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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN VECTOR-VALUED FUNCTION SPACE

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Abstract. The aim of this work is to study the existence and uniqueness of solutions of the fractional integro-differential equations $\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}(\int_{-\infty}^{s} a(s-\xi)x(\xi)d\xi)ds + f(t)$, $(\alpha > 0)$ with the periodic condition $x(0) = x(2\pi)$, where $a \in L^1(\mathbb{R}_+)$. Our approach is based on the R-boundedness of linear operators L^p-multipliers and UMD-spaces.

1. Introduction

The aim of this paper is to study the existence and uniqueness of solutions for some retarded fractional integro-differential equations with delay by using methods of maximal regularity in spaces of vector valued functions. Motivated by the fact that neutral functional integro-differential equations with finite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers ([3], [8], [14], [24]) and the references listed therein for information on this subject. One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the subjet, we refer to Weis [17], Poblete [26], Lizama [24], Keyantue [19], Hernan et al [21] et Arendt-Bu [4].

We characterize the existence of periodic solutions for the following integro-differential equations in vector-valued spaces. Our results involve UMD spaces, the concept of R-boundedness and a condition on the resolvent operator. We remark that many

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of the most powerful modern theorems are valid in UMD spaces, i.e., Banach spaces in which martingale are unconditional differences. The probabilistic definition of UMD spaces turns out to be equivalent to the L^p-boundedness of the Hilbert transform, a transformation which is, in a sense, the typical representative example of a multiplier operator. On the other hand the notion of R-boundedness has played an important role in the functional analytic approach to partial differential equations.

In this work, we study the existence of periodic solutions for the following integro-differential equations

(1.1)
$$\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}(\int_{-\infty}^{s} a(s-\xi)x(\xi)d\xi)ds + f(t)\,, \quad 0 \le t \le 2\pi\,,$$

where $A\colon D(A) \subseteq X \to X$ is a linear closed operator on Banach space $(X, \|\cdot\|)$, $\Gamma(\cdot)$ is the Euler gamma function ($\alpha > 0$). and $f \in L^p(\mathbb{T}, X)$ for all $p \ge 1$. For $r_{2\pi} := 2\pi N$ (some $N \in \mathbb{N}$) L and G are in $B(L^p([-r_{2\pi}, 0],\ X);\ \ X)$ is the space of all bounded linear operators and x_t is an element of $L^p([-r_{2\pi}, 0],\ X)$ which is defined as follows

$$x_t(\theta) = x(t+\theta) \quad \text{for} \quad \theta \in [-r_{2\pi},\ 0]\,.$$

Initially, Arendt and Bu [3] dealt with the problem $u'(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$. Maximal regularity for the evolution problem in L^p was treated earlier by Weis [28, 29] (see also [11] for a different proof of the operator-valued Mikhlin multiplier theorem using a transference principle). The study in the L^p framework (when $1 < p < \infty$) was made possible thanks to the introduction of the concept of randomized boundedness (hereafter R-boundedness, also known as Riesz-boundedness or Rademacher-boundedness). With this, necessary conditions for operator-valued Fourier multipliers were found in this context. In addition, the space X must have the UMD property. This was done initially by L. Weis [28, 29] for the evolutionary problem and then by Arendt-Bu [3] for periodic boundary conditions. For non-degenerate integro-differential equations both in the periodic and non periodic cases, operator-valued Fourier multipliers have been used by various authors to obtain well-posedness in various scales of function spaces: see [7, 8, 9, 19, 20, 21, 25, 26] and the corresponding references. The well-posedness or maximal regularity results are important in that they allow for the treatment of nonlinear problems. Earlier results on the application of operator-valued Fourier multiplier theorems to evolutionary integral equations can be found in [11]. More recent examples of second order integro-differential equations with frictional damping and memory terms have been studied in the paper [10].

In [1], Aparicio et al, studied the existence of periodic solution of degenerate integro-differential equations in function spaces described in the following form:

$$(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds = \gamma u(t) + Au(t)$$

$$+ \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t)\,,$$

and periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. Here, A, B, Λ and M are closed linear operators in a Banach space X satisfying the assumption $D(A)\cap D(B) \subset D(\Lambda)\cap D(M)$, b, $c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0, 2\pi]$, and γ is a constant.

In [22], S. Koumla, Kh. Ezzinbi, R. Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay

$$\frac{d}{dt}x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) + h(t, x_t)$$

where $A\colon D(A)X \to X$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, for $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(B) \supset D(A)$.

This work is organized as follows: In Section 2 we collect some preliminary results and definitions. In Section 3, we study the existence and uniqueness of strong L^p-solution of the Eq. (1.1) solely in terms of a property of R-boundedness for the sequence of operators $ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}$. We obtain that the following assertion are equivalent in UMD space:

(1) $(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))$ is invertible and $\{ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}, k \in \mathbb{Z}\}$ is R-bounded.

(2) For every $f \in L^p(\mathbb{T}; X)$ there exist a unique function $u \in H^{1,p}(\mathbb{T}; X)$ such that $u \in D(A)$ and equation (1.1) holds for a.e. $t \in [0, 2\pi]$.

2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let X be a complex Banach space. We denote as usual by $L^1(0, 2\pi, X)$ the space of Bochner integrable functions with values in X. For a function $f \in L^1(0, 2\pi; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the k-th Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt\,,$$

where $e_k(t) = e^{ikt}$, $t \in \mathbb{R}$.

Let $u \in L^1(0, 2\pi; X)$. We denote again by u its periodic extension to $\mathbb{R}$. Let $a \in L^1(\mathbb{R}_+)$. We consider the the function

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds, \quad t \in \mathbb{R}\,.$$

Since

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_0^{\infty} a(s)u(t-s)ds\,, \tag{2.1}$$

we have $\|F\|_{L^1} \leq \|a\|_1 \|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}_+)} \|u\|_{L^1(0,\,2\pi;X)}$ and F is periodic of period $T = 2\pi$ as u. Now using Fubini's theorem and (2.1) we obtain, for $k \in \mathbb{Z}$, that

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k)\,, \quad k \in \mathbb{Z} \tag{2.2}$$

where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt$ denotes the Laplace transform of a. This identity plays a crucial role in the paper.

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{L}(X)$.

Proposition 2.1 ([3, Fejer's Theorem]). *Let $f \in L^p(0, 2\pi; X))$, then one has*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with convergence in $L^p(0, 2\pi; Y))$.

R-boundedness-UMD space, L^p-multiplier and Riemann-Liouville fractional integral. For results on operator-valued Fourier multipliers and R-boundedness (used in the next section), as well as some applications to evolutionary partial differential equations, we refer to Bourgain [5, 6], Clément-de Pagter-Sukochev--Witvliet [12], Weis [28, 29], Girardi-Weis [17, 18], Kunstmann-Weis [23], Clément-Prüss [13], Arendt [2], Arendt-Bu [3], Ataricio-Keyantuo [1] and Suresh [27].

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$.

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = \mathrm{sgn}(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of R-bounded for a given family of bounded linear operators is defined as follows.

Definition 2.2. A family $\mathbf{T} \subset \mathcal{L}(X, Y)$ is called R-bounded if there exists $c_q \geq 0$ such that

$$\Big\| \sum_{j=1}^{n} r_j \otimes T_j x_j \Big\|_{L^q(0,1;X)} \leq c_q \Big\| \sum_{j=1}^{n} r_j \otimes x_j \Big\|_{L^q(0,1;X)} \tag{2.3}$$

for all $T_1, \ldots, T_n \in \mathbf{T}, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (2.3) holds.

Remark 2.3. Several useful properties of R-bounded families can be found in the monograph of Denk-Hieber-Prüss [16, Section 3], see also [2, 3, 12, 15, 23]. We collect some of them here for later use.

(a) Any finite subset of $\mathcal{L}(X)$ is is R-bounded.

(b) If $\mathbf{S} \subset \mathbf{T} \subset \mathcal{L}(X)$ and $\mathbf{T}$ is R-bounded, then $\mathbf{S}$ is R-bounded and $R_p(\mathbf{S}) \leq R_p(\mathbf{T})$.

(c) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S} \cdot \mathbf{T} := \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded and

$$R_p(\mathbf{S} \cdot \mathbf{T}) \leq R_p(\mathbf{S}) \cdot R_p(\mathbf{T})\,.$$

(d) Let $\mathbf{S},\mathbf{T}\subset\mathcal{L}(X)$ be R-bounded sets. Then $\mathbf{S}+\mathbf{T}:=\{S+T : S\in\mathbf{S}, T\in\mathbf{T}\}$ is R- bounded and

$$R_p(\mathbf{S}+\mathbf{T})\le R_p(\mathbf{S})+R_p(\mathbf{T}).$$

(e) If $\mathbf{T}\subset\mathcal{L}(X)$ is R- bounded, then $\mathbf{T}\cup\{0\}$ is R-bounded and $R_p(\mathbf{T}\cup\{0\})=R_p(\mathbf{T})$.

(f) If $\mathbf{S},\mathbf{T}\subset\mathcal{L}(X)$ are R- bounded, then $\mathbf{T}\cup\mathbf{S}$ is R-bounded and

$$R_p(\mathbf{T}\cup\mathbf{S})\le R_p(\mathbf{S})+R_p(\mathbf{T}).$$

(g) Also, each subset $M\subset\mathcal{L}(X)$ of the form $M=\{\lambda I:\lambda\in\Omega\}$ is R-bounded whenever $\Omega\subset\mathbb{C}$ is bounded (I denotes the identity operator on X).

The proofs of (a), (e), (f), and (g) rely on Kahane's contraction principle.

We sketch a proof of (f). Since we assume that $\mathbf{S},\mathbf{T}\subset\mathcal{L}(X)$ are R-bounded, it follows from (e) (which is a consequence of Kahane's contraction principle) that $\mathbf{S}\cup\{0\}$ and $\mathbf{T}\cup\{0\}$ are R-bounded. We now observe that $\mathbf{S}\cup\mathbf{T}\subset\mathbf{S}\cup\{0\}+\mathbf{T}\cup\{0\}$. Then using (d) and (b) we conclude that $\mathbf{S}\cup\mathbf{T}$ is R-bounded.

We make the following general observation which will be valid throughout the paper, notably in Section 4. Whenever we wish to establish R-boundedness of a family of operators $(M_k)_{k\in\mathbb{Z}}$, if at some point we make an exception such as $(k\neq 0)$, $(k\notin\{-1,0\})$ and so on, then later we recover the property for the entire family using items (a), (c) and (f) of the foregoing remark. The corresponding observation for boundedness is clear.

Definition 2.4. Let $\varepsilon\in\,]0,1[$ and $1<p<\infty$. Define the operator H_ε by: for all $f\in L^p(\mathbb{R};X)$

$$(H_\varepsilon f)(t):=\frac{1}{\pi}\int_{\varepsilon<|s|<\frac{1}{\epsilon}}\frac{f(t-s)}{s}ds$$

if $\lim\limits_{\varepsilon\to 0}H_\varepsilon f:=Hf$ exists in $L^p(\mathbb{R};X)$. Then Hf is called the Hilbert transform of f on $L^p(\mathbb{R},X)$.

Definition 2.5. A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R};\ X)$ for all $1<p<\infty$.

Definition 2.6. For $1\le p<\infty$, a sequence $\{M_k\}_{k\in\mathbb{Z}}\subset\mathbf{B}(X,Y)$ is said to be an L^p-multiplier if for each $f\in L^p(\mathbb{T},X)$, there exists $u\in L^p(\mathbb{T},Y)$ such that $\hat{u}(k)=M_k\hat{f}(k)$ for all $k\in\mathbb{Z}$.

Proposition 2.7. *Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1\le p<\infty$. Then the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.*

Theorem 2.8 (Marcinkiewicz operator-valud multiplier Theorem)**.** *Let X, Y be UMD spaces and $\{M_k\}_{k\in\mathbb{Z}}\subset B(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1<p<\infty$.*

Definition 2.9. The Riemann-Liouville fractional integral operator of order $\alpha>0$ is defined by

$$I^\alpha_{-\infty}f(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}f(s)ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} s^{\alpha-1} f(t-s)ds\,,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Those familiar with the Fourier transform know that the Fourier transform of a derivative can be expressed by the following:

$$\widehat{\frac{dx}{dt}}(k) = ik\hat{x}(k)\,, \quad \forall k \in \mathbb{Z}$$

and more generally,

$$\widehat{\frac{d^n x}{dt^n}}(k) = (ik)^n \hat{x}(k)\,, \quad \forall k \in \mathbb{Z}$$

A similar identity holds for anti-derivatives

$$\widehat{I^s_{-\infty} f}(k) = (ik)^{-s} \hat{x}(k)\,, \quad \forall k \in \mathbb{Z}$$

Remark 2.10. If we set $u(x) = e^{ikx}$ for $k \in \mathbb{Z}$ we have

1) $I^{\alpha}_{-\infty} u(t) = (ik)^{-\alpha} e^{ikx}$

2) $I^{\alpha}_{-\infty}(a * u)(t) = (ik)^{-\alpha} e^{ikx} \tilde{a}(ik)$.

3. Periodic solutions in UMD space

For $a \in L^1(\mathbb{R}_+)$, we denote by $a * x$ the function

$$(a * x)(t) := \int_{-\infty}^{t} a(t-s)x(s)ds$$

and $D\varphi = \varphi(0) - L(\varphi)$, with this notation we may rewrite Eq. (1.1) in the following way:

$$\frac{d}{dt}(Dx_t) = A(Dx_t) + I^{\alpha}_{-\infty}(a * x)(t) + G(x_t) + f(t) \quad \text{for} \quad t \in \mathbb{R} \tag{3.1}$$

we have $\widehat{a * x}(k) = \tilde{a}(ik)\hat{x}(k)$ and $\widehat{I^{\alpha}_{-\infty}(a * x)}(k) = (ik)^{-\alpha}\tilde{a}(ik)\hat{x}(k)$.

Denote by $L_k(x) := L(e_k x)$; $G_k(x) := G(e_k x)$ and $e_k(\theta) := e^{ik\theta}, D_k = I - L_k$ for all $k \in \mathbb{Z}$. We define

$\Delta_k = \big(ikD_k - AD_k - G_k - (ik)^{-\alpha}\widehat{a}(ik)\big)$and $\sigma_{\mathbb{Z}}(\Delta) = \{k \in \mathbb{Z} : \Delta_k \text{ is not bijective}\}$

the periodic vector-valued space is defined by

$$H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}$$

Lemma 3.1. *Let $f \in L^1(\mathbb{T}; X)$. If $g(t) = \int_0^t f(s)ds$ and $k \in \mathbb{Z}$, $k \neq 0$. Then*

$$\hat{g}(k) = \frac{i}{k}\hat{f}(0) - \frac{i}{k}\hat{f}(k)\,.$$

Definition 3.2. For $1 \le p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathbf{B}(X, Y)$ is an $(L^p, H^{1,p})$-multiplier, if for each $f \in L^p(\mathbb{T}, X)$ there exists $u \in H^{1,p}(\mathbb{T}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Lemma 3.3. *Let $1 \le p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:*

(i) $(M_k)_{k\in\mathbb{Z}}$ *is an $(L^p, H^{1,p})$-multiplier.*

(ii) $(ikM_k)_{k\in\mathbb{Z}}$ *is an (L^p, L^p)-multiplier.*

We begin by establishing our concept of strong solution for Eq. (3.1).

Definition 3.4. Let $f \in L^p(\mathbb{T}; X)$. A function $x \in H^{1,p}(\mathbb{T}; X)$ is said to be a 2π-periodic strong L^p-solution of Eq. (3.1) if $Dx_t \in D(A)$ for all $t \ge 0$ and Eq. (3.1) holds almost every where.

Lemma 3.5 ([24])**.** *Let $G\colon L^p(\mathbb{T}, X) \to X$ be a bounded linear operateur. Then*

$$\widehat{G(u.)}(k) = G(e_k\hat{u}(k)) := G_k\hat{u}(k) \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Proposition 3.6. *Let A be a closed linear operator defined on an UMD space X. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$. Then the following assertions are equivalent:*

(i) $\big(ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}\big)_{k\in\mathbb{Z}}$ *is an L^p-multiplier for* $1 < p < \infty$

(ii) $\big(ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}\big)_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) As a consequence of Proposition 2.7.
(ii) $\Rightarrow$ (i) Define $M_k = ik(C_k - AD_k)^{-1}$, where $C_k := ikD_k - b_k - G_k$ such that $b_k = (ik)^{-\alpha}\tilde{a}(ik)$. By Theorem (2.8) it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Since

$$\begin{aligned}
k\,[M_{k+1} - M_k] &= k\left[i(k+1)(C_{k+1} - AD_{k+1})^{-1} - ik(C_k - AD_k)^{-1}\right]\\
&= k(C_{k+1} - AD_{k+1})^{-1}\left[i(k+1)(C_k - AD_k) - ik(C_{k+1} - AD_{k+1})\right](C_k - AD_k)^{-1}\\
&= k(C_{k+1} - AD_{k+1})^{-1}\left[ik(C_k - C_{k+1}) + i(C_k - AD_k) + ik(AD_{k+1} - AD_k)\right]\\
&\quad \times (C_k - AD_k)^{-1}\\
&= k(C_{k+1} - AD_{k+1})^{-1}(C_k - C_{k+1})ik(C_k - AD_k)^{-1} + ik(C_{k+1} - AD_{k+1})^{-1}\\
&\quad + k(C_{k+1} - AD_{k+1})^{-1}(AD_{k+1} - AD_k)ik(C_k - AD_k)^{-1}
\end{aligned}$$

we have

$$\begin{aligned}
C_k - C_{k+1} &= ik(D_k - D_{k+1}) - iD_{k+1} + (G_{k+1} - G_k) - (b_{k+1} - b_k)\\
&= ik(L_{k+1} - L_k) + (G_{k+1} - G_k) + iL_k - iI - (b_{k+1} - b_k)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&k(C_{k+1} - AD_{k+1})^{-1}(AD_{k+1} - AD_k)ik(C_k - AD_k)^{-1}\\
&= k(C_{k+1} - AD_{k+1})^{-1}AD_{k+1}ik(C_k - AD_k)^{-1}\\
&\quad - k(C_{k+1} - AD_{k+1})^{-1}ikAD_k(C_k - AD_k)^{-1}\\
&= k[C_{k+1}(C_{k+1} - AD_{k+1})^{-1} - I]ik(C_k - AD_k)^{-1}\\
&\quad + k(C_{k+1} - AD_{k+1})^{-1}ik[I - C_k(C_k - AD_k)^{-1}]
\end{aligned}$$

Since products and sums of R-bounded sequences is R-bounded [24, Remark 2.2]. Then the proof is complete. □

Lemma 3.7. *Let $1 \leq p < \infty$. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and that for every $f \in L^p(\mathbb{T}; X)$ there exists a 2π-periodic strong L^p-solution x of Eq.* (3.1)*. Then x is the unique 2π-periodic strong L^p-solution.*

Proof. Suppose that x_1 and x_2 two strong L^p-solution of Eq. (3.1) then $x = x_1 - x_2$ is a strong L^p-solution of Eq. (3.1) corresponding to $f = 0$. Taking Fourier transform in (3.1), we obtain that

$$ikD_k\hat{x}(k) = AD_k\hat{x}(k) + (ik)^{-\alpha}\tilde{a}(ik)\hat{x}(k) + G_k\hat{x}(k)\ , \quad k \in \mathbb{Z}\,.$$

Then

$$(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)\hat{x}(k) = 0\,.$$

It follows that $\hat{x}(k) = 0$ for every $k \in \mathbb{Z}$ and therefore $x = 0$. Then $x_1 = x_2$. □

Theorem 3.8. *Let X be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong solution of Eq.* (3.1) *for $1 \leq p < \infty$. Then*

(1) *for every $k \in \mathbb{Z}$ the operator $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$ has bounded inverse*

(2) *$\left\{ik\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Before to give the proof of Theorem 3.8, we need the following lemma.

Lemma 3.9. *if $(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)(x) = 0$ for all $k \in \mathbb{Z}$, then $u_t(\cdot) = e^{ikt}e_k(\cdot)x$ is a 2π-periodic strong L^p-solution of the following equation*

$$\frac{d}{dt}(Du_t) = A(Du_t) + I^{\alpha}_{-\infty}(a * u)(t) + Gu_t\,.$$

Proof. $(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)(x) = 0 \Rightarrow ikD_kx = AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx$. Then

$$ikx = ikL_kx + AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx\,.$$

We have $u_t = e^{ikt}e_kx$ and by Remark 2.10 (2),

$$\begin{aligned}
u_t^{'} &= ike^{ikt}e_kx = e^{ikt}e_k(ikx)\\
&= e^{ikt}e_k[ikL_kx + AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx]\\
&= ike^{ikt}e_kL_kx + e^{ikt}e_kAD_kx + e_ke^{ikt}(ik)^{-\alpha}\tilde{a}(ik)x + e^{ikt}e_kG_kx\\
&= ike^{ikt}e_kL_kx + e^{ikt}e_kAD_kx + I^{\alpha}_{-\infty}(a * u_t) + G(e^{ikt}e_kx)\\
&= ikL(e^{ikt}e_kx) + AD(e^{ikt}e_kx) + I^{\alpha}_{-\infty}(a * u_t) + G(e^{ikt}e_kx)\\
&= ikL(e^{ikt}e_kx) + AD(e^{ikt}e_kx) + I^{\alpha}_{-\infty}(a * u_t) + G(e^{ikt}e_kx)\\
&= ikL(e^{ikt}e_kx) + AD(e^{ikt}e_kx) + I^{\alpha}_{-\infty}(a * u_t) + G(e^{ikt}e_kx)\\
&= ikL(u_t) + A(Du_t) + I^{\alpha}_{-\infty}(a * u_t) + G(u_t)\\
&= (Lu_t)^{'} + A(Du_t) + I^{\alpha}_{-\infty}(a * u_t) + G(u_t)\\
(u_t - Lu_t)^{'} &= A(Du_t) + I^{\alpha}_{-\infty}(a * u_t) + G(u_t)\,,
\end{aligned}$$

$$(Du_t)' = A(Du_t) + I^{\alpha}_{-\infty}(a * u_t) + G(u_t)$$

□

Proof of Theorem 3.8. 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^{1,p}(\mathbb{T};X)$ such that:

$$\frac{d}{dt}Dx_t = A(Dx_t) + I^{\alpha}_{-\infty}(a * x)(t) + G(x_t) + f(t)\,.$$

Taking Fourier transform, G and D are bounded. We have $\widehat{(Dx_.)'}(k) = \widehat{x'}(k) - \widehat{(Lx_.)'}(k)$ and $I^{\alpha}_{-\infty}\widehat{(a * x)}(k) = (ik)^{-\alpha}\tilde{a}(ik)\hat{x}(k)$ by Lemma 3.5, we deduce that:

$$\hat{x'}(k) - \widehat{(Lx_.)'}(k) = ik\hat{x}(k) - ikL_k\hat{x}(k) = ik(I - L_k)\hat{x}(k) = ikD_k\hat{x}(k)\,.$$

Consequently, we have

$$ikD_k\hat{x}(k) = AD_k\hat{x}(k) + (ik)^{-\alpha}\tilde{a}(ik)\hat{x}(k) + G_k\hat{x}(k) + \hat{f}(k)$$

$(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)\hat{x}(k) = \hat{f}(k) = y \Rightarrow (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$ is surjective if $(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)(u) = 0$, then by Lemma 3.9, $x_t = e^{ikt}e_k u$ is a 2π-periodic strong L^p-solution of Eq.(3.1) corresponding to the function $f(t) = 0$ Hence $x_t = 0$ and $u = 0$ then $(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$ is injective.

2) Let $f \in L^p(\mathbb{T};X)$. By hypothesis, there exists a unique $x \in H^{1,p}(\mathbb{T},X)$ such that the Eq. (3.1) is valid. Taking Fourier transforms, we deduce that

$$\hat{x}(k) = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)^{-1}\hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Hence

$$ik\hat{x}(k) = ik(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)^{-1}\hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Since $x \in H^{1,p}(\mathbb{T};X)$, then there exists $v \in L^p(\mathbb{T};X)$ such that

$$\hat{v}(k) = ik\hat{x}(k) = ik(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)^{-1}\hat{f}(k)\,.$$

Then $\left\{ik\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and $\left\{ik\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded. □

4. Main result

Our main result in this work is to establish that the converse of Theorem 3.8, are true, provided X is an UMD space.

Theorem 4.1. *Let X be an UMD space and $A\colon D(A) \subset X \to X$ be an closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*

(1) *for every $f \in L^p(\mathbb{T};X)$ there exists a unique 2π-periodic strong L^p-solution of Eq. (3.1).*
(2) *$\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\left\{ik\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Lemma 4.2 ([3]). *Let $f, g \in L^p(\mathbb{T};X)$. If $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. Then*

$$f(t) \in D(A) \quad \textit{and} \quad Af(t) = g(t) \quad \textit{for all} \quad t \in [0, 2\pi]\,.$$

Proof. 1) $\Rightarrow$ 2) see Theorem 3.8.
1) $\Leftarrow$ 2) Let $f \in L^p(\mathbb{T}; X)$. Define $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$.
By Lemma 3.3, the family $\{ik\Delta_k^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier it is equivalent to the family $\{\Delta_k^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{1,p}(\mathbb{T}; X)$, namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{x}(k) = \Delta_k^{-1}\hat{f}(k) = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)^{-1}\hat{f}(k)\,. \tag{4.1}$$

In particular, $x \in L^p(\mathbb{T}; X)$ and there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = ik\hat{x}(k)$

$$\widehat{(Dx_.)}'(k) := D_k\hat{v}(k) = ikD_k\hat{x}(k)\,. \tag{4.2}$$

By Theorem 2.1, we have

$$x_t(\theta) = x(t+\theta) = \lim_{n\to+\infty} \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}e^{ik\theta}\hat{x}(k)\,.$$

Hence in $L^p(\mathbb{T}; X)$, we obtain that

$$x_t = \lim_{n\to+\infty} \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}e^{ik\cdot}\hat{x}(k)\,.$$

Since G is bounded, then

$$\begin{aligned} Gx_t &= \lim_{n\to+\infty} \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}G(e_k\hat{x}(k)) \\ &= \lim_{n\to+\infty} \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}G_k\hat{x}(k) \end{aligned}$$

Using now (4.1) and (4.2) we have:

$$\widehat{(Dx_.)}'(k) = ikD_k\hat{x}(k) = AD_k\hat{x}(k) + I^{\alpha}_{-\infty}\widehat{(a*x)}(k) + G_k\hat{x}(k) + \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}\,.$$

Since A is closed, then $Dx_t \in D(A)$ [Lemma 4.2] and from the uniqueness theorem of Fourier coefficients, that Eq. (3.1) is valid. □

References

[1] Aparicio, R., Keyantuo, V., *Well-posedness of degenerate integro-differential equations in function spaces*, 2018, No. 79, pp. 1–31.

[2] Arendt, W., *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, Handb. Differ. Equ., vol. I, North-Holland, Amsterdam, 2004, pp. 1–85.

[3] Arendt, W., Bu, S., *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), 311–343.

[4] Arendt, W., Bu, S., *Operator-valued Fourier multipliers on periodic Besov spaces and applications*, Proc. Edinb. Math. Soc. (2) **47** (2004), 15–33.

[5] Bourgain, J., *Vector-valued Hausdorff-Young inequalities and applications*, Geometric Aspects of Functional Analysis (1986/1987),, vol. 1317, Lecture Notes in Math., Springer Verlag Berlin, 1986, pp. 239–249.

[6] Bourgain, J., *Vector-valued singular integrals and the H^1-BMO duality*, probability theory and harmonic analysis ed., Marcel Dekker, New York, 1986.

[7] Bu, S., *Maximal regularity for integral equations in Banach spaces*, Taiwanese J. Math. **15** (1) (2011), 229–240.

[8] Bu, S., Fang, F., *Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces*, Studia Math. **184** (2) (2008), 103–119.

[9] Cai, G., Bu, S., *Well-posedness of second order degenerate integro-differential equations with infinite delay in vector-valued function spaces*, Math. Nachr. **289** (2016), 436–451.

[10] Cavalcanti, M.M., Cavalcanti, V.N. Domingos, Guesmia, A., *Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory*, J. Differential Equations **259** (2015), 7540–7577.

[11] Clément, Ph., Da Prato, G., *Existence and regularity results for an integral equation with infinite delay in a Banach space*, Integral Equations Operator Theory **11** (1988), 480–500.

[12] Clément, Ph., de Pagter, B., Sukochev, F.A., Witvliet, M., *Schauder decomposition and multiplier theorems*, Studia Math. **138** (2000), 135–163.

[13] Clément, Ph., Prüss, J., *An operator-valued transference principle and maximal regularity on vector-valued Lp-spaces*, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, pp. 67–87.

[14] Da Prato, G., Lunardi, A., *Periodic solutions for linear integrodifferential equations with infinite delay in Banach spaces*, Differential Equations in Banach spaces, Lecture Notes in Math., vol. 1223, Springer, Berlin, 1986, pp. 49–60.

[15] de Pagter, B., Witvliet, H., *Unconditional decompositions and UMD spaces*, Publ. Math. Besançon, Fasc. **16** (1998), 79–111.

[16] Denk, R., Hieber, M., Prüss, Jan, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **788** (2003).

[17] Girardi, M., Weis, L., *Operator-valued Fourier multiplier theorems on Besov spaces*, Math. Nachr. **251** (2003), 34–51.

[18] Girardi, M., Weis, L., *Operator-valued Fourier multipliers and the geometry of Banach spaces*, J. Funct. Anal. **204** (2) (2003), 320–354.

[19] Keyantuo, V., Lizama, C., *Fourier multipliers and integro-differential equations in Banach spaces*, J. London Math. Soc. **69** (3) (2004), 737–750.

[20] Keyantuo, V., Lizama, C., *Periodic solutions of second order differential equations in Banach spaces*, Math. Z. **253** (2006), 489–514.

[21] Keyantuo, V., Lizama, C., Poblete, V., *Periodic solutions of integro-differential equations in vector-valued function spaces*, J. Differential Equations **246** (2009), 1007–1037.

[22] Koumla, S., Ezzinbi, Kh., Bahloul, R., *Mild solutions for some partial functional integrodifferential equations with finite delay in Fréchet spaces*, SeMA J. **74** (4) (2017), 489–501.

[23] Kunstmann, P.C., Weis, L., *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.

[24] Lizama, C., *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl. **324** (1) (2006), 921–933.

[25] Lizama, C., Poblete, V., *Periodic solutions of fractional differential equations with delay*, Journal of Evolution Equations **11** (2011), 57–70.

[26] Poblete, V., *Solutions of second-order integro-differental equations on periodic Besov space*, Proc. Edinburgh Math. Soc. (2) **50** (20) (2007), 477–492.

[27] Suresh Kumar, P., Balachandran, K., Annapoorani, N., *Controllability of nonlinear fractional Langevin delay systems*, Nonlinear Analysis: Modelling and Control **23** (3) (2017), 321–340, https://doi.org/10.15388/NA.2018.3.3.

[28] Weis, L., *A new approach to maximal L_p-regularity*, Lect. Notes Pure Appl. Math. **2115** (2001), 195–214.

[29] Weis, L., *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), 735–758.

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