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THE GROUP RING $\mathbb{K}F$ OF RICHARD THOMPSON'S GROUP F HAS NO MINIMAL NON-ZERO IDEALS

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Abstract. We use a total order on Thompson's group F to show that the group ring $\mathbb{K}F$ has no minimal non-zero ideals.

1. Introduction

We define Richard Thompson's group F to be the group of right fractions of the monoid P which is given by the presentation

$$\langle x_0, x_1, x_2, \dots \mid x_n x_m = x_m x_{n+1} \text{ for } n > m \rangle .$$

Geoghegan has conjectured that the group F is an example of a finitely presented, nonamenable group which has no free subgroup on two generators [2]. In [1], Brin and Squier show that the group F has no free subgroup on two generators. However, the question of whether or not the group F is amenable has been open for over twenty years [2].

Let $\mathbb{K}$ denote a field. It is shown in [1] that the group F is totally ordered. Using this fact we can show that the group ring $\mathbb{K}F$ is cancellative, and consequently does not have any zero-divisors. Thus, the set of all nonzero elements in $\mathbb{K}F$ forms a multiplicative monoid $\mathcal{H}$ whose identity is the identity 1_F of the group F. We leave it to the reader to check that if $\mathcal{H}$ is (left/right) amenable, then the group F is amenable.

Thus, one can ask whether or not the multiplicative monoid $\mathcal{H}$ is right amenable. In [3], Frey gives necessary conditions that any minimal ideal of a semigroup S must satisfy for S to be right amenable. In particular, Frey shows that if S is a right amenable semigroup, $\mathcal{L}$ is a minimal left ideal of S, and $\mathcal{R}$ is a minimal right ideal of S, then

(i) $\mathcal{L}$ is a two-sided ideal of S.

(ii) $\mathcal{R} \subseteq \mathcal{L}$.

(iii) $\mathcal{R}$ is a group.

(iv) There exists a semigroup T such that $\mathcal{L}$ is isomorphic to $\mathcal{R} \oplus T$, and such that for all $z_1, z_2 \in T$, $z_1 z_2 = z_1$.

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Frey also shows that if S is a semigroup containing a minimal left ideal $\mathcal{L}$ and a minimal right ideal $\mathcal{R}$, then S is right amenable if and only if $\mathcal{R}$ is an amenable group.

Thus, one can ask what the minimal ideals of $\mathcal{H}$ are, and whether or not they satisfy the conditions stated above. In this paper, we use a total ordering on the group F to show that $\mathcal{H}$ has no minimal left, right, or two-sided ideals.

2. A total ordering on the group F

We denote the set of generators $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ of P (and consequently, of F) by Σ, and we define the set $\Sigma_n = \{x_m \in \Sigma \mid m \geq n\}$. Given an element $q \in P$, we let $|q|$ denote the length of a word over Σ representing q. Every element of the group F can be represented uniquely by a normal form

$$x_{i_1}^{b_1} x_{i_2}^{b_2} x_{i_3}^{b_3} \ldots x_{i_m}^{b_m} x_{j_k}^{-d_k} \ldots x_{j_3}^{-d_3} x_{j_2}^{-d_2} x_{j_1}^{-d_1}$$

where

(i) for each t, and for each r, we have that b_t, $d_r > 0$;

(ii) $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_k$;

(iii) if there exists some i such that both x_i and x_i^{-1} are generators in the normal form, then x_{i+1} or x_{i+1}^{-1} is a generator in the normal form as well.

Given two generators x_i and x_j of P, then we define $x_i < x_j$ if and only if $i < j$. We can now use the shortlex ordering on the set of normal forms for the elements of the monoid P to get a total ordering $<_P$ on the monoid P. We use the ordering $<_P$ on P to define an ordering $<_F$ on all of the group F in the following way: Given $g \in F$ such that g has normal form xy^{-1}, with x, $y \in P$, then $g <_F 1_F$ if and only if $x <_P y$. We extend this to compare all elements of the group F by defining for each distinct pair g, $h \in F$ that $g <_F h$ if and only if $gh^{-1} <_F 1_F$. We will prove that $<_F$ is a well defined total ordering on the group F.

Let g, $h \in F$. Assume that gh^{-1} has normal form ab^{-1}, where a, $b \in P$. Since ab^{-1} is in normal form, then ba^{-1} is in normal form. Moreover, since $hg^{-1} = (gh^{-1})^{-1} = (ab^{-1})^{-1} = ba^{-1}$, then hg^{-1} has normal form ba^{-1}. Note that in case (i) below, since $a = b$ and ab^{-1} is in normal form, then a and b are empty words and consequently ab^{-1} is the identity element of F. Therefore, if gh^{-1} has normal form ab^{-1}, where a, $b \in P$, then

(i) $g = h$ if and only if $a = b$;

(ii) $gh^{-1} <_F 1_F$ if and only if $a <_P b$;

(iii) $hg^{-1} <_F 1_F$ if and only if $b <_P a$.

Since for each pair of elements a, $b \in P$, exactly one of $a = b$, $a <_P b$, or $b <_P a$ must hold, then given two elements g, $h \in F$, exactly one of $g = h$, $gh^{-1} <_F 1_F$, or $hg^{-1} <_F 1_F$ must hold. Thus, given two distinct elements g, $h \in F$, then either $gh^{-1} <_F 1_F$, in which case $g <_F h$, or else $hg^{-1} <_F 1_F$, in which case $h <_F g$.

Thus, it follows that $<_F$ is well defined and linear.

Lemma 1. *Let w_1, $w_2 \in P$ be such that $w_1 <_P w_2$. If x_m is any generator of the monoid P, then $x_m w_1 <_P x_m w_2$.*

Proof. Let $|w_1| = h$ and $|w_2| = k$. If $|w_1| < |w_2|$, then we see that $|x_m w_1| = h + 1 < k + 1 = |x_m w_2|$, which implies that $x_m w_1 <_P x_m w_2$.

Assume that $|w_1| = k = |w_2|$. Let w_1 have normal form $v x_{a_1} x_{a_2} \dots x_{a_t}$, and let w_2 have normal form $v x_{b_1} x_{b_2} \dots x_{b_t}$, where v is a (possibly empty) word over Σ, and $a_1 < b_1$.

Assume that $k = 1$. In this case, v is empty, $w_1 = x_{a_1}$, and $w_2 = x_{b_1}$, with $a_1 < b_1$. First assume that $m \le a_1 < b_1$. In this case, $x_m w_1$ has normal form $x_m x_{a_1}$ and $x_m w_2$ has normal form $x_m x_{b_1}$. Since $a_1 < b_1$, then $x_m w_1 <_P x_m w_2$. Next assume that $a_1 < m \le b_1$. In this case $x_m w_1$ has normal form $x_{a_1} x_{m+1}$, and $x_m w_2$ has normal form $x_m x_{b_1}$. Since $a_1 < m$, then $x_m w_1 <_P x_m w_2$. Finally, assume that $a_1 < b_1 < m$. In this case $x_m w_1$ has normal form $x_{a_1} x_{m+1}$, and $x_m w_2$ has normal form $x_{b_1} x_{m+1}$. Since $a_1 < b_1$, then $x_m w_1 <_P x_m w_2$.

Now assume that $k \ge 2$, and that for each $j \in \{1, \dots k-1\}$, if $u_1,\, u_2 \in P$ are such that $|u_1| = |u_2| = j$ and $u_1 <_P u_2$, then for each generator x_m of P, $x_m u_1 <_P x_m u_2$.

Assume that $|v| \ge 1$, and that $x_m v = v x_{m+|v|}$. Since $a_1 < b_1$, then it follows that $x_{a_1} x_{a_2} \dots x_{a_t} <_P x_{b_1} x_{b_2} \dots x_{b_t}$. Therefore, by our induction hypothesis we have that $x_{m+|v|} x_{a_1} x_{a_2} \dots x_{a_t} <_P x_{m+|v|} x_{b_1} x_{b_2} \dots x_{b_t}$. Thus, $x_{m+|v|} x_{a_1} x_{a_2} \dots x_{a_t}$ has normal form $\sigma x_{i_1} x_{i_2} \dots x_{i_q}$. Similarly, we see that $x_{m+|v|} x_{b_1} x_{b_2} \dots x_{b_t}$ has normal form $\sigma x_{j_1} x_{j_2} \dots x_{j_q}$, where σ is a (possibly empty) word over Σ, and $i_1 < j_1$. Therefore, $x_m w_1$ has normal form $v \sigma x_{i_1} x_{i_2} \dots x_{i_q}$, and $x_m w_2$ has normal form $v \sigma x_{j_1} x_{j_2} \dots x_{j_q}$. Since $i_1 < j_1$, then $x_m w_1 <_P x_m w_2$.

Now Assume that $|v| \ge 1$, and that $x_m v = u x_{m+|u|} z$, where z is some nonempty word over $\Sigma_{m+|u|}$, and where u is some (possibly empty) word over Σ. In this case, $x_m w_1$ has normal form $u x_{m+|u|} z x_{a_1} x_{a_2} \dots x_{a_t}$, and $x_m w_2$ has normal form $u x_{m+|u|} z x_{b_1} x_{b_2} \dots x_{b_t}$. Since $a_1 < b_1$, then it follows that $x_m w_1 <_P x_m w_2$.

Finally, assume that v is empty. In this case, w_1 has normal form $x_{a_1} x_{a_2} \dots x_{a_k}$, and w_2 has normal form $x_{b_1} x_{b_2} \dots x_{b_k}$, where $a_1 < b_1$. First assume that $m \le a_1 < b_1$. In this case, $x_m w_1$ has normal form $x_m x_{a_1} x_{a_2} \dots x_{a_k}$, and $x_m w_2$ has normal form $x_m x_{b_1} x_{b_2} \dots x_{b_k}$. Since $a_1 < b_1$, then $x_m w_1 <_P x_m w_2$. Next assume that $a_1 < m \le b_1$. In this case, $x_m w_1$ has normal form $x_{a_1} \beta$, where β is a word over Σ_{a_1} of length k, and $x_m w_2$ has normal form $x_m x_{b_1} x_{b_2} \dots x_{b_k}$. Since $a_1 < m$, then $x_m w_1 <_P x_m w_2$. Finally, assume that $a_1 < b_1 < m$. In this case, $x_m w_1$ has normal form $x_{a_1} \rho_1$, where ρ_1 is a word over Σ_{a_1} of length k, and $x_m w_2$ has normal form $x_{b_1} \rho_2$, where ρ_2 is a word over Σ_{b_1} of length k. Since $a_1 < b_1$, then $x_m w_1 <_P x_m w_2$. □

Lemma 2. *Let $w_1,\, w_2 \in P$ be such that $w_1 <_P w_2$. If x_m is any generator of the monoid P, then $w_1 x_m <_P w_2 x_m$.*

Proof. Let $|w_1| = h$ and $|w_2| = k$. If $|w_1| < |w_2|$, then we see that $|w_1 x_m| = h + 1 < k + 1 = |w_2 x_m|$, which implies that $w_1 x_m <_P w_2 x_m$.

Assume that $|w_1| = k = |w_2|$. Let w_1 have normal form $x_{a_1} x_{a_2} \dots x_{a_k}$, and let w_2 have normal form $x_{b_1} x_{b_2} \dots x_{b_k}$. First assume that $k = 1$. In this case, $w_1 = x_{a_1}$ and $w_2 = x_{b_1}$, with $a_1 < b_1$. If $m < a_1 < b_1$, then $w_1 x_m$ has normal form $x_m x_{a_1+1}$, and $w_2 x_m$ has normal form $x_m x_{b_1+1}$, which implies that $w_1 x_m <_P w_2 x_m$. Assume that $a_1 \leq m < b_1$. Thus, w_1x_m has normal form $x_{a_1}x_m$, and w_2x_m has normal form $x_mx_{b_1+1}$. If $a_1 < m$, then $w_1x_m = x_{a_1}x_m <_P x_mx_{b_1+1} = w_2x_m$. If $a_1 = m$, then $w_1x_m = x_mx_m <_P x_mx_{b_1+1} = w_2x_m$. Assume that $a_1 < b_1 \leq m$. In this case, we see that w_1x_m has normal form $x_{a_1}x_m$, and w_2x_m has normal form $x_{b_1}x_m$, which implies that $w_1x_m = x_{a_1}x_m <_P x_{b_1}x_m = w_2x_m$.

Now assume that $k \geq 2$, and that for each $j \in \{1, \ldots k-1\}$, if u_1, $u_2 \in P$ are such that $|u_1| = |u_2| = j$ and $u_1 <_P u_2$, then for each generator x_m of P, $u_1x_m <_P u_2x_m$.

Assume that $a_1 = b_1 \leq m$. In this case, w_1 has normal form $x_{a_1}\sigma_1$, and w_2 has normal form $x_{a_1}\sigma_2$, where σ_1 and σ_2 are words over Σ_{a_1}. Since $x_{a_1}\sigma_1 = w_1 <_P w_2 = x_{a_1}\sigma_2$, then it must be the case that $\sigma_1 <_P \sigma_2$. Therefore, it follows by our induction hypothesis that $\sigma_1x_m <_P \sigma_2x_m$. Thus, σ_1x_m has normal form $vx_{c_1}x_{c_2}\ldots x_{c_t}$, and σ_2x_m has normal form $vx_{e_1}x_{e_2}\ldots x_{e_t}$, where v is a word over Σ_{a_1}, and $c_1 < e_1$. Therefore, $x_{a_1}\sigma_1x_m$ has normal form $x_{a_1}vx_{c_1}x_{c_2}\ldots x_{c_t}$, and $x_{a_1}\sigma_2x_m$ has normal form $x_{a_1}vx_{e_1}x_{e_2}\ldots x_{e_t}$, where $c_1 < e_1$. Since $w_1x_m = x_{a_1}\sigma_1x_m$ and $w_2x_m = x_{a_1}\sigma_2x_m$, then it follows that $w_1x_m <_P w_2x_m$.

Now assume that $a_1 < b_1 \leq m$. In this case w_1x_m has normal form $x_{a_1}\beta_1$, where β_1 is a word over Σ_{a_1}, and w_2x_m has normal form $x_{b_1}\beta_2$, where β_2 is a word over Σ_{b_1}. Thus, $w_1x_m <_P w_2x_m$.

Assume that $a_1 = m < b_1$. In this case, w_1x_m has normal form $x_mx_m\rho$, where ρ is a word over Σ_m, and w_2x_m has normal form $x_mx_{b_1+1}x_{b_2+1}\ldots x_{b_k+1}$. Since $m < b_1 < b_1 + 1$, then $w_1x_m <_P w_2x_m$.

Assume that $a_1 < m < b_1$. In this case, w_1x_m has normal form $x_{a_1}\alpha$, where α is a word over Σ_{a_1}, and w_2x_m has normal form $x_mx_{b_1+1}x_{b_2+1}\ldots x_{b_k+1}$. Since $a_1 < m$, then $w_1x_m <_P w_2x_m$.

Finally, assume that $m < a_1 < b_1$. In this case, w_1x_m has normal form $x_mx_{a_1+1}x_{a_2+1}\ldots x_{a_k+1}$, and w_1x_m has normal form $x_mx_{b_1+1}x_{b_2+1}\ldots x_{b_k+1}$. This implies that $w_1x_m <_P w_2x_m$. ☐

Given a, b, $c \in P$, with $a <_P b$, then by using Lemmas 1 and 2 as induction base steps, one can use induction on the length $|c|$ to show that $ca <_P cb$ and $ac <_P bc$. We now extend this property to the ordering $<_F$ by showing that for all g, h, $d \in F$, if $g <_F h$, then $dg <_F dh$ and $gd <_F hd$. Again, we note that in case (i) below, since $a = b$ and ab^{-1} is in normal form, then a and b are empty words and consequently ab^{-1} is the identity element 1_P of P.

Lemma 3. *Let a, b, c, $d \in P$ be such that $ab^{-1} = cd^{-1}$, and such that ab^{-1} is in normal form. Then*

(i) *$a = b$ if and only if $c = d$;*

(ii) *$a <_P b$ if and only if $c <_P d$;*

(iii) *$b <_P a$ if and only if $d <_P c$.*

Proof. Let 1_P denote the identity element of the monoid P. Since $a = b$ if and only if $cd^{-1} = ab^{-1} = 1_P$, and since $cd^{-1} = 1_P$ if and only if $c = d$, then $a = b$ if and only if $c = d$.

Assume that $a \neq b$. When rewriting the normal form ab^{-1} to get the word cd^{-1}, we multiply a on the right by a (possibly empty) word u over Σ, and we multiply

b^{-1} on the left by the (possibly empty) word u^{-1} over Σ^{-1}. In particular, the word u consists of all the generators from c that cancel when we multiply c with d^{-1}, and then simplify to rewrite cd^{-1} in the normal form ab^{-1}. Thus, $c = au$ and $d = bu$. If $a <_P b$, then it follows by the comments above that $c = au <_P bu = d$. If $b <_P a$, then again it follows by the comments above that $d = bu <_P au = c$.

Now assume that $c <_P d$. Since $<_P$ is a linear ordering on P, then exactly one of the following is true: $a <_P b$, $a = b$, or $b <_P a$. If $a = b$, then $c = d$, a contradiction. If $b <_P a$, then it follows from the argument given above that $d <_P c$, a contradiction. Thus, $a <_P b$. A similar argument shows that if $d <_P c$, then $b <_P a$. □

Lemma 4. *Let c, $d \in P$. Then $c <_P d$ if and only if $c <_F d$.*

Proof. Let cd^{-1} have normal form ab^{-1}, where a, $b \in P$. Assume that $c <_P d$. Since $c <_P d$ and cd^{-1} has normal form ab^{-1}, then it follows by Lemma 3 that $a <_P b$. Thus, by definition of $<_F$, we have that $ab^{-1} <_F 1_F$, which implies that $cd^{-1} <_F 1_F$, which in turn implies that $c <_F d$.

Conversely, assume that $c <_F d$. By definition of $<_F$, we have that $cd^{-1} <_F 1_F$. Since $ab^{-1} = cd^{-1}$, then $ab^{-1} <_F 1_F$, which implies that $a <_P b$. Therefore, by Lemma 3, we have that $c <_P d$. □

Lemma 5. *Let g, $h \in F$ be such that $g <_F h$. Then for any $d \in F$, $gd <_F hd$.*

Proof. Since $g <_F h$, then it follows by definition of $<_F$ that $gh^{-1} <_F 1_F$. Thus, we see that $(gd)(d^{-1}h^{-1}) = gh^{-1} <_F 1_F$, which implies that $gd <_F hd$. □

Lemma 6. *Let $g \in F$ and c, $d \in P$. If $g <_F c$ and $c <_F d$, then $g <_F d$.*

Proof. Let g have normal form ab^{-1}, where a, $b \in P$. Since $ab^{-1} <_F c$, then it follows by Lemma 5 that $a <_F cb$. Since $a <_F cb$, then it follows that $a <_P cb$. Similarly, since $c <_F d$, then $c <_P d$. Since $a <_P cb$ and $c <_P d$, then $a <_P cb <_P db$. Therefore, by definition of $<_F$, we have that $ab^{-1}d^{-1} <_F 1_F$, which implies that $ab^{-1} <_F d$. Hence, $g <_F d$. □

Lemma 7. *Let g, $h \in F$. If $g <_F h$ and $h <_F 1_F$, then $g <_F 1_F$.*

Proof. Let g have normal form ab^{-1}, where a, $b \in P$, and let h have normal form cd^{-1}, where c, $d \in P$. Since $cd^{-1} <_F 1_F$, then $c <_F d$. Since $ab^{-1} <_F cd^{-1}$, then it follows by Lemma 5 that $ab^{-1}d <_F c$. Since $ab^{-1}d <_F c$ and $c <_F d$, then it follows by Lemma 6 that $ab^{-1}d <_F d$. Thus, by Lemma 5, we have that $ab^{-1} <_F 1_F$. Hence, $g <_F 1_F$. □

Lemma 8. *Let g, h, $d \in F$. If $g <_F h$ and $h <_F d$, then $g <_F d$.*

Proof. Since $g <_F h$, then it follows by Lemma 5 that $gd^{-1} <_F hd^{-1}$. Since $h <_F d$, then it follows that $hd^{-1} <_F 1_F$. Therefore, it follows by Lemma 7 that $gd^{-1} <_F 1_F$. Thus, it follows by definition of $<_F$ that $g <_F d$. □

Lemma 9. *Let g, $h \in F$, and let $b \in P$. If $g <_F h$, then $bg <_F bh$.*

Proof. Let gh^{-1} have normal form cd^{-1}, where $c,\, d \in P$. Since $g <_F h$, then it follows that $gh^{-1} <_F 1_F$. Since gh^{-1} has normal form cd^{-1}, then it follows that $cd^{-1} <_F 1_F$. Therefore, $c <_F d$, which implies that $c <_P d$. Thus, it follows that $bc <_P bd$, which implies that $bc <_F bd$, and therefore that $bcd^{-1}b^{-1} <_F 1_F$. Again, since gh^{-1} has normal form cd^{-1}, then it follows that $bgh^{-1}b^{-1} <_F 1_F$, and therefore that $bg <_F bh$. □

Lemma 10. *Let $g,\, h \in F$, and let $b \in P$. If $g <_F h$, then $b^{-1}g <_F b^{-1}h$.*

Proof. If $b^{-1}g = b^{-1}h$, then $g = h$, a contradiction. Thus, $b^{-1}g \neq b^{-1}h$. Suppose that $b^{-1}h <_F b^{-1}g$. Thus, it follows by Lemma 9 that $b(b^{-1}h) <_F b(b^{-1}g)$, which implies that $h <_F g$, a contradiction. Hence, $b^{-1}g <_F b^{-1}h$. □

Lemma 11. *Let $g,\, h,\, u \in F$. If $g <_F h$, then $ug <_F uh$.*

Proof. Let u have normal form ab^{-1}, where $a,\, b \in P$. Since $g <_F h$, then it follows by Lemma 10 that $b^{-1}g <_F b^{-1}h$. Therefore, since $b^{-1}g <_F b^{-1}h$, then it follows by Lemma 9 that $ab^{-1}g <_F ab^{-1}h$. Hence, $ug <_F uh$. □

Lemma 12. *Let $g_1,\, g_2,\, h_1,\, h_2 \in F$ be such that $g_1 <_F g_2$ and $h_1 <_F h_2$. Then it follows that $g_1h_1 <_F g_2h_2$.*

Proof. Since $g_1 <_F g_2$, then it follows by Lemma 5 that $g_1h_1 <_F g_2h_1$. Similarly, since $h_1 <_F h_2$, then it follows by Lemma 11 that $g_2h_1 <_F g_2h_2$. Therefore, since $g_1h_1 <_F g_2h_1$ and $g_2h_1 <_F g_2h_2$, then it follows by Lemma 8 that $g_1h_1 <_F g_2h_2$. □

3. The main result

Lemma 13. *Let $g_1,\, g_2 \in F$. Assume that g_1 has normal form $a_1b_1^{-1}$, and that g_2 has normal form $a_2b_2^{-1}$, where $a_1,\, a_2,\, b_1,\, b_2 \in P$. If $|a_1| + |b_2| < |a_2| + |b_1|$, then $g_1 <_F g_2$.*

Proof. Let $b_1^{-1}b_2$ have normal form cd^{-1}, where $c,\, d \in P$. Each generator x_i in the normal form of b_2 which cancels when multiplying b_1^{-1} and b_2 to put $b_1^{-1}b_2$ in normal form will cancel with exactly one of the generators x_j^{-1} in the normal form of b_1^{-1}. That is, any generators from b_2 and b_1^{-1} which cancel when transforming $b_1^{-1}b_2$ into normal form will cancel in pairs. Thus, if k generators cancel from the normal form of b_2, then k generators cancel from the normal form of b_1^{-1}. Therefore, we see that $|c| = |b_2| - k$, and that $|d| = |b_1| - k$. Since $a_1,\, c \in P$, then there is no cancellation of generators when multiplying a_1 and c. Thus, we see that $|a_1c| = |a_1| + |c|$. Similarly, we see that $|a_2d| = |a_2| + |d|$. Therefore, we have that $|a_1c| = |a_1| + |c| = |a_1| + |b_2| - k < |a_2| + |b_1| - k = |a_2| + |d| = |a_2d|$. Since $|a_1c| < |a_2d|$, then it follows that $a_1c <_F a_2d$. Thus, $a_1b_1^{-1}b_2 = a_1cd^{-1} <_F a_2$, which implies that $a_1b_1^{-1} <_F a_2b_2^{-1}$. Hence, $g_1 <_F g_2$. □

Theorem 1. *Let $\mathcal{H}$ denote the multiplicative monoid of nonzero elements in the group ring $\mathbb{K}F$. Then $\mathcal{H}$ has no minimal ideals.*

Proof. Suppose, to the contrary, that $\mathcal{I}$ is a minimal two-sided ideal of $\mathcal{H}$. Since $\mathcal{H}$ is a cancellative monoid, then $\mathcal{I}$ is a principal ideal. Let $\hat{g} = \sum_{i=1}^{m} r_i g_i \in \mathcal{H}$ be such that $\mathcal{I} = \mathcal{H}\hat{g}\mathcal{H}$. By renumbering if necessary, we may assume that $g_1 <_F g_2 <_F \cdots <_F g_m$. Since $\sum_{i=1}^{m} r_i g_i = g_1 \sum_{i=1}^{m} r_i(g_1^{-1}g_i)$, then $\mathcal{I} = \mathcal{H}\sum_{i=1}^{m} r_i(g_1^{-1}g_i)\mathcal{H}$. In particular, we may assume that $g_1 = 1_F$. Let g_m have normal form cd^{-1}, where c, $d \in P$. Let $\mathcal{J} = \mathcal{H}(\hat{g})(1_F + c)\mathcal{H} = \mathcal{H}(\sum_{i=1}^{m} r_i g_i + \sum_{i=1}^{m} r_i(g_i c))\mathcal{H}$. Since $1_F = g_1 <_F g_m = cd^{-1}$, then $1_F \leq_F d <_F c$. Thus, 1_F and $g_m c$ are the smallest and largest elements, respectively, of F used to write $(\hat{g})(1_F + c)$ as a sum in the group ring $\mathbb{K}F$. Since $(\hat{g})(1_F + c) \in \mathcal{I}$, then $\mathcal{J}$ is a subideal of $\mathcal{I}$. Since $\mathcal{I}$ is minimal, then it must be the case that $\hat{g} \in \mathcal{I} = \mathcal{J}$. Thus, there exist $\sum_{j=1}^{l} s_j h_j$, $\sum_{k=1}^{e} t_k q_k \in \mathcal{H}$ such that $\hat{g} = \Big(\sum_{j=1}^{l} s_j h_j\Big)(\hat{g})(1_F + c)\Big(\sum_{k=1}^{e} t_k q_k\Big)$. Again, by renumbering if necessary, we may assume that $h_1 \leq_F h_2 \leq_F \cdots \leq_F h_l$ and $q_1 \leq_F q_2 \leq_F \cdots \leq_F q_e$. Since F is totally ordered, and since 1_F and g_m are the smallest and largest elements, respectively, of F used to write $\hat{g}$ as a sum in the group ring $\mathbb{K}F$, then it follows that $h_1 q_1 = 1_F$, and $h_l g_m c q_e = g_m$. This implies that $h_1 = q_1^{-1}$.

First assume that $h_1 = q_1 = 1_F$. Since $1_F <_F c$, $1_F = q_1 \leq_F q_e$, and $1_F = h_1 \leq_F h_l$, then $g_m <_F h_l g_m c q_e$, a contradiction. Therefore, $q_1 \neq 1_F$. In this case we have that $1_F <_F q_1$ or $1_F <_F h_1$. We may assume that $1_F <_F q_1$ (the proof for the case that $1_F <_F h_1$ is similar). Since $1_F <_F q_1$, then $h_1 = q_1^{-1} <_F 1_F$. Let h_l have normal form ab^{-1}, and let q_e have normal form yx^{-1}, where a, b, x, $y \in P$. Since $1_F <_F q_1 \leq_F q_e$, then $|x| \leq |y|$. Similarly, since $1_F <_F c$, then $|c| \geq 1$. Since $h_l g_m c q_e = g_m$, then $h_l = g_m q_e^{-1} c^{-1} g_m^{-1} = (cd^{-1})(xy^{-1})(c^{-1})(dc^{-1})$. As in the proof of Lemma 13, when transforming h_l into normal form, any generators which cancel must cancel in pairs. Thus, if k generators cancel in c, x, or d, then k generators must cancel in d^{-1}, y^{-1}, or either of the copies of c^{-1}. Thus, $|a| = |c| + |x| + |d| - k$, and $|b| = |d| + |y| + 2|c| - k$. Therefore, we see that $|a| + |y| = (|c| + |x| + |d| - k) + |y| < |d| + |y| + 2|c| + |x| - k = |b| + |x|$. It follows by Lemma 13 that $h_l = ab^{-1} <_F xy^{-1} = q_e^{-1}$. Thus, it follows that $h_l <_F q_e^{-1} \leq_F q_1^{-1} = h_1$, which is a contradiction. Hence, $\mathcal{H}$ does not have a minimal two-sided ideal.

A similar argument shows that $\mathcal{H}$ has neither a minimal left ideal nor a minimal right ideal. □

References

[1] Brin, M., Squier, C., *Groups of piecewise linearhomeomorphisms of the real line*, Invent. Math. **79** (3) (1985), 485–498.

[2] Cannon, J.W., Floyd, W.J., Parry, W.R., *Introductory notes on Richard Thompson's groups*, Enseign. Math. **42** 3–4) (1996), 215–256.

[3] Frey, A.H., *Studies on Amenable Semigroups*, Ph.D. thesis, University of Washington, 1960.

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