, hence by (3.7) and repeated use of (3.8) we have $[w_j f_x \cdot g]_i = f_x[w_j \cdot g]_{i-x}$ and so

$$\begin{aligned}
[f \cdot g]_i &= \sum_{j=0}^{k} \sum_{x=0}^{i-k} f_x [w_j \cdot g]_{i-x} \,, \\
&= \sum_{x=0}^{i-k} f_x \sum_{j=0}^{k} [w_j \cdot g]_{i-x} \,, \\
&= \sum_{x=0}^{i-k} f_x [(1 + w_2 + \cdots + w_k) \cdot g]_{i-x} \,, \\
&= \sum_{x=0}^{i-k} f_x (g_{i-x} + w_2 g_{i-x-2} + \cdots + w_k g_{i-x-k}) \,,
\end{aligned}$$

which, by the recurrence formula (3.1), is a sum of zeros. □

Definition 3.9. Suppose $f \in \mathbb{Z}_2[w_2, w_3 \ldots, w_k]$. Let us denote $\bar{f}$ the polynomial obtained from f by replacing each instance of w_k with the sum $1 + w_2 + \cdots + w_{k-1}$.

Corollary 3.10. *Suppose $i \in \mathbb{N}$ and polynomial $f \in \mathbb{Z}_2[w_2, w_3, \ldots, w_k]$ is contained within dimension i. Then $\bar{f} \in \mathbb{Z}_2[w_2, w_3 \ldots, w_{k-1}]$ is contained within dimension i and we have*

$$[f \cdot g]_i = [\bar{f} \cdot g]_i \,.$$

Proof. The polynomial f is some finite sum of monomials

$$f = \sum_j w_2^{a_{j,2}} w_3^{a_{j,3}} \ldots w_k^{a_{j,k}} \,,$$

where for each j we have $2a_{j,2} + 3a_{j,3} \cdots + k a_{j,k} \le i$. Hence we have

$$\bar{f} = \sum_j w_2^{a_{j,2}} w_3^{a_{j,3}} \ldots w_{k-1}^{a_{j,k-1}} (1 + w_2 + \cdots + w_{k-1})^{a_{j,k}} \,.$$

Each summand is now a polynomial, which is contained in dimension $2a_{j,2} + 3a_{j,3} \cdots + (k-1)a_{j,k-1} + (k-1)a_{j,k} \le i$.

Also $f + \bar{f}$ is the following sum

$$\sum_j w_2^{a_{j,2}} w_3^{a_{j,3}} \ldots w_{k-1}^{a_{j,k-1}} \left((1 + w_2 + \cdots + w_{k-1})^{a_{j,k}} + w_k^{a_{j,k}} \right) .$$

Each summand is divisible by $(1 + w_2 + \cdots + w_{k-1})^{a_{j,k}} + w_k^{a_{j,k}}$ and consequently the polynomial $f + \bar{f}$ is divisible by the common divisor $(1 + w_2 + \cdots + w_{k-1}) + w_k$.

Hence by the Lemma 3.8 we have $[(f + \bar{f}) \cdot g]_i = 0$, which by (3.7) implies that $[f \cdot g]_i = [\bar{f} \cdot g]_i$. □

Corollary 3.11. *Suppose $f \in \mathbb{Z}_2[w_2, w_3 \dots, w_k]$ is a nonzero polynomial of the form $f = f_x + f_{x-1} + \cdots f_{x-k+1}$ where $f_x, f_{x-1}, \dots, f_{x-k+1}$ are polynomials lying in dimensions $x, x-1, \dots, x-k+1$ respectively. Then $\bar{f}$ is nonzero.*

Proof. By the same argument as before, the polynomial $f + \bar{f}$ is divisible by $1 + w_2 + \cdots + w_k$. But because of the range of dimensions in which the terms of polynomial f lie, the polynomial f is not divisible by $1 + w_2 + \cdots + w_k$. Hence the difference, $\bar{f}$, is nonzero. $\square$

Proposition 3.12. *Let $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$. For any $x \geq 0$ and n such that*

$$(k-1) \cdot 2^{t-1} + \frac{k-3}{k-1} \cdot 2^{t-1} + \frac{x}{k-1} - 1 < n \leq k \cdot 2^{t-1} - 1 - x$$

the set $N_x(G_{n,k})$ is linearly independent.

Remark. The lower bound for t is chosen just so that there exists an $n \geq 2k$ in the given interval.

Proof. Suppose the converse is true. By the earlier discussion about the form of linear combinations of elements of $N_x(G_{n,k})$ it means that there exists a nonzero polynomial $f = f_x + f_{x-1} + \cdots + f_{x-k+1}$, where $f_x, f_{x-1}, \dots, f_{x-k+1} \in \mathbb{Z}_2[w_2, w_3, \dots, w_k]$ are some polynomials in dimensions $x, x-1, \dots, x-k+1$ respectively, such that

$$f_x g_{n-k+1} + f_{x-1} g_{n-k+2} + \cdots + f_{x-k+1} g_n = 0\,,$$

or equivalently, $[f \cdot g]_{n-k+1+x} = 0$.

Since f is nonzero, by Corollary 3.11, the polynomial $\bar{f}$ is nonzero as well. By Corollary 3.10 we have $[\bar{f} \cdot g]_{n-k+1+x} = 0$.

As $\bar{f}$ is nonzero and contained in dimension x, there must exist $y \leq x$ such that

$$\bar{f} = \bar{f}_y + \cdots + \bar{f}_0\,,$$

where each $\bar{f}_i$ lies in dimension i and $\bar{f}_y \neq 0$. Hence

$$0 = [\bar{f} \cdot g]_{n-k+1+x} = \bar{f}_y g_{n-k+1+x-y} + \cdots + \bar{f}_0 g_{n-k+1+x}\,,$$

or equivalently

$$0 = \sum_{i=0}^{y} \bar{f}_i g_{n-k+1+x-i}\,. \tag{3.10}$$

In preparation to expand the RHS of (3.10) using Lemma 3.6 let us denote $a_i = (n-k+1+x-i) - ((k-1) \cdot 2^{t-1} - k)$ for $i = 0, \dots, j$, so that we can write

$$g_{n-k+1+x-i} = g_{(k-1) \cdot 2^{t-1} - k + a_i}\,.$$

Since $i \leq y \leq x$ we have

$$a_i = (n-k+1+x-i) - ((k-1) \cdot 2^{t-1} - k) \geq n - (k-1) \cdot 2^{t-1} + 1 \geq 1$$

and on the other hand from $n \leq k.2^{t-1} - 1 - x$ we deduce

$$a_i = (n-k+1+x-i) - ((k-1) \cdot 2^{t-1} - k) \leq n - (k-1) \cdot 2^{t-1} + 1 + x \leq 2^{t-1}\,.$$

So the conditions of Lemma 3.6 are satisfied and we have

$$g_{n-k+1+x-i} = \sum_{j=2}^{k-1} (w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a_i}) + w_{k-1}^{2^{t-1}-a_i} w_k^{a_i-1}$$
$$+ \text{ terms divisible by } w_k^{a_i} .$$

Hence we observe, noting that $a_i = a_0 - i$ is a decreasing function of $i \in \{0, \dots, y\}$, that the RHS of (3.10) is a sum of three polynomials

$$A = \sum_{i=0}^{y} \sum_{j=2}^{k-1} (\bar{f}_i w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a_i}) ,$$
$$B = \sum_{i=0}^{y} \bar{f}_i w_{k-1}^{2^{t-1}-a_i} w_k^{a_i-1} ,$$
$$C = \text{ sum of terms divisible by } w_k^{a_y} .$$

But the polynomial B also contains some terms divisible by $w_k^{a_y}$, since

$$B = \bar{f}_y w_{k-1}^{2^{t-1}-a_y} w_k^{a_y-1} + \sum_{i=0}^{y-1} \bar{f}_i w_{k-1}^{2^{t-1}-a_i} w_k^{a_i-1}$$

and $a_i - 1 \geq a_{y-1} - 1 = a_y$. So let us denote $B' = \bar{f}_y w_{k-1}^{2^{t-1}-a_y} w_k^{a_y-1}$ to write

$$B + C = B' + C'$$

where C' is some polynomial divisible by $w_k^{a_y}$.

Substituting into (3.10) yields

$$0 = A + B' + C',$$

or more explicitly

$$\sum_{i=0}^{y} \sum_{j=2}^{k-1} (\bar{f}_i w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a_i}) = \bar{f}_y w_{k-1}^{2^{t-1}-a_y} w_k^{a_y-1} + C'. \tag{3.11}$$

Since $\bar{f}_y \neq 0$, it is sufficient to show that in the LHS the indeterminate w_k never appears in a power of $a_y - 1$ or higher in order to reach a contradiction. Let us consider the summand

$$\bar{f}_i w_j^{2^{t-1}} g_{(k-j-1)\cdot 2^{t-1}-k+a_i} .$$

Every $\bar{f}_i$ is a polynomial in indeterminates $w_2, \dots, w_{k-1}$ and j is always distinct from k, so the only way for any power of w_k to appear in this summand is by that power occuring in the polynomial $g_{(k-j-1)\cdot 2^{t-1}-k+a_i}$.

By our assumptions n satisfies

$$(k-1) \cdot 2^{t-1} + \frac{k-3}{k-1} \cdot 2^{t-1} + \frac{x}{k-1} - 1 < n .$$

With a series of implications utilizing the facts that $j \geq 2$, $a_i \leq a_0$, $y \leq x$ and $a_y = a_0 - y$ we conclude

$$
\begin{aligned}
(k-1)\cdot 2^{t-1} + \frac{k-3}{k-1}\cdot 2^{t-1} + \frac{x}{k-1} - 1 &< n\,,\\
\frac{k-3}{k-1}\cdot 2^{t-1} + \frac{x}{k-1} - 1 &< n - (k-1)\cdot 2^{t-1}\,,\\
\frac{k-3}{k-1}\cdot 2^{t-1} + \frac{x}{k-1} + x &< n - (k-1)\cdot 2^{t-1} + 1 + x\,,\\
\frac{k-3}{k-1}\cdot 2^{t-1} + \frac{x}{k-1} + x &< a_0\,,\\
(k-3)\cdot 2^{t-1} + x &< (k-1)(a_0 - x)\,,\\
(k-3)\cdot 2^{t-1} + a_0 &< k(a_0 - x)\,,\\
(k-3)\cdot 2^{t-1} - k + a_0 &< k(a_0 - x - 1)\,,\\
(k-j-1)\cdot 2^{t-1} - k + a_i &< k(a_y - 1)\,.
\end{aligned}
$$

Therefore the polynomial $g_{(k-j-1)\cdot 2^{t-1}-k+a_i}$ lies in a dimension lower than $k(a_y-1)$ and consequently none of its terms may contain $w_k^{a_y-1}$. With that we have shown that (3.11) is impossible. □

4. The results

We are now ready to prove our main result. Our aim is to extend the statement of [7, Theorem 3.1 (2)], which can be reformulated (by shifting t by one, substitution $x = s+1$ and including cases $x = 0$ and $x = 1$ corresponding to [5, Theorem 2.1]) as follows

For any non-negative $x \leq 7$ and $n \geq 6$ such that there exists t sufficing $2^t + \frac{x}{2} - 1 < n \leq 2^{t+1} - 1 - x$ we have $\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geq n - 3 + x$.

Combining Proposition 3.12 and Proposition 2.1 we obtain following lower bounds for characteristic rank of $\widetilde{\gamma}_{n,k}$.

Theorem 4.1. *Let $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$. For any $x \geq 0$ and n such that*

$$(k-1)\cdot 2^{t-1} + \frac{k-3}{k-1}\cdot 2^{t-1} + \frac{x}{k-1} - 1 < n \leq k\cdot 2^{t-1} - 1 - x \tag{4.1}$$

we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \geq n - k + x.$$

Proof. The assumptions of the theorem are the same as in the Proposition 3.12. Since the LHS of (4.1) is increasing function of x, while the RHS is decreasing, if n satisfies this inequality for some x, it also satisfies it for all $y \leq x$. Thus for all $y \leq x$, each set $N_y(G_{n,k})$ is linearly independent.

The LHS of (4.1) is at least $(k-1)\cdot 2^{t-1}$, so $y \leq x < 2^{t-1}$ and also

$$(n-k-1) - 2^{t-1} \geq (k-2)\cdot 2^{t-1} - k - 1 = 2^{t-1} + (k-3)\cdot 2^{t-1} - k - 1\,.$$

Since $k \geq 3$ and $t \geq 3$ we have

$$(n-k-1) - 2^{t-1} \geq 2^{t-1} + (k-3) - k - 1 = 2^{t-1} - 4 \geq 0\,.$$

Combining the inequalities, we obtain $y < 2^{t-1} \leq n-k-1$.

Proposition 2.1 now implies that for all $y \leq x$ the homomorphisms

$$w_1\colon H^{n-k+y}(G_{n,k}) \longrightarrow H^{n-k+y+1}(G_{n,k})$$

are injective. Hence $\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \geq n-k+x$. □

Now, let us explore the implications of Theorem 4.1. The oriented Grassmann manifold $\widetilde{G}_{n,k}$ is a connected closed smooth manifold of dimension $k(n-k)$. We can apply Theorem 1.2 with $j = n-k+x$, since all monomials $w_{i_1}(\widetilde{\gamma}_{n,k}) \dots w_{i_r}(\widetilde{\gamma}_{n,k})$ in the top cohomology group $H^{k(n-k)}(\widetilde{G}_{n,k})$ vanish due to the self-annihilating property of the subring $\operatorname{Im}(p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k}))$.

First, let us consider the case for $k=3$. The inequality (4.1) simplifies to

$$2^t + \frac{x}{2} - 1 < n \leq 2^t + 2^{t-1} - 1 - x\,. \tag{4.2}$$

Although the characteristic rank of $\widetilde{\gamma}_{n,3}$ is already known [10], together with the corresponding result on cup-length, we will explicitly derive the values implied by Theorem 4.1 in this case. The intent is to allow us to discuss certain aspects of the proof, which generalize to the case of arbitrary k.

Theorem 4.2 ([10, Theorem 2])**.** *For any $a \geq 0$ and t sufficiently large so that it satisfies $2^{t-1} \geq 3a+2$ we have*

$$\operatorname{cup}(\widetilde{G}_{2^t+a,3}) = 2^t - 3\,.$$

Proof. Denote $n = 2^t + a$ and set $x = 2a+1$, so that $2^t + \frac{x}{2} - 1 < n$. Since $2^{t-1} \geq 3a+2$ we have

$$\begin{aligned} a &\leq 2^{t-1} - 2a - 2\,,\\ 2^t + a &\leq 2^t + 2^{t-1} - 2a - 2\,,\\ n &\leq 2^t + 2^{t-1} - 1 - x\,, \end{aligned}$$

hence the inequality (4.2) is satisfied.

By Theorem 4.1 we have $\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geq n-3+x$. As discussed before, we can apply Theorem 1.2 to obtain

$$\begin{aligned} \operatorname{cup}(\widetilde{G}_{n,3}) &\leq 1 + \frac{3(n-3) - (n-3+x) - 1}{2}\,,\\ \operatorname{cup}(\widetilde{G}_{n,3}) &\leq 1 + \frac{3(2^t+a-3) - (2^t+a-3+2a+1) - 1}{2}\,,\\ \operatorname{cup}(\widetilde{G}_{n,3}) &\leq 1 + \frac{2^{t+1}-8}{2}\,,\\ \operatorname{cup}(\widetilde{G}_{n,3}) &\leq 2^t - 3\,. \end{aligned}$$

On the other hand, we know that $w_2^{2^t-4}(\widetilde{\gamma}_{2^t,3}) \neq 0$ [4, Theorem 1.2] and there is an "inclusion" $\widetilde{j}\colon \widetilde{G}_{2^t,3} \longrightarrow \widetilde{G}_{2^t+a,3}$, such that the canonical bundle $\widetilde{\gamma}_{2^t,3}$

is isomorphic to the pullback $\widetilde{j}^*(\widetilde{\gamma}_{2^t+a,3})$. Hence, in cohomology, the induced homomorphism $\widetilde{j}^*\colon H^*(\widetilde{G}_{2^t+a,3}) \longrightarrow H^*(\widetilde{G}_{2^t,3})$ maps the class $w_2^{2^t-4}(\widetilde{\gamma}_{2^t+a,3})$ to the non-zero class $w_2^{2^t-4}(\widetilde{\gamma}_{2^t,3})$. Thus the class $w_2^{2^t-4}(\widetilde{\gamma}_{2^t+a,3})$ itself is non-zero and by Poincaré duality we have $\mathrm{cup}(\widetilde{G}_{n,3}) \geq 2^t - 3$. □

Now, let us proceed with the general case. Since the LHS of (4.1) might not be an integer, we need to handle it carefully. For any $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$ let us denote $x_{k,t}$ the smallest positive integer such that $2^t - x_{k,t}$ is divisible by $k-1$. We define an integer

$$n_{k,t} = k \cdot 2^{t-1} - \frac{2^t - x_{k,t}}{k-1} - 1 \,. \tag{4.3}$$

Theorem 4.3. *For any $k \geq 3$, $t \geq \max\{3, \log_2(k-1)\}$ and $a \geq 0$, such that a satisfies $ka + x_{k,t} - 1 \leq \frac{2^t - x_{k,t}}{k-1}$ we have*

$$\mathrm{cup}(\widetilde{G}_{n_{k,t}+a,k}) \leq 1 + \frac{k(n_{k,t} - k) - (n_{k,t} - k + x_{k,t})}{2} \,.$$

Proof. Denote $n = n_{k,t} + a$ and set $x = (k-1)a + x_{k,t} - 1$. For the LHS of (4.1) we have

$$(k-1) \cdot 2^{t-1} + \frac{k-3}{k-1} \cdot 2^{t-1} + \frac{x}{k-1} - 1 = k \cdot 2^{t-1} - \frac{2^t - x}{k-1} - 1$$

and so by (4.3), the LHS of (4.1) is equal to

$$n_{k,t} + a - \frac{1}{k-1} = n - \frac{1}{k-1} \,.$$

On the other hand, since $ka + x_{k,t} - 1 \leq \frac{2^t - x_{k,t}}{k-1}$ we have

$$\begin{aligned} a &\leq \frac{2^t - x_{k,t}}{k-1} - (k-1)a - x_{k,t} + 1 \,, \\ -\frac{2^t - x_{k,t}}{k-1} - 1 + a &\leq -(k-1)a - x_{k,t} \,, \\ k \cdot 2^t - \frac{2^t - x_{k,t}}{k-1} - 1 + a &\leq k \cdot 2^t - (k-1)a - x_{k,t} \,, \\ n &\leq k \cdot 2^t - 1 - x \,. \end{aligned}$$

Hence the inequality (4.1) is satisfied and by Theorem 4.1 we have $\mathrm{charrank}(\widetilde{\gamma}_{n,k}) \geq n - k + x$. Theorem 1.2 now yields

$$\begin{aligned} \mathrm{cup}(\widetilde{G}_{n,k}) &\leq 1 + \frac{k(n-k) - (n-k+x) - 1}{2} \,, \\ \mathrm{cup}(\widetilde{G}_{n,k}) &\leq 1 + \frac{k(n_{k,t}+a-k) - (n_{k,t}+a-k+(k-1)a + x_{k,t}-1)-1}{2} \,, \\ \mathrm{cup}(\widetilde{G}_{n,k}) &\leq 1 + \frac{k(n_{k,t} - k) - (n_{k,t} - k + x_{k,t})}{2} \,. \end{aligned}$$

since a appears k times in both parentheses. □

As should be clear from juxtaposing the proofs of Theorem 4.2 and Theorem 4.3, the only difference is in having information about the height of the second Stiefel-Whitney class of $\widetilde{\gamma}_{n_{k,t},k}$. For $k = 3$ we had $x_{3,t+1} = 2$ and $n_{3,t+1} = 2^t$ and the height of $w_2(\widetilde{\gamma}_{2^t,3})$ was known. If we knew that the height of $w_2(\widetilde{\gamma}_{n_{k,t},k})$ is $\frac{k(n_{k,t}-k)-(n_{k,t}-k+x_{k,t})}{2}$, we would be able to make inferences analogous to those in the second part of the proof of Theorem 4.2 to reach a conclusion that the cup-length of $\widetilde{G}_{n,k}$ for n in the corresponding interval is actually equal to the upper bound given by Theorem 4.3.

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