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OPERATOR CONNES-AMENABILITY OF COMPLETELY BOUNDED MULTIPLIER BANACH ALGEBRAS

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Abstract. For a completely contractive Banach algebra B, we find conditions under which the completely bounded multiplier algebra $\mathcal{M}_{cb}(B)$ is a dual Banach algebra and the operator amenability of B is equivalent to the operator Connes-amenability of $\mathcal{M}_{cb}(B)$. We also show that, in this case, these are equivalent to the existence of a normal virtual operator diagonal.

1. Introduction

The notion of amenability of Banach algebras was introduced by B.E. Johnson in [9]. He showed that the group algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable. The Johnson's result fails to be true for the Fourier algebras $A(G)$. In other words, there are some compact groups G, e.g. $G = SO(3)$, the group of all rotations about the origin of three-dimensional Euclidean space $\mathbb{R}^3$ for which $A(G)$ is not amenable [10]. In [14], Ruan introduced a variant of amenability which is called operator amenability for operator algebras and showed that the Fourier algebra $A(G)$ is operator amenable if and only if G is amenable.

When there is a natural weak*-topology on the algebra, it is suggested to restrict the attention to those derivations which enjoy certain weak*-continuity. This is successfully done by B.E. Johnson, R.V. Kadison and J.V. Ringrose for von Neumann algebras [11]. Due to some important contribution of A. Connes, A.Ya. Helemskii coined the term Connes-amenability for this concept [7]. Later V. Runde extended this notion to the setting of dual Banach algebras [15]. Examples of dual Banach algebras (besides von Neumann algebras) include the measure algebra $M(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G. Runde [17] showed that a locally compact group G is amenable if and only if its measure algebra $M(G)$ is Connes-amenable, see also [18]. On the other hand, the Fourier-Stieltjes algebra $B(G)$ is operator Connes-amenable if and only if G is almost abelian [21]. The first and third authors in [5, 6] investigated two possible setups in which one could guarantee that the multiplier algebra $\mathcal{M}(B)$ of a Banach algebra B is a dual Banach algebra, and found conditions for the equivalence of

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amenability of B with Connes-amenability of $\mathcal{M}(B)$. The first paper [6] was based on a paper by E. O. Oshobi and J. S. Pym [13] which gives conditions for $\mathcal{M}(B)$ to be a dual algebra (c.f. the paragraph after [13, Theorem 1.0.1].) The same task was done in [5] using a different condition based on the work of M. Daws in [2] (which was weaker than that of Oshobi-Pym).

In this paper we use the latter setup and complementary results of M. Daws in [1] to do the same for a completely contractive Banach algebra B and its completely bounded multiplier algebra $\mathcal{M}_{cb}(B)$. We find conditions under which the operator amenability of B is equivalent to the operator Connes-amenability of $\mathcal{M}_{cb}(B)$.

2. Notations and preliminary results

Let A be a Banach algebra and X be a Banach A-bimodule. A continuous linear map $D\colon A \longrightarrow X$ such that

$$D(ab) = D(a)\cdot b + a\cdot D(b) \qquad\qquad (a, b \in A)$$

is called a *derivation* from A into X. The space of all derivations of A into X is denoted by $\mathcal{Z}^1(A, X)$. For each $x \in X$, the map $a \mapsto a\cdot x - x\cdot a$ is a derivation, and these maps form the space $\mathcal{N}^1(A, X)$ of *inner* derivations. The quotient space $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$ is the first *cohomology group* of A with coefficients in X. A Banach algebra A is called *amenable* if $\mathcal{H}^1(A, X^*) = \{0\}$, for every Banach A-bimodule X, where X^* is the dual Banach A-bimodule with a canonical action [9].

A Banach algebra A is said to be a *dual Banach algebra* if it is dual as a Banach A-bimodule. It is easily checked that a Banach algebra which is also a dual space is a dual Banach algebra if and only if the multiplication map is separately w^*-continuous [15]. Examples of dual Banach algebras include all von Neumann algebras, the algebra $\mathfrak{B}(E) = (E \otimes_\gamma E^*)^*$ of all bounded operators on a reflexive Banach space E where $\otimes_\gamma$ stands for the projective tensor product. Also the measure algebra $M(G) = C_0(G)^*$, the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, and the second dual of Arens regular Banach algebras are dual Banach algebras.

Let A be a Banach algebra. We use the notations $A \otimes_\gamma A$ and $\mathfrak{B}^2(A; \mathbb{C})$ to denote the projective tensor product of A with itself and the space of bounded bilinear maps on $A \times A$, respectively. A dual Banach A-bimodule X is called *normal* if for each $x \in X$, the maps $a \mapsto a\cdot x$ and $b \mapsto x\cdot b$ from A into X are w^*-continuous, and *Connes-amenable* if for every normal dual Banach A-bimodule X, every w^*-continuous derivation $D\colon A \longrightarrow X$ is inner [15].

Let $\Delta_A\colon A \otimes_\gamma \mathcal{A} \longrightarrow A$ be the diagonal operator induced by $a \otimes b \mapsto ab$, a, $b \in A$. Since the multiplication in A is separately w^*-continuous, $\Delta_A^* A_* \subset \mathfrak{B}^2_{w^*}(A; \mathbb{C}) \subset \mathfrak{B}^2(A; \mathbb{C}) \cong (A \otimes_\gamma A)^*$, where A_* is a closed submodule of A^* such that $A = A_*^*$, and $\mathfrak{B}^2_{w^*}(A; \mathbb{C})$ is the set of all w^*-continuous bilinear maps from $A \otimes_\gamma A$ into $\mathbb{C}$. Taking the adjoint of $\Delta_A^* \mid_{A_*}$, we may extend Δ_A to a A-bimodule homomorphism Δ_{w^*} on $\mathfrak{B}^2_{w^*}(A; \mathbb{C})^*$.

A *multiplier* of an algebra A is a pair (L, R) of linear maps of A into itself such that $aL(b) = R(a)b$ for all a, $b \in A$. This is also called a centralizer in the literature; see [8] and [12]. Every element of A by the natural map, $a \mapsto (L_a, R_a)$, gives rise to a multiplier where L_a and R_a are left and right multiplications, respectively. The set of all multipliers on A is denoted by $\mathcal{M}(A)$ which is an algebra (under composition of maps) with identity $1_{\mathcal{M}(A)} = (\mathrm{id}_A, \mathrm{id}_A)$. We write $\mathcal{L}(A)$ for the collection of left multipliers of A, that is, maps from A into itself with $L(ab) = L(a)b$ for all a, $b \in A$. Similarly, we define $\mathcal{R}(A)$, the collection of right multipliers, those maps from A into itself with with $R(ab) = aR(b)$ for all a, $b \in A$.

An algebra A is called *faithful* if the only element $b \in A$ such that $abc = 0$, for all a, $c \in A$, is $b = 0$. When A is faithful, the natural map from A into $\mathcal{M}(A)$ is injective and A is an ideal of $\mathcal{M}(A)$. One can see that a normed algebra A with a bounded approximate identity is faithful. In fact, assume that (e_α) is a bounded approximate identity for A. We have $bc = \lim_\alpha e_\alpha bc = 0$ for all a, $c \in A$, and so $b = \lim_\alpha be_\alpha = 0$.

From now on, we denote the space of all bounded operators on A into itself by $\mathfrak{B}(A)$. Suppose that A is faithful. It can be shown that if $(L, R) \in \mathcal{M}(A)$, then $L, R \in \mathfrak{B}(A)$. Indeed, if $a_n \to a$ and $L(a_n) \to b$ in A, then

$$cb = \lim_n cL(a_n) = \lim_n R(c)a_n = R(c)a = cL(a) \qquad (c \in A)\,.$$

Hence, $L(a) = b$, and so we conclude that L is bounded. Similarly, R is bounded. A norm on $\mathcal{M}(A)$ is defined by considering $\mathcal{M}(A)$ as a subset of $\mathfrak{B}(A) \oplus_\infty \mathfrak{B}(A)$. Therefore

$$\|(L, R)\|_{\mathcal{M}(A)} = \max\{\|L\|, \|R\|\}\,.$$

The inclusion map $A \longrightarrow \mathcal{M}(A)$ is norm non-increasing and if moreover A has a bounded approximate identity $(e_\alpha)_\alpha$ with bound K, then

$$\|b\|_A = \lim_\alpha \|e_\alpha b\|_A \leq K\|b\|_{\mathcal{L}(A)}\,.$$

Note that here b can be regarded as an element of $\mathcal{L}(A)$ with range $\{ba \mid a \in A\}$.

Let E be a linear space. A *matricial norm* on E is a family $(\|\cdot\|_n)_{n=1}^\infty$ such that $\|\cdot\|_n$ is a norm on $\mathbb{M}_n(E)$ for $n \in \mathbb{N}$ with the following properties:

i) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \quad (x \in \mathbb{M}_n(E),\ y \in \mathbb{M}_m(E))$,

ii) $\|\alpha x \beta\|_n \leq |\alpha|_{m,n}\|x\|_n|\beta|_{n,m} \quad (x \in \mathbb{M}_n(E),\ \alpha \in \mathbb{M}_{m,n},\ \beta \in \mathbb{M}_{m,n})$,
where $x \oplus y := \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in \mathbb{M}_{n+m}(E)$

A linear space E equipped with a matricial norm $(\|\cdot\|_n)_{n=1}^\infty$ is called a *matricially normed space*; for more details of matricial norms we refer the reader to [4] (see also [3] for a similar notion on the space of matrices). If each space $(\mathbb{M}_n(E), \|\cdot\|_n)$ is a Banach space, then E is called an (abstract) *operator space*.

Let E_1, E_2 and F be operator spaces. A bilinear map $T\colon E_1 \times E_2 \longrightarrow F$ is said to be *completely contractive* if

$$\|T\|_{cb} := \sup_{n_1, n_2 \in \mathbb{N}} \|T^{(n_1,n_2)}\| \leq 1\,,$$

where

$$T^{(n_1,n_2)}\colon \mathbb{M}_{n_1}(E_1) \times \mathbb{M}_{n_2}(E_2) \longrightarrow \mathbb{M}_{n_1 n_2}(F)$$

is defined by

$$\big((x_{i,j}),(y_{k,l})\big) \mapsto \big(T(x_{i,j},y_{k,l})\big)\,.$$

Next we need the notion of a completely contractive Banach algebra [20]. The main idea is to consider both the algebra and operator space structures at the same time.

Definition 2.1. A *completely contractive Banach algebra* (CCBA) is a Banach algebra which is also an operator space such that multiplication is a completely contractive bilinear map.

For a Hilbert space H, every closed subalgebra of $\mathfrak{B}(H)$ is a completely contractive Banach algebra. For a locally compact group, the group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ are completely contractive Banach algebras.

Let E and F be operator spaces and let $T \in \mathfrak{B}(E,F)$. Then

i) T is *completely bounded* if

$$\|T\|_{cb} \;:= \sup_{n\in\mathbb{N}} \|T^{(n)}\|_{\mathfrak{B}(\mathbb{M}_n(E),\mathbb{M}_n(F))} < \infty\,.$$

ii) T is *complete contraction* if $\|T\|_{cb} \leq 1$.

iii) T is *complete isometry* if $T^{(n)}$ is an isometry for each $n \in \mathbb{N}$.

We denote the space of all completely bounded maps between operator spaces E and F by $\mathfrak{B}_{cb}(E,F)$, and denote $\mathfrak{B}_{cb}(E,E)$ by $\mathfrak{B}_{cb}(E)$.

Definition 2.2. A *dual* CCBA is a Banach algebra which is a dual operator space such that multiplication is completely contractive and separately w^*-continuous.

One should note that there are operator spaces that have predual Banach spaces, but do not have predual operator spaces. Dual operator spaces are those operator spaces for which there exist predual operator spaces. Let E be a reflexive operator space, then every w^*-closed subalgebra of $\mathfrak{B}_{cb}(E)$ is a dual completely contractive Banach algebra. Also for a locally compact group G, $M(G)$, $B(G)$, and the reduced Fourier-Stieltjes algebra $B_r(G)$, are dual completely contractive Banach algebras.

Definition 2.3. Let B be a CCBA. An operator space E which is also a left B-module is called *left operator B-module* if the bilinear map

$$B \times E \longrightarrow E\,, \qquad (b,x) \mapsto b\cdot x$$

is completely bounded.

Right operator B-modules and operator B-bimodules are defined analogously. One can see that if E is an operator B-module (left, right, or bi-) then E^* with the corresponding dual action is again an operator B-module (right, left, or bi-, respectively).

Definition 2.4 ([14])**.** Let B be a CCBA. Then B is called *operator amenable* if for every operator B-bimodule E, every completely bounded derivation $D\colon B \longrightarrow E^*$ is inner.

Clearly, if a CCBA is amenable as a Banach algebra then it is operator amenable. The example of the Fourier algebra shows that the converse is not true [16, Chapter 7].

Let X and Y be operator spaces. Then the matricial norm $(\|\cdot\|_{\pi,n})_{n=1}^{\infty}$ obtained through the embedding

$$X \otimes Y \hookrightarrow \mathfrak{B}_{cb}(X, Y^*)^*$$

is called the *operator projective* tensor norm on $X \otimes Y$. The completion of $X \otimes Y$ with respect to this matricial norm is called the operator projective tensor product of X and Y and is denoted by $X \widehat{\otimes} Y$.

For a CCBA such as B, let $\Delta_B \colon B \widehat{\otimes} B \longrightarrow B$ be the continuous map defined on elementary tensors by $a \otimes b \mapsto ab$.

Definition 2.5 ([19])**.** Let B be a CCBA. $M \in (B \widehat{\otimes} B)^{**}$ is called a *virtual operator diagonal* for B if

$$b \cdot M = M \cdot b\,, \quad b \cdot \Delta_B^{**} M = b \qquad (b \in B)\,.$$

A bounded net $(m_\alpha)_\alpha$ in $B \widehat{\otimes} B$ is called an *approximate operator diagonal* for B if

$$b \cdot m_\alpha - m_\alpha \cdot b \to 0\,, \quad b \Delta_B m_\alpha \to b \qquad (b \in B)\,.$$

One can see that the operator amenability, the existence of a virtual operator diagonal and the existence of an approximate operator diagonal are all equivalent for a CCBA [16, Theorem 7.4.3].

3. Operator Connes-amenability of $\mathcal{M}_{cb}(B)$

Let B be a completely contractive Banach algebra (CCBA). We write $\mathcal{M}_{cb}(B)$ for the subalgebra of $\mathcal{M}(B)$ consisting of those pairs (L, R) with L, $R \in \mathfrak{B}_{cb}(B)$. We get an operator space structure on $\mathcal{M}_{cb}(B)$ by embedding it in $\mathfrak{B}_{cb}(B) \oplus_\infty \mathfrak{B}_{cb}(B)$, that is,

$$\|(L, R)\|_n = \max\{\|L\|_n, \|R\|_n\}\,, \qquad (L, R \in \mathbb{M}_n(\mathcal{M}_{cb}(B)), n \geq 1)\,.$$

When A is a dual CCBA with predual A_*, we sometimes refer to the pair (A, A_*) as a dual CCBA (to specify that we have fixed a predual). There are conditions on B, forcing $\mathcal{M}_{cb}(B)$ to be a dual CCBA, whenever B is a CCBA. The next result is proved in [2, Theorem 8.6].

Theorem 3.1. *Let B be a CCBA with dense products, let (A, A_*) be a dual CCBA, and let $\imath \colon B \longrightarrow A$ be a complete isometry with $\imath(B)$ as an ideal in A. Suppose further that the induced map $\theta \colon A \longrightarrow \mathcal{M}_{cb}(B)$ is injective. Then there is a unique operator space X such that $\mathcal{M}_{cb}(B)$ is completely isometrically isomorphic to X^*, turning $\mathcal{M}_{cb}(B)$ into a dual CCBA, and such that for a bounded net (a_α) in A, $a_\alpha \to a$ weak* in A if and only if $\theta(a_\alpha) \to \theta(a)$ in $\mathcal{M}_{cb}(B)$.*

In the light of the proofs of [2, Theorems 7.1, 8.6], we review the structure of the operator space predual of $\mathcal{M}_{cb}(B)$ as follows:

We consider $(B \widehat{\otimes} A_*) \oplus_1 (B \widehat{\otimes} A_*)$ with dual space $\mathfrak{B}_{cb}(B, A) \oplus_\infty \mathfrak{B}_{cb}(B, A)$ and take

$$X = \operatorname{span}\{(b \otimes a_* \cdot \imath(a) \oplus (-a \otimes \imath(b) \cdot a_*) :\ a, b \in B, a_* \in A_*\}\,.$$

Then, the closure $\bar{X}$ of X is a closed linear subspace of $(B\widehat{\otimes}A_*)\oplus_1(B\widehat{\otimes}A_*)$ and $\bar{X}^\perp = X^\perp$ is a weak*-closed subspace of $\mathfrak{B}_{cb}(B,A)\oplus_\infty\mathfrak{B}_{cb}(B,A)$. For each element $(S,T)\in X$, there exist $L,R\in\mathfrak{B}_{cb}(B)$ such that $T=\iota\circ L$ and $S=\iota\circ R$ and $(L,R)\in\mathcal{M}_{cb}(B)$. $\mathcal{M}_{cb}(\mathcal{B})$ and $X^\perp$ are completely isometric as operator spaces, so a predual is $(B\widehat{\otimes}A_*\oplus_1 B\widehat{\otimes}A_*)/\bar{X}$, where the weak*-topology is induced by the embedding $\mathcal{M}_{cb}(B)\to(B\widehat{\otimes}A_*\oplus_1 B\widehat{\otimes}A_*)^*$ given by

$$\langle (L,R),(a\otimes a_*)\oplus(b\otimes b_*)\rangle = \langle \iota L(a),a_*\rangle + \langle \iota R(b),b_*\rangle\,,$$

for L, $R\in\mathfrak{B}_{cb}(B)$, a, $b\in B$ and $a_*,b_*\in A_*$.

Example 3.2. We note that Theorem 3.1 actually turns $\mathcal{M}_{cb}(A(G))$ into a dual CCBA. Here, we present a concrete operator space predual for $\mathcal{M}_{cb}(A(G))$, constructed by Spronk in Section 6.2 of [22]. We use the usual convolution product on $L^1(G)$, and $\otimes^h$ which is the completed Haagerup tensor product. Let $\mathcal{K}$ be the closure of the set

$$\{\sum_j f_j\otimes g_j\in L^1(G)\otimes L^1(G):\sum_j f_j * g_j=0\}$$

in $L^1(G)\otimes^h L^1(G)$. Then, $\mathcal{Q}(G)=(L^1(G)\otimes^h L^1(G))/\mathcal{K}$ is an operator space in which $\mathcal{Q}(G)^*$ is isometrically isomorphic to $\mathcal{M}_{cb}(A(G))$. The dual pairing is given via

$$\langle (f\otimes g)+\mathcal{K},\psi\rangle = \int_G (f*g)(t)\psi(t)dt \qquad (f,g\in L^1(G),\psi\in\mathcal{M}_{cb}(A(G)))\,.$$

A similar argument as in [2, Example 7.5] shows that as an operator space, $\mathcal{Q}(G)$ is completely isometrically isomorphic to the predual constructed by Theorem 3.1

By [2, Lemma 10.1] and similar to the proof of [2, Theorem 7.2], one can show that under a natural assumption the weak*-topology in Theorem 3.1 is unique as follows:

Theorem 3.3. *Let B and (A,A_*) be as above, and $\theta\colon A\longrightarrow\mathcal{M}_{cb}(B)$ be the induced map. There is one and only one weak*-topology on $\mathcal{M}_{cb}(B)$ such that*

(i) *$\mathcal{M}_{cb}(B)$ is a dual Banach algebra;*

(ii) *For a bounded net $(a_\alpha)_\alpha$ in A, we have $a_\alpha\to a$ weak* in A if and only if $\theta(a_\alpha)\to\theta(a)$ weak* in $\mathcal{M}_{cb}(B)$.*

Let B be a dual CCBA. A dual operator B-bimodule E is called *normal* if, for each $x\in E$, the maps $b\mapsto b\cdot x$ and $b\mapsto x\cdot b$ from B into E are w^*-continuous.

Definition 3.4 ([20])**.** A dual CCBA, say B, is called *operator Connes-amenable* if every w^*-continuous, completely bounded derivation from B into every normal dual operator B-bimodule is inner.

Let B be a Banach algebra satisfying conditions of Theorem 3.1. Each element $a_*\in A_*$ may be considered as an element of B^* via

$$\langle a_*,b\rangle_B = \langle \iota(b),a_*\rangle_{A_*} \quad (b\in B)\,.$$

One can easily see that A_* is a B-submodule of B^* with the following module actions

$$\langle a_* \cdot a, b\rangle_B = \langle \iota(ab), a_*\rangle_{A_*}, \quad \langle a \cdot a_*, b\rangle_B = \langle \iota(ba), a_*\rangle_{A_*} \qquad (a, b \in B,\ a_* \in A_*)\,.$$

Lemma 3.5. *Let E and F be operator spaces. Then*

$$\mathfrak{B}_{cb}(E, F; \mathbb{C}) \cong \mathfrak{B}_{cb}(E, F^*).$$

Proof. By [4, Proposition 7.1.2] and Smith's Lemma, we have

$$\begin{aligned}\mathfrak{B}_{cb}(E, F; \mathbb{C}) &\cong \mathfrak{B}_{cb}(E\widehat{\otimes}F, \mathbb{C})\\ &\cong \mathfrak{B}_{cb}(E, \mathfrak{B}_{cb}(F, \mathbb{C}))\\ &\cong \mathfrak{B}_{cb}(E, \mathfrak{B}(F, \mathbb{C}))\\ &\cong \mathfrak{B}_{cb}(E, F^*)\,.\end{aligned}$$

□

We denote the set of all separately w^*-continuous elements of $\mathfrak{B}_{cb}(B, B; \mathbb{C})$ by $\mathfrak{B}^2_{w^*-cb}(B; \mathbb{C})$.

Let $\Delta_B \colon B\widehat{\otimes}B \longrightarrow B$ be defined as above, then Δ_B^* is a map from B^* into $(B\widehat{\otimes}B)^* \cong \mathfrak{B}_{cb}(B, B; \mathbb{C})$ if B is a dual CCBA with an operator space B_* as a predual, we have $\Delta_B^*(B_*) \subseteq \mathfrak{B}^2_{w^*-cb}(B; \mathbb{C})$, since multiplication in B is separately w^*- continuous. Now consider the map $\Delta_B^*|_{B_*} : B_* \longrightarrow \mathfrak{B}^2_{w^*-cb}(B; \mathbb{C})$; we denote the adjoint of this map by Δ_{w^*}.

Definition 3.6. For a dual CCBA such as B, a *normal virtual operator diagonal* is an element $M \in \mathfrak{B}^2_{w^*-cb}(B; \mathbb{C})^*$ such that

$$a \cdot M = M \cdot a\ , \quad a\Delta_{w^*}M = a \qquad (a \in B)\,.$$

The following result is proved in [2]

Theorem 3.7. *Let A be a CCBA with a bounded approximate identity (e_α), and let $\Phi_0 \in A^{**}$ be a weak*-accumulation point of (e_α). Then*

(i) *$\mathcal{M}_{cb}(A) \subseteq \mathfrak{B}_{cb}(A) \times \mathfrak{B}_{cb}(A)$ is closed in the strict topology,*

(ii) *A is a closed ideal in $\mathcal{M}_{cb}(A)$ which is strictly dense,*

(iii) *$\sigma : \mathcal{M}_{cb}(A) \longrightarrow (A^{**}, \Diamond)$, defined by $(L, R) \mapsto L^{**}(\Phi_0)$, is an algebra homomorphism and a complete isomorphism onto its range, with $\sigma(a) = a$ for all $a \in A$.*

In analogy with [6, Lemma 3.1], we have the following lemma for completely bounded multiplier Banach algebras. The proof is similar, but we include it for the sake of completeness.

Lemma 3.8. *Let B and (A, A_*) be as in Theorem 3.1, $\sim : \mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B), \mathbb{C}) \to \mathfrak{B}^2_{cb}(B; \mathbb{C})$ a map defined by $\tilde{\psi} := \psi|_{B\times B}$, for $\psi \in \mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B), \mathbb{C}))$, then*

(i) *$\sim$ is a continuous linear map,*

(ii) *$[\Delta^*_{\mathcal{M}_{cb}(B)}((a\otimes a_*)\oplus(b\otimes b_*)+\bar{X})]^\sim = \Delta_B^*(a\cdot a_* + b_*\cdot b)$ $\quad(a, b \in B,\ a_*, b_* \in A_*\ )$,*

(iii) $(\psi\cdot\tau)^{\sim} = \tilde{\psi}\cdot\tau,\ \ (\tau\cdot\psi)^{\sim} = \tau\cdot\tilde{\psi},\ \ (\ \tau \in \mathcal{M}_{cb}(B),\ \psi \in \mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B);\mathbb{C})\ ).$

Proof. (i) It is easily verified that $\sim$ is a continuous linear map.

(ii) Let a, b, c, $d \in B$ and a_*, $b_* \in A_*$. We denote the canonical map of B in $\mathcal{M}_{cb}(B)$ by ϕ. We have

$$\begin{aligned}
&\langle [\Delta^*_{\mathcal{M}_{cb}(B)}((a\otimes a_*)\oplus(b\otimes b_*)+\bar{X})]^{\sim}, (c,d)\rangle_{B\times B} \\
&\qquad = \langle \Delta^*_{\mathcal{M}_{cb}(B)}((a\otimes a_*)\oplus(b\otimes b_*)+\bar{X}), (\phi(c),\phi(d)\rangle_{\mathcal{M}_{cb}(B)\times\mathcal{M}_{cb}(B)} \\
&\qquad = \langle (a\otimes a_*)\oplus(b\otimes b_*)+\bar{X}, \phi(cd)\rangle_{\mathcal{M}_{cb}(B)} \\
&\qquad = \langle (a\otimes a_*)\oplus(b\otimes b_*), (L_{cd},R_{cd})\rangle_{\mathcal{M}_{cb}(B)} \\
&\qquad = \langle (L_{cd},R_{cd}), (a\otimes a_*)\oplus(b\otimes b_*),\rangle_{(B\widehat{\otimes}A)\oplus_1(B\widehat{\otimes}A_*)} \\
&\qquad = \langle \iota L_{cd}(a), a_*\rangle_{A_*} + \langle \iota R_{cd}(b), b_*\rangle_{A_*} \\
&\qquad = \langle \iota(cda), a_*\rangle_{A_*} + \langle \iota(cdb), b_*\rangle_{A_*} \\
&\qquad = \langle a\cdot a_*, cd\rangle_B + \langle b\cdot b_*, cd\rangle_B \\
&\qquad = \langle a\cdot a_* + b\cdot b_*, \Delta_B(c,d)\rangle_B \\
&\qquad = \langle \Delta^*_B(a\cdot a_* + b_*\cdot b), (c,d)\rangle_{B\times B}.
\end{aligned}$$

(iii) Since B is a closed ideal in $\mathcal{M}_{cb}(B)$, the argument of this part carries over verbatim from part (iii) of [5, Lemma 3.2]. $\square$

As in [6], we say that a (τ_α) in $\mathcal{M}(B)$ converges in weak strictly topology (wst) to τ in $\mathcal{M}(B)$ if $\langle\phi, (\tau_\alpha-\tau)b\rangle_B \to 0$ for each $\phi\in B^*,\ b\in B$.

The proof of the upcoming result is similar to the proof of [6, Theorem 3.3], and so we omit it.

Theorem 3.9. *Let B and (A, A_*) be as in Theorem 3.1. Then, the followings are equivalent:*

(i) *$\theta\colon A \longrightarrow \mathcal{M}_{cb}(B)$ is w^*-wst-continuous;*

(ii) *$\imath(B)\cdot A^* \subseteq A_*$;*

(iii) *For each $b\in B$ and $b^*\in B^*$, there is $a_*\in A_*$ with*

$$\langle b^*, \imath^{-1}(a\imath(b))\rangle_B = \langle a, a_*\rangle_{A_*}\quad (a\in A)\,.$$

The next theorem is the main result of this paper which shows that for a completely contractive Banach algebra B, the operator Connes-amenability of $\mathcal{M}_{cb}(B)$ is equivalent to the operator amenability of B under some mild conditions.

Theorem 3.10. *Let B and (A, A_*) be as in Theorem 3.1 and one of the conditions of Theorem 3.9 hold. Assume B has a bounded approximate identity. Then the following assertions are equivalent:*

(i) *B is operator amenable,*

(ii) *$\mathcal{M}_{cb}(B)$ has a normal virtual operator diagonal,*

(iii) *$\mathcal{M}_{cb}(B)$ is operator Connes-amenable.*

Proof. (i) $\Rightarrow$ (ii) By assumption, B has a virtual operator diagonal $\mathfrak{M} \in \mathfrak{B}^2_{cb}(B;\mathbb{C})^*$ such that

$$a \cdot \mathfrak{M} = \mathfrak{M} \cdot a\ , \quad a\Delta_{w^*}\mathfrak{M} = a \qquad (a \in B)\,.$$

Define $\tilde{\mathfrak{M}}$ by

$$\langle \tilde{\mathfrak{M}}, \psi\rangle = \langle \mathfrak{M}, \tilde{\psi}\rangle \qquad (\psi \in\ \mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B);\mathbb{C})).$$

Then $\tilde{\mathfrak{M}}$ is linear and $\tilde{\mathfrak{M}} \in \mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B);\mathbb{C})^*$ by Lemma 3.8 (i). Let $\tau \in \mathcal{M}_{cb}(B)$. By Theorem 3.7, B is strictly dense in $\mathcal{M}_{cb}(B)$, hence there exists a net $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \to \tau$ in the strict topology. Since B has a bounded approximate identity, by Cohen factorization theorem, $\mathfrak{B}^2_{cb}(B;\mathbb{C})^*$ is a pseudo unital Banach B-bimodule (see also [2, Theorem 2.4]). Hence, there exist $a \in B$ and $\mathfrak{M}' \in \mathfrak{B}^2_{cb}(B;\mathbb{C})^*$ such that $\mathfrak{M} = a \cdot \mathfrak{M}'$. Thus, $b_\alpha a \to \tau a$ in the norm topology, and so $b_\alpha a \cdot \mathfrak{M}' \to \tau a \cdot \mathfrak{M}'$ in the weak*-topology. Similarly there exist $b \in B$ and $\mathfrak{M}'' \in \mathfrak{B}^2_{cb}(B;\mathbb{C})^*$ such that $\mathfrak{M} = \mathfrak{M}'' \cdot b$. We have $\mathfrak{M}'' \cdot bb_\alpha \to \mathfrak{M}'' \cdot b\tau$ in the weak*-topology, and thus $\tau \cdot \mathfrak{M} = \mathfrak{M} \cdot \tau$. Now, the part (iii) of Lemma 3.8 implies that $\tau \cdot \tilde{\mathfrak{M}} = \tilde{\mathfrak{M}} \cdot \tau$. Let $(a \otimes a_*) \oplus (b \otimes b_*) + \bar{X}$ be an arbitrary element of Y. By Lemma 3.8 (ii), we get

$$\begin{aligned}
\langle \Delta_{w^*}(\tilde{\mathfrak{M}}), &(a \otimes a_*) \oplus (b \otimes b_*) + \bar{X}\rangle_Y \\
&= \langle \tilde{\mathfrak{M}}, \Delta^*_{\mathcal{M}_{cb}(B)}((a \otimes a_*) \oplus (b \otimes b_*) + \bar{X})\rangle_{\mathfrak{B}^2_{w^*-cb}(\mathcal{M}_{cb}(B);\mathbb{C})} \\
&= \langle \mathfrak{M}, [\Delta^*_{\mathcal{M}_{cb}(B)}((a \otimes a_*) \oplus (b \otimes b_*) + \bar{X})]^{\sim}\rangle_{\mathfrak{B}^2_{cb}(B;\mathbb{C})} \\
&= \langle \mathfrak{M}, \Delta^*_B(a \cdot a_* + b \cdot b_*)\rangle_{\mathfrak{B}^2_{cb}(B;\mathbb{C})} \\
&= \langle \Delta^{**}_B(\mathfrak{M}), (a \cdot a_* + b \cdot b_*)\rangle_{B^*} \\
&= \langle \Delta^{**}_B(\mathfrak{M}) \cdot a, a_*\rangle_{A_*} + \langle \Delta^{**}_B(\mathfrak{M}) \cdot b, b_*\rangle_{A_*} \\
&= \langle a, a_*\rangle_{A_*} + \langle b, b_*\rangle_{A_*} \\
&= \langle \iota(a), a_*\rangle_{A_*} + \langle \iota(b), b_*\rangle_{A_*} \\
&= \langle (id_B, id_B), (a \otimes a_*) \oplus (b \otimes b_*) + \bar{X}\rangle_Y \\
&= \langle 1_{\mathcal{M}_{cb}(B)}, (a \otimes a_*) \oplus (b \otimes b_*) + \bar{X}\rangle_Y\,.
\end{aligned}$$

The above relations show that $\Delta_{w^*}(\tilde{\mathfrak{M}})$ is the identity for $\mathcal{M}_{cb}(B) = Y^*$. This completes the proof of this part.

(ii)$\Rightarrow$(iii) For a dual CCBA such as B, $\mathfrak{B}^2_{w^*-cb}(B;\mathbb{C}) \subseteq \mathfrak{B}^2_{w^*}(B;\mathbb{C})$, and this implication is proved similar to [16, Theorem 4.4.15].

(iii)$\Rightarrow$(i) Without loss of generality and in view of [16, Proposition 2.1.5], we assume that E is a neo-unital Banach operator B-bimodule and $D : B \longrightarrow E^*$ is a completely bounded derivation. By Theorem 3.7, B is a closed ideal of $\mathcal{M}_{cb}(B)$ which is strictly dense. For $x \in E$, let $b \in B$ and $y \in E$ be such that $x = b \cdot y$. For $\tau \in \mathcal{M}_{cb}(B)$, define $\tau \cdot x := \tau b \cdot y$. Let $(e_\alpha)_\alpha$ be an approximate identity for B bounded by K. If $x = b' \cdot y'$ where $b' \in B$ and $y' \in E$, then

$$\tau b' \cdot y' = \lim_\alpha \tau e_\alpha b' \cdot y' = \lim_\alpha \tau e_\alpha b \cdot y = \tau b \cdot y\,.$$

Hence, this action is well-defined. Similarly, one can define a right Banach $\mathcal{M}_{cb}(B)$--module structure on E, and so E is a Banach $\mathcal{M}_{cb}(B)$-bimodule. We show that the above action is completely bounded. For this, suppose that $[\tau_{ij}] \in \mathbb{M}_n(\mathcal{M}_{cb}(B))$ and $[x_{kl}] \in \mathbb{M}_n(E)$. We have

$$\begin{aligned}\|[\tau_{ij} \cdot x_{kl}]\| &= \| \lim_{\alpha}[\tau_{ij} \cdot (e_\alpha \cdot x_{kl})]\| = \lim_{\alpha} \|[\tau_{ij} e_\alpha \cdot x_{kl}]\| \\ &\leq \limsup_{\alpha} C\|[\tau_{ij}e_\alpha]\|\|[x_{kl}]\| \\ &\leq CK\|[\tau_{ij}]\|\|[x_{kl}]\|\,,\end{aligned}$$

where C is completely bounded norm of left action B on E. A similar inequality holds for the right action. By the proof of [16, Proposition 2.1.6], it follows that there exists a unique extension of D to a derivation

$$\tilde{D}\colon \mathcal{M}_{cb}(B) \longrightarrow E^*; \quad \tau \mapsto w^* - \lim_{\alpha} \big(D(\tau e_\alpha) - \tau \cdot D(e_\alpha)\big)$$

such that $\tilde{D}$ is continuous with respect to the strict topology on $\mathcal{M}_{cb}(B)$ and the w^*-topology on E^*. We wish to show that E^* is a normal, dual operator Banach $\mathcal{M}_{cb}(B)$-bimodule, and $\tilde{D} : \mathcal{M}_{cb}(B) \longrightarrow E^*$ is w^*-w^*-continuous, completely bounded derivation. To prove that E^* is a normal, we can simply follow the proof of [6, Theorem 3.5].

To show that $\tilde{D}$ is w^*-w^*-continuous, let $\tau_j \xrightarrow{w^*} 0$ in $\mathcal{M}_{cb}(B)$. We note that for any $b \in B, \phi \in E^*$, we have $\tau_j \cdot \phi \xrightarrow{w^*} 0$ and $\tau_j b \xrightarrow{w} 0$. For $x \in E$, take $b \in B$ and $y \in E$ such that $x = b \cdot y$. We get

$$\begin{aligned}\langle \tilde{D}(\tau_j), x\rangle &= \langle \tilde{D}(\tau_j) \cdot b, y\rangle = \langle \tilde{D}(\tau_j b) - \tau_j \cdot \tilde{D}(b), y\rangle \\ &= \langle D(\tau_j b), y\rangle - \langle \tau_j \cdot D(b), y\rangle \\ &\to 0.\end{aligned}$$

Let we see that $\tilde{D}$ is completely bounded. Consider the nth amplification of $\tilde{D}$ as $\tilde{D}^{(n)} : \mathbb{M}_n(\mathcal{M}_{cb}(B)) \longrightarrow \mathbb{M}_n(E^*)$ for each $n \in \mathbb{N}$. If $[\tau_{ij}] \in \mathbb{M}_n(\mathcal{M}_{cb}(B))$, we have

$$\begin{aligned}\|\tilde{D}\|_{cb} &= \sup\{\|\tilde{D}^{(n)}([\tau_{ij}])\| :\ \|[\tau_{ij}]\| \leq 1\}\} \\ &= \sup\{\|[w^* - \lim_{\alpha}(D(\tau_{ij}e_\alpha) - \tau_{ij} \cdot D(e_\alpha))]\| :\ \|[\tau_{ij}]\| \leq 1\}\} \\ &\leq \sup\{\lim_{\alpha}(\|[(D(\tau_{ij}e_\alpha)]\| + \|[\tau_{ij} \cdot D(e_\alpha)]\|) :\ \|[\tau_{ij}]\| \leq 1\}\} \\ &\leq \sup\{\lim_{\alpha}(\|D\|\|[\tau_{ij}]\|\|e_\alpha\| + M\|D\|\|e_\alpha\|) :\ \|[\tau_{ij}]\| \leq 1\}\} \\ &\leq (1+M)K\|D\|,\end{aligned}$$

where M is completely bounded norm of left action $\mathcal{M}_{cb}(B)$ on E^*. Due to the operator Connes-amenability of $\mathcal{M}_{cb}(B)$, the derivation $\tilde{D}$ and thus D is inner. □

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