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CONTACT QUANTIZATION: QUANTUM MECHANICS = PARALLEL TRANSPORT

G. Herczeg, E. Latini, and Andrew Waldron

Abstract. Quantization together with quantum dynamics can be simultaneously formulated as the problem of finding an appropriate flat connection on a Hilbert bundle over a contact manifold. Contact geometry treats time, generalized positions and momenta as points on an underlying phase-spacetime and reduces classical mechanics to contact topology. *Contact quantization* describes quantum dynamics in terms of parallel transport for a flat connection; the ultimate goal being to also handle quantum systems in terms of contact topology. Our main result is a proof of local, formal gauge equivalence for a broad class of quantum dynamical systems—just as classical dynamics depends on choices of clocks, local quantum dynamics can be reduced to a problem of studying gauge transformations. We further show how to write quantum correlators in terms of parallel transport and in turn matrix elements for Hilbert bundle gauge transformations, and give the path integral formulation of these results. Finally, we show how to relate topology of the underlying contact manifold to boundary conditions for quantum wave functions.

1. Introduction

To understand why a study of contact geometry is fundamental to quantum mechanics, it is useful to think about the standard Copenhagen interpretation in a novel way: According to the Copenhagen interpretation, one prepares an initial quantum state, allows it to evolve for some time, and then calculates the probability of observing some choice of final state. The basic data here is a Hilbert space and a one parameter family of unitary operators that determine time evolution. This parameter typically corresponds to time intervals as measured in a classical laboratory. Two modifications of this standard paradigm will lead us to a—rather propitious—reformulation of quantum mechanics as a theory of flat connections on a Hilbert bundle over a contact manifold:

(i) Because it ought be possible to describe quantum dynamics for any choice of laboratory time coordinate (for example one may conceive of notions of time that mix varying combinations of classical-laboratory measurements),

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we replace the time interval with a classical "phase-spacetime" manifold Z, which can be thought of as a classical phase space augmented by a timelike direction that enjoys general coordinate (diffeomorphism) invariance.

(ii) Instead of viewing quantum dynamics as trajectories in a single given Hilbert space $\mathcal{H}$, we associate—in a manner reminiscent of gauge theories and general relativity—a copy of the Hilbert space to every point in the phase-spacetime Z. This structure is a *Hilbert bundle* $Z \ltimes \mathcal{H}$, *viz.* a vector bundle whose fibers are Hilbert spaces [8]. We use the warped product notation $Z \ltimes \mathcal{H}$ to indicate that, locally in Z, the Hilbert bundle is a direct product, although this need not globally be the case.

Given the geometric data of the vector bundle $Z \ltimes \mathcal{H}$, we wish to compare Hilbert space states at distinct points in Z. For that we need a connection ∇. Concretely

$$\nabla = d + \widehat{A}\,,$$

where d is the exterior derivative on Z and $i\widehat{A}$ is a one-form taking values in the space of hermitean operators on $\mathcal{H}$. In particular, if $\mathcal{H}$ is simply $L^2(\mathbb{R}^n)$, we may consider $\widehat{A}$ to take values in the self-adjoint subspace of the corresponding Weyl algebra.

To construct the connection ∇, additional data is required. In Section 2, we will show that giving the phase-spacetime manifold a strict contact stucture endows the Hilbert bundle $Z \ltimes \mathcal{H}$ with a flat connection. Physically, this strict contact data corresponds to specifying classical dynamics on Z. The construction we give is partly motivated by earlier BRST studies of Fedosov quantization [9] for symplectic manifolds [14]. Solutions to the quantum Schrödinger equation are then parallel sections of the Hilbert bundle—quantum dynamics amounts to parallel transport of states from one Hilbert space fiber to another. The main theorem of Section 2 establishes that solutions for connections obeying the flatness condition are locally and formally gauge equivalent. The method of proof is close to that employed in Fedosov's original work on deformation quantization of Poisson structures [9]. The key advantage is that our contact approach not only incorporates dynamics, but also establishes a very general local gauge equivalence between dynamical quantum systems.

In Section 3, we focus on the description of dynamics in terms of parallel sections of the Hilbert bundle. In particular we show how to reduce the problem of computing quantum correlators to that of finding the matrix element of a gauge transformation. We also give a path integral description of correlators in terms of paths in a novel extended phase-spacetime description of contact Reeb dynamics. We also show how topology of the underlying contact manifold determines boundary conditions for quantum wavefunctions. Open problems and future prospects are discussed in Section 4.

2. Strict contact structures and quantization

Contact geometry may be viewed as a unification of Hamiltonian dynamics and symplectic geometry. Therefore, before discussing quantization, we introduce the salient features of contact structures [13, 22].

2.1. **Contact geometry.** A *strict contact structure* is the data (Z,α) where Z is a $2n+1$ dimensional manifold and α is a *contact one-form*, meaning that the volume form

$$\mathrm{Vol}_\alpha := \alpha \wedge \varphi^{\wedge n} \tag{2.1}$$

is nowhere vanishing[1], where the two form

$$\varphi := d\alpha\,,$$

determines the *Levi-form* along the distribution; we therefore also term φ the Levi.

The data (Z,α) allows us to formulate classical dynamics via the action principle

$$S = \int_\gamma \alpha\,, \tag{2.2}$$

defined by integrating the contact one-form along *unparameterized* paths γ in Z. Requiring S to be extremal under compact variations of the embedding $\gamma \hookrightarrow Z$ yields equations of motion

$$\varphi(\dot\gamma,\boldsymbol{\cdot}) = 0\,. \tag{2.3}$$

Since the Levi-form necessarily has maximal rank, the above condition determines the tangent vector to γ up to an overall scale. The choice of solution $\dot\gamma = \rho$ to Equation (2.3) with normalization $\alpha(\rho) = 1$ is called the *Reeb vector*. Classical evolution is therefore governed by flows of the Reeb vector; and in this context is dubbed *Reeb dynamics*. It is not difficult to verify that these obey a contact analog of the classical Liouville theorem, namely that the volume form is preserved by Reeb dynamics:

$$\mathcal{L}_\rho \mathrm{Vol}_\alpha = 0\,,$$

where $\mathcal{L}_{\boldsymbol{\cdot}}$ denotes the Lie derivative.

The contact Darboux theorem is particularly powerful; it ensures that locally there exists a diffeomorphism on Z that brings any contact form to the normal form

$$\alpha = \pi_A d\chi^A - d\psi\,, \tag{2.4}$$

where (π_A,χ^A,ψ) are $2n+1$ local coordinates for Z. On this coordinate patch the Reeb vector $\rho = -\frac{\partial}{\partial\psi}$ so that dynamics are locally trivial. Observe that in the

[1]A *contact structure* is the data of a maximally non-integrable hyperplane distribution; the kernel of α (viewed as a map on tangent spaces $T_PZ \to \mathbb{R}$) determines precisely such a distribution (as does any $f\alpha$ where $0 < f \in C^\infty Z$). Note also, that it is interesting to consider models for which the Levi-form $\varphi = d\alpha$ has maximal rank, but Vol_α may vanish (either locally or globally). The massless relativistic particle falls into this class.

worldline diffeomorphism gauge $\psi = \tau$, where τ is a worldline parameter along γ, the action (2.2) becomes

$$S = \int d\tau \big[\pi_a \dot{\chi}^a - 1\big]\,.$$

This is the Hamiltonian action principle for a system with Darboux symplectic form $d\pi_a \wedge d\chi^a$ and trivial Hamiltonian $H = 1$.

2.2. **Constraint analysis.** Our quantum BRST treatment of Reeb dynamics requires that we examine the constraint structure of the model (2.2). Firstly observe that the action principle (2.2) is worldline diffeomorphism invariant, and in a choice of coordinates z^i for Z reads $S = \int \alpha_i(z)\dot{z}^i d\tau$. Therefore the canonical momenta p_i for $\dot{z}^i$ obey $2n+1$ constraints

$$C_i := p_i - \alpha_i(z) = 0\,,$$

of which $2n$ are second class (because these constraints Poisson commute to give the maximal rank Levi-form: $\{C_i, C_j\}_{\rm PB} = \varphi_{ij}$) and one is first class (corresponding to worldline diffeomorphisms). By introducing $2n$ "fiber coordinates" s^a (see [2]), local classical dynamics can be described by an equivalent extended action principle for paths Γ in $\mathcal{Z} := Z \times \mathbb{R}^{2n}$ for which all constraints are first class[2]:

$$S_{\rm ext} = \int_\Gamma \Big[\tfrac{1}{2} s^a J_{ab} ds^b + A(s)\Big]\,. \tag{2.5}$$

In the above J_{ab} is a constant, maximal rank antisymmetric matrix (and therefore an invariant tensor for the Lie algebra $\mathfrak{sp}(2n)$). The one-form A is given by

$$A(s) = \alpha + e^a J_{ab} s^b + \omega(s)\,,$$

where the *soldering forms* e^a together with the contact one-form α are a basis for T^*Z such that the Levi-form decomposes as

$$\varphi = \frac{1}{2}\, J_{ab} e^a \wedge e^b\,,$$

and $e^a(\rho) = 0$. The extended action (2.5) enjoys $2n+1$ gauge invariances (and hence $2n+1$, abelian, first class constraints) when A obeys the zero curvature type condition[3]

$$dA + \frac{1}{2}\{A \wedge A\}_{\rm PB} = 0\,.$$

This condition can always be solved for a one-form $\omega(s)$ to any order in a formal power series in s (and therefore exactly for contact forms expressible as polynomials in some coordinate system). The main ingredients for quantization are now ready.

[2]To analyze global dynamics one ought promote $\mathcal{Z}$ to a bundle $Z \ltimes \mathbb{R}^{2n}$.

[3]For a pair of one-forms A and B, we denote $\{A(s) \wedge B(s)\}_{\rm PB} := J^{ab} \frac{\partial A}{\partial s^a} \wedge \frac{\partial B}{\partial s^b}$ where the inverse matrix J_{ab} obeys $J_{ab} J^{bc} = \delta_a^c$.

2.3. **Flat connections.** Because the constraints are now abelian and first class, it is straightforward to quantize the *extended Reeb dynamics* defined by the action (2.5) using the Hamiltonian BRST technology of [12]. The resultant nilpotent BRST charge may be interpreted as a flat connection ∇ on the Hilbert bundle $Z \ltimes \mathcal{H}$. [An analogous connection has been constructed for symplectic manifolds in [19].] In detail,

$$\nabla = d + \widehat{A}\,,$$

where $\widehat{A}$ is a one-form taking hermitean values in the enveloping algebra $\mathcal{U}(\mathfrak{heis})$ of the Heisenberg algebra

$$\mathfrak{heis} = \operatorname{span}\{1, \hat{s}^a\}\,, \qquad [\hat{s}^a, \hat{s}^b] = i\hbar J^{ab}\,. \tag{2.6}$$

In particular

$$i\widehat{A} = \frac{\alpha}{\hbar} + \frac{e^a J_{ab}\hat{s}^b}{\hbar} + i\,\widehat{\Omega}\,,$$

where $\hbar\widehat{\Omega}$ is a hermitean operator, potentially involving higher powers of the generators $\hat{s}^a$, that is expressible as a formal power series in $\hbar$. It is formally determined by the zero curvature condition

$$\nabla^2 = 0\,. \tag{2.7}$$

Example 2.1 (Hamiltonian dynamics)**.** Let $Z = \mathbb{R}^3 = \{p, q, t\}$ and

$$\alpha = pdq - H(p, q, t)dt\,,$$

with Hamiltonian H given by a (possibly time-dependent) polynomial in p and q. Notice that $\varphi = e \wedge f$ where $e := dp + \frac{\partial H}{\partial q}dt$ and $f := dq - \frac{\partial H}{\partial p}dt$, so we make a choice of soldering $e^a = (f, e)$ which we use to construct the flat connection:

$$\nabla = d + \frac{i}{\hbar}\Big[dp\, S - dq\,\Big(p + \frac{\hbar}{i}\frac{\partial}{\partial S}\Big)\Big] + \frac{i}{\hbar}\, dt\widehat{H}\,, \tag{2.8}$$

where the operator

$$\widehat{H} := \Big(H\big(q + S, p + \frac{\hbar}{i}\frac{\partial}{\partial S}\big)\Big)_{\rm Weyl}$$

is given by Weyl ordering the operators[4] $\hat{s}^a := (S, \frac{\hbar}{i}\frac{\partial}{\partial S})$. (This ensures formal self-adjointness of the operator $\widehat{H}$.) The Schrödinger equation (2.9) may be solved by setting $\Psi = \exp(-\frac{i}{\hbar}pS)\,\psi(q + S, t)$, where $\psi(Q, t)$ obeys the standard time dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(Q, t)}{\partial t} = \Big(H\big(Q, \frac{\hbar}{i}\frac{\partial}{\partial Q}\big)\Big)_{\rm Weyl}\psi(Q, t)\,.$$

This example therefore shows how contact quantization recovers standard quantum mechanics. ❋

[4]Note that we have made the choice of Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ here as well as a polarization for the space of wavefunctions. Different choices of polarization differ only by gauge transformations—recall that in its metaplectic representation, compact elements of $sp(2n)$ act by Fourier transform on Schwartz functions.

To better understand the space of flat connections ∇, we further organize the expansion in powers of operators $\hat{s}$ by assigning a grading gr to the operators $\hat{s}$ and $\hbar$ where[5]

$$\mathsf{gr}(\hbar) = 2\,, \qquad \mathsf{gr}(\hat{s}^a) = 1\,.$$

Thus, arranging the connection in terms of this grading we have

$$\nabla = \underbrace{\frac{\alpha}{i\hbar}}_{-2} + \underbrace{\frac{e^a J_{ab}\hat{s}^b}{i\hbar}}_{-1} + \underbrace{d_w}_{0} + \underbrace{\widehat{\omega}}_{\geqslant 1}\,,$$

where

$$d_\omega := d + \frac{1}{2i\hbar}\,\omega_{ab}\hat{s}^a\hat{s}^b\,.$$

Here the symmetric part of ω_{ab} gives an $\mathfrak{sp}(2n)$-valued one-form (or connection) while the antisymmetric part is necessarily pure imaginary in order that $\widehat{\Omega}$ is hermitean. Also, the terms with strictly positive grading are $\widehat{\omega} := \widehat{\Omega} - \frac{1}{2i\hbar}\,\omega_{ab}\hat{s}^a\hat{s}^b$.

Observe that this grading is invariant under rewritings of products of the operators $\hat{s}$ given by quantum reorderings, for example

$$\mathsf{gr}(\hat{s}^a\hat{s}^b) = \mathsf{gr}\Big(\hat{s}^b\hat{s}^a + i\hbar J^{ab}\Big)\,.$$

In other words, gr filters $\mathcal{U}(\mathfrak{heis})$. The projection of an element in $\mathcal{U}(\mathfrak{heis})$ to the part of grade k is denoted by[6] $\mathsf{gr}_{\mathsf{k}}(\boldsymbol{\cdot})$.

In Theorem 2.2 we shall show that locally, every solution to the flatness condition 2.7 is formally[7] gauge equivalent[8] to a connection where $\widehat{\Omega} = 0$. Moreover the latter such solutions always exist.

Realizing $\hat{s}^a$ by hermitean operators representing the Heisenberg algebra acting on $\mathcal{H}$, the (principal) connection ∇ gives a connection on the (associated) Hilbert bundle $Z \ltimes \mathcal{H}$. The Schrödinger equation is then simply the parallel transport condition

$$\nabla\Psi = 0 \tag{2.9}$$

on Hilbert bundle sections $\Psi \in \Gamma(Z \ltimes \mathcal{H})$. Indeed, modulo (non-trivial) global issues, the problem of quantizing a given classical system now amounts to solving the above flat connection problem (2.7), while quantum dynamics amounts to parallel transport.

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[5]When applied to sums of terms inhomogeneous in the grading, we define gr by the grade of the lowest grade term.

[6]We also employ $\mathsf{gr}_{\mathsf{K}}(\boldsymbol{\cdot})$, where $K \subset \mathbb{Z}$, to denote projection to subspaces with the corresponding grades. For the exterior derivative, we define $\mathsf{gr}(d) = 0$.

[7]The terms *formally equivalent* here are defined to mean that gauge transformations exist giving connections that are equal to any chosen order in the grading gr.

[8]To be sure, we are not claiming that this means all quantum dynamics on a given Hilbert space are equivalent, rather having identified the physical meaning of variables for a given connection ∇, the "gauge equivalent" (in the bundle sense) connection $\nabla' = \widehat{U}\,\nabla\,\widehat{U}^\dagger$ will in general describe different dynamics. This is much like the case of active diffeomorphisms for a theory in a fixed generally curved background. Moreover, it is a highly useful feature, because at least locally, it allows complicated dynamics to be described in terms of simpler ones.

Theorem 2.2. *Any two flat connections $\nabla = d + \widehat{A}$ and $\nabla' = d + \widehat{A}'$ where*

$$\mathsf{gr}_{-2}(\widehat{A}) = \frac{\alpha}{i\hbar} = \mathsf{gr}_{-2}(\widehat{A}')\,,$$

are locally, formally gauge equivalent.

Proof. The contact Darboux theorem ensures that locally, there exists a set of *closed* one-forms $dE^a = 0$, such that

$$\varphi = \tfrac{1}{2} J_{ab} E^a \wedge E^b \text{ and } \iota_\rho E^a = 0\,.$$

(In the normal form (2.4), $E^a = (d\chi^A, d\pi_A)$.) Hence the connection

$$\nabla_{\mathrm{D}} := \frac{\alpha}{i\hbar} + \frac{E^a J_{ab} \hat{s}^b}{i\hbar} + d \tag{2.10}$$

solves the flatness condition (2.7). Our strategy is to construct the gauge transformation bringing a general flat ∇ to this "Darboux form".

Firstly, the flatness condition of a general $\nabla = d + \widehat{A}$ at grade -2 implies that

$$\frac{d\alpha}{i\hbar} + \Big(\mathsf{gr}_{-1}(\widehat{A})\Big)^2 = 0\,.$$

This is solved, as discussed earlier, by

$$i\hbar\, \mathsf{gr}_{-1}\widehat{A} = e^a J_{ab} \hat{s}^b\,,$$

where

$$\varphi = \tfrac{1}{2} J_{ab} e^a \wedge e^b \text{ and } \iota_\rho e^a = 0\,.$$

Comparing the line above with the first display of this proof, we see there must (pointwise in some neighborhood in Z) exist an invertible linear transformation $U \in GL(2n)$ such that

$$E^a = U^a{}_b e^b\,.$$

Moreover, U must preserve J and hence is in fact $Sp(2n)$-valued with unit determinant. Thus, we may write $U = \exp(u)$. In turn it follows that

$$\mathsf{gr}_{\{-2,-1\}}\big(\exp(\hat{u}_0)\widehat{A}\exp(-\hat{u}_0)\big) = \frac{\alpha}{i\hbar} + \frac{E^a J_{ab}\hat{s}^b}{i\hbar}\,,$$

where

$$\hat{u}_0 = \frac{J_{ac} u^c{}_b \hat{s}^a \hat{s}^b}{2i\hbar}\,.$$

Essentially, we have just intertwined U in the fundamental representation of $Sp(2n)$ to its metaplectic representation.

We now observe that

$$\mathsf{gr}_0\big(\exp(\hat{u}_0)(d + \widehat{A})\exp(-\hat{u}_0)\big) = d - i\alpha_1 + \frac{\omega_{ab}\hat{s}^a\hat{s}^b}{2i\hbar}\,, \tag{2.11}$$

where α_1 is some real-valued, $\hbar$-independent one-form and the one-form $\omega_{ab} = \omega_{ba}$ (the Heisenberg algebra (2.6) may be used to absorb an antisymmetric part of ω_{ab} in α_1).

We now again employ flatness of ∇ and closedness of the E^a's to obtain

$$0 = \mathsf{gr}_{-1}\Big(\big(\exp(\hat{u}_0)(d + \widehat{A})\exp(-\hat{u}_0)\big)^2\Big) = \frac{\omega_{ab} \wedge E^a\, \hat{s}^b}{i\hbar}\,.$$

We decompose the one-form ω^{ab} with respect to the (local) basis (α, e^a) for T^*Z as $\omega_{ab} = W_{ab}\,\alpha + W_{abc}E^c$. The above display then implies that the functions W_{ab} must vanish and

$$W_{abc}E^a \wedge E^c = 0\,.$$

Hence W_{abc} is totally symmetric in the indices a, b, c.

We now gauge away the term $\omega_{ab}\hat{s}^a\hat{s}^b/(2i\hbar) = W_{abc}\hat{s}^a\hat{s}^bE^c/(2i\hbar)$ in Equation (2.11). For that we work formally order by order in the grading employing the adjoint action $\exp(\hat{u})\,\widehat{W}\exp(-\hat{u}) = \exp([\hat{u},\boldsymbol{\cdot}])(\widehat{W})$. In particular this gives

$$\mathsf{gr}_0\Big(\exp(\hat{u}_1)\frac{E^aJ_{ab}\hat{s}^b}{i\hbar}\exp(-\hat{u}_1)\Big) = -\frac{W_{abc}\hat{s}^a\hat{s}^bE^c}{2i\hbar}\,,$$

for the choice $\hat{u}_1 = W_{abc}\hat{s}^a\hat{s}^b\hat{s}^c/(3!i\hbar)$. Hence we have achieved

$$\mathsf{gr}_{\{-2,-1,0\}}\Big(\exp(\hat{u}_1)\exp(\hat{u}_0)(d+\widehat{A}\,)\exp(-\hat{u}_0)\exp(-\hat{u}_1)\Big) = \frac{\alpha}{i\hbar}+\frac{E^aJ_{ab}\hat{s}^b}{i\hbar}+d-i\alpha_1\,.$$

At this juncture, we have established the base case for an induction. Proceeding recursively we now assume that the flat connection $\nabla = d + \widehat{A}$ obeys

$$\mathsf{gr}_{\{-2,\dots,k\}}(\widehat{A}\,) = \frac{\alpha + \hbar\alpha_1 + \cdots + \hbar^{[(k+1)/2]}\alpha_{[(k+1)/2]}}{i\hbar} + \frac{E^aJ_{ab}\hat{s}^b}{i\hbar} + d + \hat{\omega}_k\,,$$

where α_i are $\hbar$-independent one-forms and, without loss of generality, take $\mathsf{gr}(\hat{\omega}_k) = k$.

Employing the flatness condition for ∇ along the same lines explained above to $\hat{\omega}_k$ shows that

$$i\hbar\hat{\omega}_k = \begin{cases} \frac{1}{(k+2)!}\,W_{a_1\dots a_{k+3}}\hat{s}^{a_1}\cdots\hat{s}^{a_{k+2}}E^{a_{k+3}} + \frac{\hbar}{k!}\,W_{a_1\dots a_{k+1}}\hat{s}^{a_1}\cdots\hat{s}^{a_k}E^{a_{k+1}} + \cdots \\ \qquad\qquad + \hbar^{(k+1)/2}W_{a_1a_2}\hat{s}^{a_1}E^{a_2}\,, \quad k \text{ odd}\,, \\[2ex] \frac{1}{(k+2)!}\,W_{a_1\dots a_{k+3}}\hat{s}^{a_1}\cdots\hat{s}^{a_{k+2}}E^{a_{k+3}} + \frac{\hbar}{k!}\,W_{a_1\dots a_{k+1}}\hat{s}^{a_1}\cdots\hat{s}^{a_k}E^{a_{k+1}} + \cdots \\ \qquad\qquad + \hbar^{k/2}W_{a_1a_2a_3}\hat{s}^{a_1}\hat{s}^{a_2}E^{a_3} + \hbar^{(k+2)/2}\,\alpha_{(k+2)/2}\,, \quad k \text{ even}\,, \end{cases}$$

where the tensors W are totally symmetric and $\alpha_{(k+2)/2}$ is some one-form. Both the W's and $\alpha_{(k+2)/2}$ are $\hbar$-independent. Indeed, all terms save the one-form $\alpha_{(k+2)/2}$ can—*mutatis mutandis*—be removed by higher order analogs of the gauge transformation $\exp(\hat{u}_1)$ employed in the base step above. Hence we have now proven that locally, gauge transformations achieve the form (formally to any power in the grading)

$$\nabla = \nabla_{\mathrm{D}} - i\sum_{j>1}\hbar^{j-1}\alpha_j\,.$$

It only remains to apply the flatness condition one more time to show that the one-form $\alpha_\hbar := \sum_{j>1}\hbar^{j-1}\alpha_j$ is closed and therefore locally $\alpha_\hbar = d\beta_\hbar$ for some function $\beta_\hbar$. Thus $\exp(i\beta_\hbar)\nabla\exp(-i\beta_\hbar) = \nabla_{\mathrm{D}}$. □

Example 2.3 (The harmonic oscillator)**.** Let $Z = \mathbb{R}^3 = \{p, q, t\}$ and

$$\alpha = pdq - \frac{1}{2}(p^2+q^2)dt\,.$$

The Levi form

$$\varphi = d\pi \wedge d\chi\,,$$

where

$$\pi = \frac{1}{2}(p^2+q^2)\,,\quad \chi = -t - \arctan(p/q)\,.$$

Indeed, setting $\psi = -\frac{1}{2}pq$, we have $\alpha = \pi d\chi - d\psi$, so (π,χ,ψ) are local Darboux coordinates and (denoting $\hat{s}^a := (\hat{S},\hat{P})$) the Darboux normal form (2.10) for the connection becomes

$$\nabla_{\mathrm{D}} := \frac{\pi d\chi - d\psi}{i\hbar} + \frac{\hat{S}d\pi - \hat{P}d\chi}{i\hbar} + d\,. \tag{2.12}$$

Let us now run the steps of the above proof in *reverse* to show how to find gauge transformations bringing ∇_{D} to the Hamiltonian dynamics form of 2.8.

The closed soldering forms $E^a = (d\chi, d\pi)$ are related to those of the Hamiltonian dynamics Example 2.1 (given here by $e^a = (dq - pdt, dp + qdt) =: (f,e)$) according to the $Sp(2)$ transformation

$$E^a := \begin{pmatrix} d\chi \\ d\pi \end{pmatrix} = \begin{pmatrix} \frac{p}{2\pi} & -\frac{q}{2\pi} \\ q & p \end{pmatrix} \begin{pmatrix} dq - pdt \\ dp + qdt \end{pmatrix} =: U^a{}_b e^b\,.$$

Writing $U = \exp(u)$ and then intertwining to its metaplectic representation $\widehat{U} := \exp\big(\frac{J_{ac}u^c{}_b\hat{s}^a\hat{s}^b}{2i\hbar}\big)$, we have $\widehat{U}^{-1}\big(\frac{\alpha}{i\hbar} + \frac{E^a J_{ab}\hat{s}^a}{i\hbar}\big)\widehat{U} = \frac{\alpha}{i\hbar} + \frac{e^a J_{ab}\hat{s}^a}{i\hbar}$, while a short computation shows that the $\mathfrak{sp}(2)$-valued one-form $U^{-1}dU$ is given explicitly by

$$U^{-1}dU = \begin{pmatrix} 0 & -dt \\ dt & 0 \end{pmatrix} + \begin{pmatrix} -\frac{(p^2-q^2)(pe+qf)}{4\pi^2} & \frac{(3p^2+q^2)qe-(p^2-q^2)pf}{4\pi^2} \\ \frac{(p^2-q^2)qe+(p^2+3q^2)pf}{4\pi^2} & \frac{(p^2-q^2)(pe+qf)}{4\pi^2} \end{pmatrix}.$$

It is not difficult to verify that the last term in the above display can be re-expressed as $W^a{}_{bc}e^c$ where the tensor W_{abc} (moving indices with the antisymmetric bilinear form J) is totally symmetric[9]. Moreover, intertwining the first term to the metaplectic representation gives the standard harmonic oscillator Hamiltonian $\frac{i}{2\hbar}\,dt(\hat{P}^2+\hat{S}^2)$. Hence the difference between the gauge transformed Darboux connection and the Hamiltonian dynamics connection of Equation 2.8 is

$$\widehat{U}^{-1}\nabla_{\mathrm{D}}\widehat{U} - \nabla = \frac{\hat{s}^a\hat{s}^b W_{abc}e^c}{2i\hbar}\,.$$

The above term is order 0 in the grading gr and therefore seeds the recursion described in the proof of Theorem 2.2. It is removed by a grade 1 gauge transformation $\exp(\hat{u}_1)$ with $\hat{u}_1 = \frac{\hat{s}^a\hat{s}^b\hat{s}^c W_{abc}}{3!i\hbar}$. It would be desirable to have an efficient recursion to compute all higher terms with respect to the grading gr for the gauge transformation between ∇ and ∇_{D}, because in a general setting this would facilitate computation of quantum correlators. ❋

[9]Note that

$$W_{222} = \frac{(3p^2+q^2)p}{4\pi^2}\,,\quad W_{221} = -\frac{(p^2-q^2)q}{4\pi^2}\,,\quad W_{211} = -\frac{(p^2-q^2)p}{4\pi^2}\,,\quad W_{111} = \frac{(3p^2+q^2)q}{4\pi^2}\,.$$

2.4. **Contact deformation quantization.** The above proof of gauge equivalence of flat connections is very close in spirit to Fedosov's formal quantization for symplectic and Poisson structures[10]. That work is concerned with constructing a quantum deformation of the Moyal star product, while here we wish to describe both dynamics and quantization. Nonetheless, we can employ Fedosov's method to our quantized contact connection ∇, to find a quantum deformation of the commutative algebra of classical solutions.

To study the algebra of operators, instead of the Hilbert bundle over Z, we consider a *Heisenberg bundle* $Z \ltimes \mathcal{U}(\mathfrak{heis})$, defined in the same way as the Weyl bundle, except that instead of working with fibers given by functions of $\mathbb{R}^{2n}$ with a non-commutative Moyal star product, we work directly with operators[11]. For our purposes, the key point is that local sections $\hat{a}$ of the Heisenberg bundle are functions of Z taking values in $\mathcal{U}(\mathfrak{heis})$, which can be expressed with respect to the grading gr as

$$\hat{a} = \underbrace{\frac{a^{(-2)}}{i\hbar}}_{-2} + \underbrace{\frac{a^{(-1)}_a \hat{s}^a}{i\hbar}}_{-1} + \underbrace{\frac{a^{(0)}_{ab}\hat{s}^a\hat{s}^b}{2i\hbar} - ia^{(0)}}_{0} + \cdots .$$

Importantly, $a^{(k)}$ are $\hbar$ independent, and we do not allow negative powers of $\hbar$ greater than one.

Requiring total symmetry of the tensors $a^{(k)}_{a_1\ldots a_{j\leqslant k}}$ appearing in the above expansion uniquely determines a function of $\hbar$ which—following Fedosov—we call the *abelian part* of $\hat{a}$ and denote by

$$\sigma(\hat{a}) := a^{(-2)} + \hbar a^{(0)} + \hbar^2 a^{(2)} + \cdots .$$

We call $\hat{a} - \frac{1}{i\hbar}\sigma(\hat{a})$ the *non-abelian* part of $\hat{a}$.

The flat connection ∇ acts on sections of the Heisenberg bundle by the adjoint action

$$\nabla \hat{a} := d\hat{a} + [\widehat{A}, \hat{a}] .$$

The following lemma locally characterizes parallel sections.

Lemma 2.4. *Let* $f_\hbar \in C^\infty Z[[\hbar]]$ *obey*

$$\mathcal{L}_\rho f_\hbar = 0 .$$

Then locally, there is a unique section $\hat{a} \in \Gamma(Z \ltimes \mathcal{U}(\mathfrak{heis}))$ *such that*

$$\nabla \hat{a} = 0 \text{ and } \sigma(\hat{a}) = f_\hbar .$$

Proof. By virtue of Theorem 2.2 we know that locally

$$\nabla = \exp(\hat{u}) \circ \nabla_{\mathrm{D}} \circ \exp(-\hat{u}) ,$$

[10]Deformation quantization dates back to the seminal work of Bayen *et al* [3], see also [4] for a review of symplectic connections.

[11]Recall that the Moyal star product amounts simply to coordinatizing the space of operators $\mathcal{U}(\mathfrak{heis})$ in terms of functions of $\mathbb{R}^{2n}$ by employing a Weyl-ordered operator basis, and then encoding their algebra using a non-commutative $\star$-multiplication of functions.

for some $\hat{u} \in \Gamma(Z \ltimes \mathcal{U}(\mathfrak{heis}))$ and ∇_{D} is given by Equation (2.10). Therefore we begin by establishing that the equation

$$\nabla_{\mathrm{D}} \hat{b} = 0 \tag{2.13}$$

has a solution such that

$$\sigma(\exp(\hat{u})\, \hat{b}\, \exp(-\hat{u})) = f_\hbar\,, \tag{2.14}$$

because $\hat{a} = \exp(\hat{u})\, \hat{b} \exp(-\hat{u})$ will then solve $\nabla \hat{a} = 0$ with the correct boundary condition $\sigma(\hat{a}) = f_\hbar$. (We deal with uniqueness at the end of this proof.)

We now work order by order in the grading gr. Firstly, we must solve

$$0 = \mathsf{gr}_{-2}(\nabla_{\mathrm{D}} \hat{b}) = \frac{db^{(-2)} + b_a^{(-1)} E^a}{i\hbar}\,.$$

From Equation (2.14) we have $b^{(-2)} = a^{(-2)} = \mathsf{gr}_{-2} f_\hbar$, but by assumption $\mathcal{L}_\rho f_\hbar = 0$ so Cartan's magic lemma gives $\iota_\rho db^{(-2)} = 0$, whence $db^{(-2)} \in \operatorname{span}\{E^a\}$. Hence we can solve the equation in the above display (uniquely) for $b_a^{(-1)}$.

At the next order in the grading we must now solve

$$0 = \mathsf{gr}_{-1}(\nabla_{\mathrm{D}} \hat{b}) = \frac{db_a^{(-1)} \hat{s}^a + b_{ab}^{(0)} E^a \hat{s}^b}{i\hbar}\,.$$

By virtue of the Darboux coordinate system, $b_a^{(-1)}$ cannot depend on ψ so $\iota_\rho db_a^{(-1)} = 0$. Hence the above display (uniquely) determines $b_{ab}^{(0)}$ (and once again $\iota_\rho db_{ab}^{(0)} = 0$). The abelian term $-ib^{(0)}$ is at this point not determined. However for that we impose Equation (2.14) to the order 0 in the grading, which now determines $b^{(0)}$ in terms of $f_\hbar$ and other ψ-independent quantities. This establishes the pattern for an obvious recursion, which completes the existence part of this proof.

To show uniqueness, suppose $\hat{a}'$ also obeys $\nabla \hat{a}' = 0$ such that $\sigma(\hat{a}' - \hat{a}) = 0$. Now, let

$$\nabla = \frac{\alpha}{i\hbar} + \frac{e^a J_{ab} \hat{s}^b}{i\hbar} + \cdots\,.$$

Then

$$0 = \mathsf{gr}_{-2}\big(\nabla(\hat{a}' - \hat{a})\big) = \frac{(a_a'^{(-1)} - a_a^{(-1)}) e^a}{i\hbar} \;\Leftrightarrow\; a_a'^{(-1)} = a_a^{(-1)}\,.$$

Indeed, the same pattern holds at all higher orders in the grading gr, so that $\hat{a}' = \hat{a}$, as required. □

Remark 2.5. Calling $\xi^a = (\chi^i, \pi_i)$, the Darboux connection (2.10) obeys

$$[\nabla_{\mathrm{D}}, \hat{s}^a - \xi^a] = 0\,.$$

So taking $\hat{b}$ equal to any polynomial $\mathcal{P}(\hat{s}^a - \xi^a)$ solves the parallel section condition (2.13). This in turn immediately solves the parallel section problem for $f_\hbar$ expressible as a polynomial in Darboux coordinates. Note however, that in general, replacing $\mathcal{P}$ by a formal power series in $\hat{s}^a - \xi^a$, may not give a well defined formal power series in Weyl ordered symbols of $\hat{s}^a$. (Quantum reordering terms potentially involve infinite, non-convergent, sums of the coefficients of the original power series.) ❋

Let us denote by σ^{-1} the map $C^\infty Z[[\hbar]] \cap \ker(\mathcal{L}_\rho) \ni f_\hbar \mapsto \hat{a}$ as defined by the above lemma. Now consider a pair of solutions $f_\hbar, g_\hbar \in C^\infty Z[[\hbar]]$ to the classical equations of motion:

$$\mathcal{L}_\rho f_\hbar = 0 = \mathcal{L}_\rho g_\hbar \,.$$

Then we have a pair of parallel sections $\sigma^{-1}(f_\hbar)$ and $\sigma^{-1}(g_\hbar)$ of $Z \ltimes \mathcal{U}(\mathfrak{heis})$. These may be multiplied pointwise along Z using the operator product on fibers. Therefore, *a lá* Fedosov [9], we may define a $\star$-multiplication of functions $f_\hbar$ and $g_\hbar$ by[12]

$$f_\hbar \star g_\hbar = \sigma\big(\sigma^{-1}(f_\hbar)\sigma^{-1}(g_\hbar)\big)\,.$$

This gives a contact analog of deformation quantization. Observe that it reduces the deformation problem to a gauge transformation. However, unlike Fedosov's work, this means that the above uniqueness proof for flat sections is local. It ought however be possible to improve this to a global statement and preliminary results indicate that this is the case; we reserve those results for a later publication, where we also plan to detail the precise map between the above display and Fedosov's deformation formula for symplectic structures.

3. Flat sections and dynamics

As discussed in the previous section, solving for a flat connection ∇ on the Hilbert bundle $Z \ltimes \mathcal{H}$ is analogous to finding an operator quantizing a classical Hamiltonian, while the parallel transport equation (2.9) is the analog of the Schrödinger equation which controls quantum dynamics. We now turn our attention to solving the latter and computing correlators.

3.1. **Parallel transport.** Let us suppose we have prepared a state $|\mathcal{E}_{\rm i}\rangle \in \mathcal{H}_{z_{\rm i}}$ where $\mathcal{H}_{z_{\rm i}}$ is the Hilbert space associated with a point $z_{\rm i} \in Z$ (one may think of $z \in Z$ as a generalized laboratory time coordinate). We would like to compute the probability of measuring a state $|\mathcal{E}_{\rm f}\rangle \in \mathcal{H}_{z_{\rm f}}$ at some other point $z_{\rm f} \in Z$. For that, observe that we can parallel transport the "initial" state $|\mathcal{E}_{\rm i}\rangle$ from the Hilbert space $\mathcal{H}_{z_{\rm i}}$ to any other Hilbert space $\mathcal{H}_z$ using a line operator

$$|\mathcal{E}(z)\rangle = \Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z} \widehat{A}\,\big)\Big)|\mathcal{E}_{\rm i}\rangle \in \mathcal{H}_z\,, \tag{3.1}$$

where P_γ denotes path ordering and γ is any path in Z joining $z_{\rm i}$ and z. Since $\nabla = d + \hat{A}$, it follows that the section $\Psi(z) = |\mathcal{E}(z)\rangle$ of $Z \ltimes \mathcal{H}$ solves the Schrödinger equation (2.9). Since the connection ∇ is flat, if the fundamental group $\pi_1(Z)$ is trivial, this solution is independent of the choice of path γ between $z_{\rm i}$ and z. When this is not the case, we must be more careful with the choice of Hilbert space fibers.

[12] Fedosov constructs a deformation of the Moyal star product for Weyl ordered operators in the Weyl algebra given the data of a symplectic manifold. Here we skip the Moyal star and work directly with operators in the Weyl algebra.

We discuss this further below. Modulo this issue, the probability $P_{\rm f,i}$ of observing $|\mathcal{E}_{\rm f}\rangle \in \mathcal{H}_{z_{\rm f}}$ having prepared $|\mathcal{E}_{\rm i}\rangle \in \mathcal{H}_{z_{\rm i}}$ is

$$P_{\rm f,i} = \frac{\Big|\langle\mathcal{E}_{\rm f}|\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z_{\rm f}}\widehat{A}\big)\Big)|\mathcal{E}_{\rm i}\rangle\Big|^2}{\langle\mathcal{E}_{\rm f}|\mathcal{E}_{\rm f}\rangle\,\langle\mathcal{E}_{\rm i}|\mathcal{E}_{\rm i}\rangle}\,.$$

In [16] we showed how to extract quantum mechanical Wigner functions from correlators

$$\mathcal{W}_{\mathcal{E}_{\rm f},\mathcal{E}_{\rm i}}(z_{\rm f},z_{\rm i}) := \langle\mathcal{E}_{\rm f}|\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z_{\rm f}}\widehat{A}\big)\Big)|\mathcal{E}_{\rm i}\rangle\,. \tag{3.2}$$

This correlator is gauge covariant. In particular, in a contractible local patch around the path γ, by virtue of Theorem 2.2, we can find a gauge transformation $\widehat{U}$ such that $\widehat{U}\nabla\widehat{U}^{-1} = \nabla_{\rm D}$, where the Darboux normal form is given in Equation (2.4). Hence the line operators for these two connections are related by

$$\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z_{\rm f}}\widehat{A}\big)\Big) = \widehat{U}(z_{\rm f})^{-1}\circ\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z}\widehat{A}_{\rm D}\big)\Big)\circ\widehat{U}(z_{\rm i})\,. \tag{3.3}$$

Inserting resolutions of unity $\int dS|S\rangle\langle S| = 1 = \int dP|P\rangle\langle P|$ for $\mathcal{H}$ (where $\hat{s}^a = (\hat{S}^A,\hat{P}_A)$ and $\hat{S}_A|S\rangle = S_A|S\rangle$, $\hat{P}_A|P\rangle = P_A|P\rangle$) in the above identity, and putting this in the correlator (3.2) gives [13]

$$\mathcal{W}_{\mathcal{E}_{\rm f},\mathcal{E}_{\rm i}}(z_{\rm f},z_{\rm i}) := \int dSdP\,\langle\mathcal{E}_{\rm f}|\widehat{U}(z_{\rm f})^{-1}|P\rangle\,\langle P|\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_{\rm i}}^{z_{\rm f}}\widehat{A}_{\rm D}\big)\Big)|S\rangle\,\langle S|\widehat{U}(z_{\rm i})|\mathcal{E}_{\rm i}\rangle\,. \tag{3.4}$$

Since the line operator for the connection $\widehat{A}_{\rm D}$ in the Darboux frame is essentially trivial (see directly below), knowledge of the gauge transformations $\widehat{U}$ determines the correlator.

Example 3.1 (The Darboux correlator)**.** Consider a pair of points $z_{\rm i} = (\pi_{\rm i},\chi_{\rm i},\psi_{\rm i})$ and $z_{\rm f} = (\pi_{\rm f},\chi_{\rm f},\psi_{\rm f})$ in the contact three-manifold $Z = (\mathbb{R}^3, \pi d\chi - d\psi)$. Since here we want to study a line operator for a flat connection $\nabla_{\rm D}$ on a trivial manifold, we may choose any path between these two points, so take $\gamma = \gamma_\psi \cup \gamma_\pi \cup \gamma_\chi$ where

$$\begin{aligned}
\gamma_\pi &:= \{(1-t)\pi_{\rm i} + t\pi_{\rm f}, \chi_{\rm i}, \psi_{\rm i})\}\,,\\
\gamma_\chi &:= \{(\pi_{\rm f}, (1-t)\chi_{\rm i} + t\chi_{\rm f}, \psi_{\rm i})\}\,,\\
\gamma_\psi &:= \{(\pi_{\rm f}, \chi_{\rm f}, (1-t)\psi_{\rm i} + t\psi_{\rm f})\}\,,
\end{aligned}$$

where $t \in [0,1]$. Then, along these three paths the potential $\widehat{A}$ for the Darboux connection (see Equation 2.12) takes the form

$$\widehat{A}_{\gamma_\pi} = \frac{1}{i\hbar}dt(\pi_{\rm f}-\pi_{\rm i})\hat{S}\,,\quad \widehat{A}_{\gamma_\chi} = \frac{1}{i\hbar}dt(\chi_{\rm f}-\chi_{\rm i})(\pi_{\rm f}-\hat{P})\,,\quad \widehat{A}_{\gamma_\psi} = -\frac{1}{i\hbar}dt(\psi_{\rm f}-\psi_{\rm i})\,.$$

[13]Of course, one could equally well insert other resolutions of unity, for example, replacing $\int dP|P\rangle\langle P|$ with $\int dS'|S'\rangle\langle S'|$ is a propitious choice used in the next example.

Hence the correlator in Darboux frame is simply

$$\langle P|\Big(\mathrm{P}_\gamma \exp\big(-\int_{z_\mathrm{i}}^{z_\mathrm{f}} \widehat{A}_\mathrm{D}\,\big)\Big)|S\rangle = \exp\Big(-\frac{(\chi_\mathrm{f}-\chi_\mathrm{i})(\pi_\mathrm{f}-P)+(\pi_\mathrm{f}-\pi_\mathrm{i})S-\psi_\mathrm{f}+\psi_\mathrm{i}}{i\hbar}\Big)\,.$$

The above result combined with Equation 3.4 indeed shows that knowledge of the gauge transformation $\widehat{U}$ bringing a connection to its Darboux form determines correlators. ❋

3.2. **Path integrals.** In general, one does not have access to the explicit diffeomorphism bringing the contact form to its Darboux normal form (let alone the gauge transformation $\widehat{U}$). Instead correlators can be computed in terms of path integrals. For that, per its definition, we split the path ordered exponential of the integrated potential $\widehat{A}$ into infinitesimal segments dz^i along the path γ, and insert successive resolutions of unity. In particular, using that, for dz^i small,

$$\langle P|\exp(-\widehat{A}_i(\hat{S},\hat{P})dz^i)|S\rangle \approx \exp\Big(\frac{i}{\hbar}P_A S^A - A_\mathrm{N}(S,P)\Big)\,,$$

where $A_\mathrm{N}(S,P)$ is the normal ordered symbol[14] of the operator $\hat{A}$, we have the operator relation

$$\exp(-\widehat{A}_i dz^i) \approx \int dSdP\,|P\rangle \exp\Big(\frac{i}{\hbar}P_A S^A - A_\mathrm{N}(S,P)\Big)\langle S|\,.$$

Concatentating this expression along the path γ gives the path integral formula for the correlator between states $|S_\mathrm{i}\rangle$ and $\langle P_\mathrm{f}|$

$$\mathcal{W}_{P_\mathrm{f},S_\mathrm{i}}(z_\mathrm{f},z_\mathrm{i}) = \int_{S(z_\mathrm{i})=S_\mathrm{i}}^{P(z_\mathrm{f})=S_\mathrm{f}} [dPdS]\exp\Big(-\frac{i}{\hbar}\int_\gamma \big(P_A dS^A + A_\mathrm{N}(S,P)\big)\Big)\,.$$

In the above γ is any path in Z connecting z_i and z_f. When ∇ has trivial holonomy (otherwise see below), neither the correlator nor its path integral representation depends on this choice. Notice that the path integration in the above formula is only performed fiberwise. We do not integrate over paths γ in Z, but rather paths in the total space $\mathcal{Z} = Z \ltimes \mathbb{R}^{2n}$ above the path γ in Z. Indeed, calling $s^a := (S^A, P_A)$ and writing $P_A dS^A = \frac{1}{2}s^a J_{ab} ds^b$ we see that the action appearing in the exponent of the above path integral is the quantum corrected analog of the extended action of Equation (2.5)(computing the operator $\widehat{A}$ and its normal ordered symbol A_N will in general produce terms proportional to powers of $\hbar$).

3.3. **Topology.** Finally, we discuss the case when the fundamental group $\pi_1(Z)$ is non-trivial[15]. The holonomy of the connection ∇ may then be non-trivial, and the parallel transport solution (3.1) to the Schrödinger equation can depend on the homotopy class of the path γ. *A priori* this seems to be a *bug* leading to loss of predictivity, however remembering that the topology of a system can influence its

[14] To be precise, $\widehat{A}$ is recovered by writing $A(S,P)$ as a power series in P and S and then replacing monomials P^kS^l by the operator $\hat{P}^k\hat{S}^l$.

[15] We owe the key idea of this section of modding out the Hilbert space fibers by the holonomy of ∇ to Tudor Dimofte.

quantum spectrum (consider a free particle in a box, for example), we have in fact hit upon a *feature*. Our quantization procedure is not complete until we impose that the holonomy of the connection ∇ acts trivially on the Hilbert space fibers. To explain this point better, as a running example consider the contact form

$$\alpha = \pi d\theta - d\psi\,,$$

on the manifold $Z = C\times\mathbb{R}$ where C is a cylinder with periodic coordinate $\theta \sim \theta+2\pi$. Now let us study the quantizaton determined by the flat connection $\nabla = d + \widehat{A}$ where

$$\widehat{A} = \frac{\alpha}{i\hbar} + d\pi\frac{S}{i\hbar} + d\theta\frac{\partial}{\partial S}\,.$$

Here we have picked some polarization for the Hilbert space fibers such that elements are given by wavefunctions $\psi(S)$.

Along the path $\gamma = \{\theta = \theta_o + \theta, \pi = \pi_o, \psi = \psi_o \ :\ \theta \in [0, 2\pi)\}$, we have $\widehat{A}_\gamma = \frac{1}{i\hbar}d\theta\big(\pi_o - \frac{\hbar}{i}\frac{\partial}{\partial S}\big)$. Hence the holonomy of ∇ at basepoint $z_o = (\theta_o, \pi_o, \psi_o)$ is

$$\mathrm{hol}_{z_o}(\widehat{A}_\gamma) = \exp\Big(-\frac{2\pi i}{\hbar}\Big(\pi_o - \frac{\hbar}{i}\frac{\partial}{\partial S}\Big)\Big)\,.$$

Requiring that this holonomy acts trivially on the Hilbert space $\mathcal{H}$ over the base point $z_o \in Z$, we impose that elements $\psi_{z_o}(S)$ of that space obey

$$\exp\Big(-\frac{2\pi i}{\hbar}\Big(\pi_o - \frac{\hbar}{i}\frac{\partial}{\partial S}\Big)\Big)\,\psi_{z_o}(S) = \psi_{z_o}(S)\,.$$

Hence

$$\psi_{z_o}(S + 2\pi) = e^{\frac{2\pi i\pi_o}{\hbar}}\,\psi_{z_o}(S)\,.$$

So, up to a basepoint dependent phase, wavefunctions are periodic. In effect, the classical topology of the contact base manifold Z has enforced the desired boundary conditions on quantum wavefunctions.

4. Discussion and conclusions

Just as contact geometry reduces classical mechanics to a problem of contact topology (all dynamics is locally trivial by virtue of the contact Darboux theorem), the contact quantization we have presented does the same for quantum dynamics. Moreover, since our approach is completely generally covariant, even seemingly disparate systems can be related by appropriate choices of clocks. This gives a concrete setting for quantum cosmology-motivated studies of the "clock ambiguity" of quantum dynamics [1, 20].

Beyond providing a solid mathematical framework for philosophical questions of time and measurement in quantum mechanics, it is very interesting to probe to which extent the gauge freedom characterized in Theorem 2.2 can be used to solve or further the study of concrete quantum mechanical systems. As discussed in Section 3, knowledge of the gauge transformation bringing the connection ∇ to its Darboux form can be used to compute correlators, which begs the question whether methods—perturbative, exact when symmetries are present, or numerical—can be developed to calculate these transformations.

Along similar lines to the above remark, symmetries and integrability play a central *rôle* in the analysis of quantum systems. Again contact geometry and its quantization ought be an ideal setting for analyzing quantum symmetries and relating them to contact topology. Preliminary results show that this is the case, and we plan to report on such questions elsewhere.

Lattice spin models and models with Fermi statistics are crucial for the description of physical systems. Here one needs to study supercontact structures (see [21, 23, 5]); it is indeed not difficult to verify that our flat connection/quantizaton and parallel section/dynamics methodology can be applied directly in the supercontact setting; again we plan to report on this interesting direction in the near future.

In Section 2.4 we showed how to relate contact quantization to Fedosov's deformation quantization. It would also be interesting to relate our approach to other quantization methods. In particular, it would be interesting to study the relation to Kontsevich's explicit deformation quantization formula for Poisson structures [18] and its Cattaneo–Felder sigma model derivation [7]. In addition, it would be interesting to study when we can go beyond formal deformation quantization, perhaps along the lines of the A-model approach of Gukov–Witten to quantization [15], or geometric quantization in general. Indeed, Fitzpatrick has made a rigorous geometric quantization study of contact structures [10] based on the proposal by Rajeev [22] to quantize Lagrange brackets (these are the contact analog of the Poisson bracket). Note also that earlier work by Kashiwara [17] studies sheaves of pseudodifferential operators over contact manifolds, and Yoshioka has performed a contact analog of Fedosov quantization where the base manifold is a symplectic manifold and the fibers carry a contact structure [24].

Finally, we mention that our construction of the connection ∇ is in spirit rather close to the Cartan normal connection in parabolic geometries, see [6] for the general theory and [11] for its application to contact structures compatible with a projective structure. These geometric methods may also end up being directly relevant to quantum mechanics.

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G. Herczeg,
Department of Physics, University of California,
Davis, CA 95616, USA
E-mail: `Herczeg@ms.physics.ucdavis.edu`

E. Latini,
Dipartimento di Matematica, Università di Bologna,
Piazza di Porta S. Donato 5, and INFN, Sezione di Bologna,
Via Irnerio 46, I-40126 Bologna, Italy
E-mail: `emanuele.latini@UniBo.it`

A. Waldron,
Center for Quantum Mathematics and Physics (QMAP),
Department of Mathematics, University of California,
Davis, CA95616, USA
E-mail: `wally@math.ucdavis.edu`