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ON THE ADJOINT MAP OF HOMOTOPY ABELIAN DG-LIE ALGEBRAS

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Dedicated to the memory of Paolo de Bartolomeis

Abstract. We prove that a differential graded Lie algebra is homotopy abelian if its adjoint map into its cochain complex of derivations is trivial in cohomology. The converse is true for cofibrant algebras and false in general.

1. Introduction

Let L be a differential graded (DG) Lie algebra over a field $\mathbb{K}$ of characteristic 0. We shall say that L is homotopy abelian if it is quasi-isomorphic to a DG-Lie algebra with trivial bracket.

Notice that a necessary condition for a DG-Lie algebra to be homotopy abelian is that the cohomology Lie algebra $H^*(L)$ is abelian: by Künneth formula, this is equivalent to the fact that the adjoint morphism

$$\mathrm{ad}\colon L \to \mathrm{Hom}^*_{\mathbb{K}}(L, L)\,, \qquad \mathrm{ad}_x(y) = [x, y]\,,$$

is trivial in cohomology. However, it is well known that this condition is generally not sufficient (Example 1.3). By Jacobi identity every adjoint endomorphism is a derivation and then the adjoint map takes values in the DG-Lie subalgebra $\mathrm{Der}^*_{\mathbb{K}}(L, L) \subset \mathrm{Hom}^*_{\mathbb{K}}(L, L)$. The aim of this short note is to prove the following theorem.

Theorem 1.1. *Let L be a DG-Lie algebra over a field $\mathbb{K}$ of characteristic 0, and consider the adjoint map*

$$\mathrm{ad}\colon L \to \mathrm{Der}^*_{\mathbb{K}}(L, L), \qquad \mathrm{ad}_x(y) = [x, y]\,. \tag{1.1}$$

Then:

(1) *if the adjoint map* (1.1) *is trivial in cohomology, then L is homotopy abelian;*

(2) *if L is cofibrant and homotopy abelian, then the adjoint map* (1.1) *is trivial in cohomology;*

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(3) *L is homotopy abelian if and only if the derived adjoint map* $\mathrm{ad}\colon L \to \mathrm{RDer}^*_{\mathbb{K}}(L, L)$ *is trivial in cohomology.*

We say that L is CTA (cohomologically trivial adjoint), if the adjoint map is trivial in cohomology. The following simple examples show that the cofibrant assumption is essential in the item (2).

Example 1.2. Consider for instance the DG-Lie algebra L generated as graded vector space by the basis x, y, z, with $\overline{x} = \overline{y} = 0$, $\overline{z} = 1$ (here the overline denotes the degree of a homogeneous element), equipped with the differential $dx = z$, $dy = 0$ and the bracket

$$[y, x] = y\,, \qquad [x, z] = [y, z] = [z, z] = 0\,.$$

Cleary L is homotopy abelian since the inclusion $\mathbb{K}\,y \to L$ is a quasi-isomorphism. This DG-Lie algebra is not CTA, since the derivation ad_y is not a coboundary in $\mathrm{Der}^*_{\mathbb{K}}(L, L)$. In fact, if there exists $\phi \in \mathrm{Hom}^{-1}_{\mathbb{K}}(L, L)$ such that $d\phi + \phi d = ad_y$, we should have

$$\phi(z) = \phi(dx) = d\phi(x) + \phi(dx) = ad_y(x) = y\,;$$

this implies that ϕ is not a derivation since

$$\phi([z, x]) = 0, \qquad [\phi(z), x] - [z, \phi(x)] = [y, x] = y\,.$$

Example 1.3. For every integer $n \geq 2$, consider the DG-Lie algebra $L = \oplus_i L^i$ defined in the following way: $L^i = 0$ for every $i \neq 1, 2$, $L^1 = \mathbb{K}^n$ with basis $e_1, \ldots, e_n$ and $L^2 = \mathbb{K}^n$ with basis $h_1, \ldots, h_n$. The differential and the bracket are defined by the formulas:

(1) $de_1 = 0$ and $[e_1, e_1] = -2h_1$;

(2) $de_i = h_{i-1}$ and $[e_1, e_i] = [e_i, e_1] = -h_i$ for every $i > 1$;

(3) $[e_i, e_j] = 0$ for every $i, j > 1$.

It is immediate to see that $H^*(L)$ is an abelian graded Lie algebra. The Maurer-Cartan functor is equal to

$$\mathrm{MC}_L(A) = \left\{ \sum_{i=1}^{n} e_i \otimes x^i \mid x \in \mathfrak{m}_A,\ x^{n+1} = 0 \right\},$$

where A is a local Artin $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and $\mathfrak{m}_A$ denotes the maximal ideal of A. Thus, the functor MC_L is obstructed and represented by $\mathbb{K}[t]/(t^{n+1})$ and then L is not homotopy abelian. Notice that the inner derivation $[e_1, -]$ is closed but not exact in $\mathrm{Der}^*_{\mathbb{K}}(L, L)$.

In many concrete cases, the first part of Theorem 1.1 is applied as in the following corollary.

Corollary 1.4. *Let L be a DG-Lie algebra and assume that there exists a morphism of cochain complexes $f\colon C \to L$ such that:*

(1) *$f\colon H^*(C) \to H^*(L)$ is surjective;*

(2) *for every $c \in C$ there exists $x \in L$ such that $[f(c) + dx, y] = 0$ for every $y \in L$.*

Then, the adjoint map $\mathrm{ad}\colon L \to \mathrm{Der}^*_{\mathbb{K}}(L, L)$ *is trivial in cohomology, and* L *is homotopy abelian.*

Proof. This is clear since every cohomology class of L can be represented by an element of type $f(c) + dx$ for some $c \in C$ and $x \in L$. Notice that if $[f(c), f(d)] = 0$ for every $c, d \in C$ then f becomes a morphism of DG-Lie algebras, where C is equipped with the trivial bracket and then the homotopy abelianity of L follows by the well known basic criterion (see e.g. [4, Lemma 2.11]). □

Example 1.5. Let $(V, \partial, \overline{\partial})$ be a double cochain complex, i.e., a graded vector space equipped with two linear operators $\partial, \overline{\partial} \in \mathrm{Hom}^1_{\mathbb{K}}(V, V)$ satisfying the equalities:

$$\partial^2 = \overline{\partial}^2 = 0\,, \qquad [\partial, \overline{\partial}] = \partial\overline{\partial} + \overline{\partial}\partial = 0\,.$$

Thus $(V, d = \partial + \overline{\partial})$ is a DG-vector space and ∂ is a cocycle in $\mathrm{Hom}^*_{\mathbb{K}}(V, V)$: $d\partial + \partial d = 0$.

The derived bracket of ∂ on the shifted Hom complex $\mathrm{Hom}^*_{\mathbb{K}}(V, V)[-1]$ is defined as:

$$\mathrm{Hom}^{i-1}_{\mathbb{K}}(V, V) \times \mathrm{Hom}^{j-1}_{\mathbb{K}}(V, V) \xrightarrow{[-,-]_\partial} \mathrm{Hom}^{i+j-1}_{\mathbb{K}}(V, V),$$

$$[f, g]_\partial = f\partial g - (-1)^{ij} g\partial f\,.$$

Denoting by δ the canonical differential on $\mathrm{Hom}^*_{\mathbb{K}}(V, V)[-1]$:

$$\delta(f) = -df - (-1)^i f d, \qquad f \in \mathrm{Hom}^{i-1}_{\mathbb{K}}(V, V),$$

it is straightforward to check that $L = (\mathrm{Hom}^*_{\mathbb{K}}(V, V)[-1], \delta, [-,-]_\partial)$ is a differential graded Lie algebra.

Lemma 1.6. *In the above situation, the following conditions are equivalent:*

(1) ∂V *is an acyclic subcomplex of* $(V, \overline{\partial})$*;*

(2) $\mathrm{Im}\,\overline{\partial}\partial = \ker \overline{\partial} \cap \mathrm{Im}\,\partial$.

If the above conditions hold, then the adjoint map $\mathrm{ad}\colon L \to \mathrm{Der}^*_{\mathbb{K}}(L, L)$ *is trivial in cohomology.*

Proof. The equivalence of the two items is clear. If ∂V is an acyclic subcomplex of $(V, \overline{\partial})$, then it is also an acyclic subcomplex of (V, d) and therefore the natural maps $\ker \partial \to V$ and $V \to \mathrm{coker}\,\partial$ are quasi-isomorphisms. By Künneth formula the obvious morphism of complexes

$$C = \mathrm{Hom}^*_{\mathbb{K}}(\mathrm{coker}\,\partial, \ker \partial)[-1] \to L = \mathrm{Hom}^*_{\mathbb{K}}(V, V)[-1]$$

satisfies the conditions of Corollary 1.4. □

2. The case of classical adjoint map

The goal of this section is to prove in Corollary 2.3 the first item of Theorem 1.1.

Let $(L, d, [\ ,\ ])$ be a DG-Lie algebra. Denote by $L[1]$ the shifted graded vector space, $L[1]^i = L^{i+1}$, and by $s\colon L[1] \to L$ the tautological isomorphism of degree $+1$. Let $L[1]^{\odot n}$ be the n-th symmetric power of $L[1]$; note that $1 \in L[1]^{\odot 0} = \mathbb{K}$. Then, we can consider the associated differential graded cocommutative coalgebra $(SL[1], \Delta, Q)$, where $SL[1] = \bigoplus_{n\geq 0} L[1]^{\odot n}$, Δ is the usual coproduct and Q the coderivation associated with d and $[\ ,\ ]$. More explicitly, we have:

$$q_1\colon L[1] \to L[1], \qquad q_1(v) = -s^{-1}ds(v)\,,$$

$$q_2\colon L[1]^{\odot 2} \to L[1], \qquad q_2(v_1 \odot v_2) = -(-1)^{\overline{v_1}} s^{-1}[sv_1, sv_2]\,,$$

where $\overline{v_1}$ denotes the degree of v_1 in $L[1]$; $q_i = 0$, for all $i \geq 3$, and

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)},$$

where $S(p, q)$ denotes the set of unshuffles of type (p, q) and $\epsilon(\sigma)$ is the Koszul sign. It turns out that $Q^2 = 0$.

Finally, denote by $(\operatorname{Coder}^*_{\mathbb{K}}(SL[1]), [Q,\ ], [\ ,\ ])$ the DG-Lie algebra of coderivations of $SL[1]$ and by $p\colon SL[1] \to L[1]^{\odot 1} = L[1]$ the natural projection. The following facts are well known (see e.g. [1, Section 2]):

(1) the morphism induced by the composition with p:

$$\operatorname{Coder}^*_{\mathbb{K}}(SL[1]) \to \operatorname{Hom}^*_{\mathbb{K}}(SL[1], L[1]), \qquad \alpha \mapsto p\alpha\,,$$

is an isomorphism of graded vector spaces. For every $\alpha \in \operatorname{Coder}^*_{\mathbb{K}}(SL[1])$ we shall denote by

$$\alpha_n\colon L[1]^{\odot n} \to L[1], \qquad n \geq 0\,,$$

the components of $p\alpha$.

(2) The linear map

$$\operatorname{Coder}^*_{\mathbb{K}}(SL[1]) \xrightarrow{b} L[1], \qquad b(\alpha) = \alpha_0(1)\,, \tag{2.1}$$

is a surjective morphism of DG-vector spaces.

Theorem 2.1 (Bandiera). *In the above setup, if $b\colon \operatorname{Coder}^*_{\mathbb{K}}(SL[1]) \to L[1]$ is surjective in cohomology, then L is homotopy abelian.*

Proof. See [2, Theorem 2.4] for the original proof based on the theory of derived brackets, or [4, Corollary 4.17] for a different proof based on the Abstract Bogomolov-Tian-Todorov Theorem. □

We can restate Theorem 2.1 in terms of the Chevalley-Eilenberg complex

$$CE(L, L) = \operatorname{Coder}^*_{\mathbb{K}}(SL[1])[-1]$$

of L [6, Section 2]. By standard décalage isomorphisms, $CE(L, L)$ can be described as the product total complex of the complex of DG-vector spaces

$$0 \to L \xrightarrow{\delta_0} \operatorname{Hom}^*_{\mathbb{K}}(L, L) \xrightarrow{\delta_1} \operatorname{Hom}^*_{\mathbb{K}}(L^{\wedge 2}, L) \xrightarrow{\delta_2} \cdots$$

and δ_0, δ_1 are defined as:

(1) for $x, y \in L$ we have $(\delta_0 x)(y) = (-1)^{\overline{x}}[x, y]$;
(2) for $\phi \in \operatorname{Hom}^*_{\mathbb{K}}(L, L)$ and $x, y \in L$ we have

$$\begin{aligned}(\delta_1\phi)(x \wedge y) &= (-1)^{\overline{\phi}+1}\left([\phi(x), y] - (-1)^{\overline{x}\,\overline{y}}[\phi(y), x] - \phi([x, y])\right)\\ &= (-1)^{\overline{\phi}}(\phi([x, y]) - [\phi(x), y] - (-1)^{\overline{x}\,\overline{\phi}}[x, \phi(y)]) .\end{aligned}$$

Corollary 2.2. *In the notation above, assume that for every cocycle $x \in L$ there exists a linear map $\phi \in \operatorname{Hom}^{\overline{x}-1}_{\mathbb{K}}(L, L)$ such that $\delta_0 x = [d, \phi]$ and $\delta_1\phi = 0$, then L is homotopy abelian.*

Proof. The condition on x implies that the element $x - \phi$ is a cocycle in $CE(L, L)$ and $p(x - \phi) = x$, where $p\colon CE(L, L) \to L$ is the natural projection. It is now sufficient to observe that p is the same, up to degree shifting, as the map b and apply Theorem 2.1. □

We are now ready to show the first item of Theorem 1.1.

Corollary 2.3. *Let L be a differential graded Lie algebra such that the adjoint map*

$$\operatorname{ad}\colon L \to \operatorname{Der}^*_{\mathbb{K}}(L, L),$$

is trivial in cohomology. Then L is homotopy abelian.

Proof. Let $x \in L^n$ such that $dx = 0$, then there exists a derivation $\psi \in \operatorname{Der}^{n-1}_{\mathbb{K}}(L, L)$ such that $[d, \psi] = \operatorname{ad}_x$. Observe now that $\delta_0 x = (-1)^n \operatorname{ad}_x$, $\delta_1\psi = 0$, since ψ is a derivation, and then $\phi = (-1)^n\psi$ satisfies the hypothesis of Corollary 2.2. □

3. The case of derived adjoint map

In this section, we conclude the proof of Theorem 1.1. We recall that a DG-Lie algebra is CTA if the adjoint map

$$\operatorname{ad}\colon L \to \operatorname{Der}^*_{\mathbb{K}}(L, L), \qquad \operatorname{ad}_x(y) = [x, y] .$$

is trivial in cohomology.

Recall (see e.g. [5, Prop. 2.1.10]) that in the category of DG-Lie algebras over a field $\mathbb{K}$ of characteristic 0, the projective model structure is defined by setting:

(1) as weak equivalences the quasi-isomorphisms;
(2) as fibrations the surjective morphisms;
(3) as cofibrations the morphisms of DG-Lie algebras that have the left lifting property with respect all trivial fibrations.

Equivalently, a cofibration is a retract of a semifree extension: a semifree extension is defined as the countable composition of extensions of type $L \hookrightarrow M = L \otimes \mathbb{L}(H)$, where $\mathbb{L}(H)$ is the free graded Lie algebra generated by a graded vector space H, $\otimes$ is the tensor product of graded Lie algebras and the differential d on M satisfies the condition $d(H) \subset L$. By a cofibrant resolution of a DG-Lie algebra L we mean a trivial fibration $R \to L$ such that R is cofibrant in the projective model structure.

For every morphism $f\colon L \to M$ of DG-Lie algebras we shall denote by $\operatorname{Der}^*(L, M, f)$ the DG-vector space of derivations $L \to M$, where the structure of L-module on M is induced by f:

$$\operatorname{Der}^*_{\mathbb{K}}(L, M, f) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Der}^k_{\mathbb{K}}(L, M, f),$$
$$\operatorname{Der}^k_{\mathbb{K}}(L, M, f) = \{\alpha \in \operatorname{Hom}^k_{\mathbb{K}}(L, M) \mid \alpha([x, y]) = [\alpha(x), f(y)] + (-1)^{k\overline{x}}[f(x), \alpha(y)]\}\,.$$

Lemma 3.1. *Let $R \xrightarrow{p} L \xrightarrow{q} M$ be morphisms of DG-Lie algebras.*

(1) *If R is cofibrant and q is a trivial fibration, then*

$$q_*\colon\ \operatorname{Der}^*_{\mathbb{K}}(R, L, p) \to \operatorname{Der}^*_{\mathbb{K}}(R, M, qp), \qquad \alpha \mapsto q\alpha\,,$$

is a surjective quasi-isomorphism of DG-vector spaces.

(2) *If p is a trivial cofibration, then*

$$p^*\colon\ \operatorname{Der}^*_{\mathbb{K}}(L, M, q) \to \operatorname{Der}^*_{\mathbb{K}}(R, M, qp)\,, \qquad \alpha \mapsto \alpha p\,,$$

is an injective quasi-isomorphism of DG-vector spaces.

Proof. This is well known to experts, see e.g. [3, 8.2, 8.3], and in any case easy to prove by observing that, for every morphism $f\colon L \to M$ of DG-Lie algebras and every integer n, there exists a natural bijection between $\operatorname{Der}^n_{\mathbb{K}}(L, M, f)$ and the set of commutative diagrams of DG-Lie algebras

$$\begin{tikzcd} & M \oplus \operatorname{cone}(\operatorname{Id}_{M[n]}) \arrow[d] \\ L \arrow[ur] \arrow[r, "f"] & M \end{tikzcd}$$

where the vertical arrow is the natural projection and $M \oplus \operatorname{cone}(\operatorname{Id}_{M[n]})$ is the trivial extension of the DG-Lie algebra M by the acyclic M-module $\operatorname{cone}(\operatorname{Id}_{M[n]}) = M[n] \oplus M[n+1]$. The differential $\operatorname{Der}^n_{\mathbb{K}}(L, M, f) \to \operatorname{Der}^{n+1}_{\mathbb{K}}(L, M, f)$ is given by composition with the morphism of DG-Lie algebras

$$M \oplus \operatorname{cone}(\operatorname{Id}_{M[n]}) \to M \oplus \operatorname{cone}(\operatorname{Id}_{M[n+1]})$$

induced by the natural morphisms of M-modules

$$\operatorname{cone}(\operatorname{Id}_{M[n]}) \to M[n+1] \to \operatorname{cone}(\operatorname{Id}_{M[n+1]})\,.$$

The surjectivity of $q_*\colon \operatorname{Der}^n_{\mathbb{K}}(R,L,p) \to \operatorname{Der}^n_{\mathbb{K}}(R,M,qp)$ follows by the lifting property applied to the diagram

$$\begin{tikzcd} & L\oplus \operatorname{cone}(\mathrm{Id}_{L[n]}) \arrow[d] \\ R \arrow[r] \arrow[ur, dashed] & L\times_M (M\oplus \operatorname{cone}(\mathrm{Id}_{M[n]})) \end{tikzcd}$$

and the other properties are proved in a similar way. □

Proposition 3.2. *Let $R\to L$, $S\to L$ be two cofibrant resolutions of a DG-Lie algebra L:*

(1) *if L is CTA, then also R, S are CTA;*

(2) *if R is CTA, then also S is CTA.*

Proof. 1) For every cofibrant resolution $p\colon R\to L$ we have a commutative diagram of differential graded vector spaces:

$$\begin{tikzcd} R \arrow[r, "\mathrm{ad}"] \arrow[d, "p"] & \operatorname{Der}^*_{\mathbb{K}}(R,R) \arrow[r, "p_*"] & \operatorname{Der}^*_{\mathbb{K}}(R,L,p) \\ L \arrow[r, "\mathrm{ad}"] & \operatorname{Der}^*_{\mathbb{K}}(L,L) \arrow[ur, "p^*"] & \end{tikzcd}$$

and p_* is injective in cohomology by Lemma 3.1. Therefore, if $\mathrm{ad}\colon L\to \operatorname{Der}^*_{\mathbb{K}}(L,L)$ is trivial in cohomology, then also $\mathrm{ad}\colon R\to \operatorname{Der}^*_{\mathbb{K}}(R,R)$ is trivial in cohomology.

2) Given two cofibrant resolutions $R\to L$, $S\to L$, consider a factorization of the colimit map $R\amalg S \xrightarrow{h} T \xrightarrow{g} L$, with h a cofibration and g a trivial fibration. By the 2 out of 3 property, both the induced maps $r\colon R\to T$ and $s\colon S\to T$ are trivial cofibrations of cofibrant DG-Lie algebras. Assume that R is CTA, then the lifting diagram

$$\begin{tikzcd} R \arrow[r, "id"] \arrow[d, "r"] & R \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

gives a surjective quasi-isomorphism $T\to R$ and then also T is CTA by previous item. Similarly, there exists a surjective quasi-isomorphism $p\colon T\to S$ such that $ps=Id_S$, and we have a commutative diagram of cochain complexes

$$\begin{tikzcd} T \arrow[r, "\mathrm{ad}"] \arrow[d, "p"] & \operatorname{Der}^*_{\mathbb{K}}(T,T) \arrow[r, "s^*"] & \operatorname{Der}^*_{\mathbb{K}}(S,T,s) \arrow[d, "p_*"] \\ S \arrow[rr, "\mathrm{ad}"] & & \operatorname{Der}^*_{\mathbb{K}}(S,S)\,. \end{tikzcd}$$

This implies that S is also CTA. □

Notice that the composition $p_* s^*$ is not a morphism of DG-Lie algebras; however $\operatorname{Der}^*_{\mathbb{K}}(T,T)$ and $\operatorname{Der}^*_{\mathbb{K}}(S,S)$ are also quasi-isomorphic as DG-Lie algebras, but this requires a different proof [3, Section 8].

Theorem 3.3. *For a DG-Lie algebra L the following conditions are equivalent:*

(1) *L is homotopy abelian;*

(2) *there exists a cofibrant resolution of L which is CTA;*

(3) *every cofibrant resolution of L is CTA.*

Proof. By Proposition 3.2 it is sufficient to prove that every cofibrant homotopy abelian DG-Lie algebra R is CTA. By assumption, there exists an abelian DG-Lie algebra H and a span of trivial fibrations $R \xleftarrow{r} P \xrightarrow{p} H$. Replacing possibly P with a cofibrant resolution it is not restrictive to assume that P is a cofibrant DG-Lie algebra and therefore both r and p are cofibrant resolutions. Since H is CTA, Proposition 3.2 implies that P is CTA and so R is also CTA. □

Note that Theorem 3.3 directly implies the second item of Theorem 1.1. The last item of Theorem 1.1 follows immediately from the definition of the derived adjoint map as the morphism in the homotopy category represented by the span

$$\begin{tikzcd} R \arrow[r, "\mathrm{ad}"] \arrow[d, "p"] & \operatorname{Der}^*_{\mathbb{K}}(R,R) \arrow[r, "p_*"] & \operatorname{Der}^*_{\mathbb{K}}(R,L,p) \\ L & & \end{tikzcd} \qquad p \text{ cofibrant resolution}\,.$$

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References

[1] Bandiera, R., *Nonabelian higher derived brackets*, J. Pure Appl. Algebra **219** (2015), 3292–3313.

[2] Bandiera, R., *Homotopy abelian L_∞ algebras and splitting property*, Rend. Mat. Appl. **37** (2016), 105–122, `http://www1.mat.uniroma1.it/ricerca/rendiconti/37_1-2_(2016)_105-122.html`.

[3] Hinich, V., *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), 3291–3323.

[4] Iacono, D., *On the abstract Bogomolov-Tian-Todorov theorem*, Rend. Mat. Appl. **38** (2017), 175–198, `http://www1.mat.uniroma1.it/ricerca/rendiconti/38_2_(2017)_175-198.html`.

[5] Lurie, J., *Derived algebraic geometry X: Formal moduli problems*, 2011, Preprint available at `http://www.math.harvard.edu/~lurie/papers/DAG-X.pdf`.

[6] Manetti, M., *On some formality criteria for DG-Lie algebras*, J. Algebra **438** (2015), 90–118.

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