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A NEW CHARACTERIZATION OF SUZUKI GROUPS

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Abstract. One of the important questions that remains after the classification of the finite simple groups is how to recognize a simple group via specific properties. For example, authors have been able to use graphs associated to element orders and to number of elements with specific orders to determine simple groups up to isomorphism. In this paper, we prove that Suzuki groups $Sz(q)$, where $q \pm \sqrt{2q} + 1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of orders of elements in G. If $k \in \pi_e(G)$, then we denote the set of the number of elements of order k in G by $m_k(G)$. Define $nse(G) := \{m_k(G) \mid k \in \pi_e(G)\}$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \dots, t(G)$. In the case where G is of even order, we assume that $2 \in \pi_1$.

One of the important problems in finite groups theory is the characterization of groups by special properties. We say that a group G is characterized by property M if any other groups exhibiting property M are isomorphic to G. In [8] Shao and Shi considered a characterization via the group order and nse. In particular, it was shown that if S is a simple group and $|\pi(S)| = 4$, then $|S|$ and $nse(G)$ characterize S. Following this result, in [6, 5, 7, 2], it is proved that sporadic simple groups, projective special linear groups $PSL_2(q)$ and also, Suzuki groups $Sz(q)$, where $q - 1$ is a prime number and ${}^2G_2(q)$ where $q \pm \sqrt{3q} + 1$ is a prime number can be uniquely determined by order of group and $nse(G)$. In this paper, we prove that Suzuki groups $Sz(q)$, where $q \pm \sqrt{2q} + 1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order. In fact, we prove the following theorem.

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Main Theorem. *Let G be a group with $|G| = |Sz(q)|$ and $nse(G) = nse(Sz(q))$, where $q \pm \sqrt{2q} + 1$ is a prime number. Then $G \cong Sz(q)$.*

2. Notation and preliminaries

Lemma 2.1 ([4]). *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*

(b) *$|H|$ divides $|K| - 1$;*

(c) *K is nilpotent.*

Definition 2.2. *A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.*

Lemma 2.3 ([1]). *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*

(b) *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$.*

Lemma 2.4 ([10]). *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group;*

(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 2.5 ([3]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group and φ denote the Euler phi function. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. □

Lemma 2.7. *Let S be a Suzuki groups $Sz(q)$, where $q \pm \sqrt{2q} + 1$ is a prime number. Then $m_p(S) = (p-1)|S|/(4p)$ and for every $i \in \pi_e(S) - \{1, p\}$, p divides $m_i(S)$.*

Proof. In [9], it has been shown that the cyclic subgroups of order $q \pm \sqrt{2q} + 1$ have index 4 in their normalizers. Therefore $n_p(S) = |S|/(4p)$ and hence $m_p(S) = \varphi(p)n_p(S) = (p-1)n_p(G) = (p-1)|S|/(4p)$. Furthermore, in [9], it has also been shown that all other elements have conjugacy class size divisible by p. Thus p divides $m_i(S)$ for all $i \in \pi_e(S) - \{1, p\}$. □

3. Proof of the Main Theorem

In this section, we prove the main theorem by the following lemmas. We denote the suzuki groups $Sz(q)$, where $q = 2^{2m+1}$, $m \geq 1$ by S and prime number $q \pm \sqrt{2q} + 1$ by p. Recall that G is a group with $|G| = |S|$ and $nse(G) = nse(S)$.

Lemma 3.1. *$m_2(G) = m_2(S)$, $m_p(G) = m_p(S)$, $n_p(G) = n_p(S)$, p is an isolated vertex of $\Gamma(G)$ and $p \mid m_i(G)$ for every $i \in \pi_e(G) - \{1, p\}$.*

Proof. By Lemma 2.6, for every $1 \neq r \in \pi_e(G)$, $r = 2$ if and only if $m_r(G)$ is odd. Thus we deduce that $m_2(G) = m_2(S)$. According to Lemma 2.6, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence Lemma 2.7 implies that $m_p(G) \in \{m_1(S), m_2(S), m_p(S)\}$. Moreover, $m_p(G)$ is even, so we conclude that $m_p(G) = m_p(S)$. Since G_p and S_p are cyclic groups of order p and $m_p(G) = m_p(S)$, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(S) = m_p(S)$, so $n_p(G) = n_p(S)$. Now we prove that p is an isolated vertex of $\Gamma(G)$. Assume the contrary. Then there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \varphi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(S)$, it follows that $\mathsf{m}_{tp}(G) = (t-1)(p-1)|S|k/(4p)$. If $m_{tp}(G) = m_p(R)$, then $t = 2$ and $k = 1$. Furthermore, Lemma 2.5 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(S)$ and $p \mid m_2(S)$, we have $p \mid m_{2p}(G)$, which is a contradiction. So Lemma 2.7 implies that $p \mid m_{tp}(G)$. Hence $p \mid t - 1$ and since $m_{tp}(G) < |G|$, we deduce that $p - 1 \leq 4$. But this is impossible because $p = q \pm \sqrt{2q} + 1$ and $q = 2^{2m+1}$, $m \geq 1$.
Let $i \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, $p \nmid i$ and $pi \notin \pi_e(G)$.Thus G_p acts fixed point freely on the set of elements of order i by conjugation and hence $|G_p| \mid m_i(G)$. So we conclude that $p \mid m_i(G)$. $\square$

Now since p is an isolated vertex, then $t(G) > 1$. Now we consider Lemma 2.4, then we show that the following lemma is considerable.

Lemma 3.2. *G is not a Frobenius group and 2-Frobenius group.*

Proof. **(i)** Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1(a), $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated vertex of $\Gamma(G)$, we have (i) $|H| = |G|/p$ and $|K| = p$, or (ii) $|H| = p$ and $|K| = |G|/p$. Assume that $|H| = |G|/p$ and $|K| = p$. Then Lemma 2.1(b) implies that $|G|/p$ divides $p - 1$ and hence $|G|/p \leq p - 1$ which is impossible. So the case $|H| = p$ and $|K| = |G|/p$ can be occure. Lemma 2.1(b) implies that p divides $|G|/p - 1$. Now we show that it is impossible. If $p = q \pm \sqrt{2}q + 1$, then $q \pm \sqrt{2}q + 1 \mid q^2(q \mp \sqrt{2}q + 1)(q - 1) - 1$. So we deduce that $q \pm \sqrt{2}q + 1 \mid (q \pm \sqrt{2}q + 1)(q^3 \mp 2\sqrt{2q}q^2 + 3q^2 - 4q \pm 4\sqrt{2q} - 4) + 3$. Hence $q \pm \sqrt{2}q + 1 \mid 3$, which is a contradiction.
(ii) Let G be 2-Frobenius group. Then by Lemma 2.3, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively, $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$ and $\pi(K/H) = \pi_2$, G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathsf{Aut}(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce that $\pi_2 = \{p\}$ and $|K/H| = p$. If $p = q \pm \sqrt{2q} + 1$, then since $(q + \sqrt{2q})(q - \sqrt{2q}) = q(q - 2)$, we can see easily $(q - 1, q(q - 2)) = 1$, in other

word $(p-1, q-1) = 1$. Now since $|G/K| \mid p-1$, we deduce that $q-1$ divides $|H|$. So $K/H \rtimes H_{q-1}$ is a Frobenius group with kenel H_{q-1}. Hence Lemma 2.1(b) implies that $p \mid q-2$. In conclude $q \pm \sqrt{2q}+1 \leq q-2$, which is a contradiction. Hence G is not a 2-Frobenius group. $\square$

Lemma 3.3. *The group G is isomorphic to the group S.*

Proof. Now by Lemma 3.2 and also 2.4 we have G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups K/H is a non-abelian simple group. As 3 does not divide $|K/H|$ and the Suzuki groups are the only simple groups whose order is not divisible by 3, we deduce that $K/H \cong Sz(q')$, where $q' = 2^{2m'+1}$, $m' \geq 1$. We know that $H \trianglelefteq K \trianglelefteq G$, hence $2^{2m'+1} \leq 2^{2m+1}$. So $m' \leq m$. On the other hand, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that $p \mid |K/H|$. Hence $q - \sqrt{2q}+1 \leq p \leq q' + \sqrt{2q'}+1 = \pi(K/H)$. Thus we deduce that $m \leq m'$ and $Sz(q') = S$. Now since $|K/H| = |S|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong S$. $\square$

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