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# INVARIANT SYMBOLIC CALCULUS FOR COMPACT LIE GROUPS

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Abstract. We study the invariant symbolic calculi associated with the unitary irreducible representations of a compact Lie group.

## 1. Introduction

In the context of covariant quantization, the main tool is the notion of invariant symbolic calculus that has been used to extend the usual Weyl correspondence to the general situation of a Lie group acting on a homogeneous space [1], [3]. Some important examples of invariant symbolic calculi are

(1) The Berezin symbolic calculus on complex domains, defined via coherent states [6], [7];
(2) The Weyl calculus for symmetric domains that constitutes the direct generalization of the classical Weyl correspondence on $\mathbb{R}^{2n}$ [2], [3];
(3) The Stratonovich-Weyl correspondence which was intensively studied, see [8], [13], [15], [19], [20], [29].

The following definition is adapted from [3] and [19].

**Definition 1.1.** Let $G$ be a Lie group and $\pi$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Let $M$ be a homogeneous $G$-space and $\mu$ a (suitably normalized) $G$-invariant measure on $M$. Then an invariant symbolic calculus for the triple $(G,\pi,M)$ is a linear map $W$ from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on $M$ satisfying the following properties:

(1) $W$ is one-to-one;
(2) $W$ maps the identity operator of $\mathcal{H}$ to the constant function 1;
(3) Reality: the function $W(A^*)$ is the complex conjugate of $W(A)$;
(4) Invariance: we have $W(\pi(g)\,A\,\pi(g)^{-1})(x) = W(A)(g^{-1}\cdot x)$.

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Moreover, if $W$ is unitary in the sense that we have

$$\int_M W(A)(x)W(B)(x)\,d\mu(x) = \mathrm{Tr}(AB)\,.$$

for each Hilbert-Schmidt operators $A$ and $B$ in the domain of $W$, then $W$ is called a Stratonovich-Weyl correspondence [19].

In Definition 1.1, $M$ is generally taken to be a coadjoint orbit of $G$ which is associated with $\pi$ by the Kirillov-Kostant method of orbits [22], [24]. The basic example is the case when $G$ is the $(2n+1)$-dimensional Heisenberg group. Each non-degenerate coadjoint orbit $M$ of $G$ is diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a unitary irreducible representation $\pi$ of $G$ on $L^2(\mathbb{R}^n)$. Then the classical Weyl correspondence provides an invariant symbolic calculus for the triple $(G,\pi,M)$, which is also a Stratonovich-Weyl correspondence [18], [19].

In this note, we consider the case when $G$ is a compact Lie group acting transitively on a manifold $M$ and $\pi$ is a unitary representation of $G$ on a finite-dimensional Hilbert space of functions on $M$. Then, by considering the Berezin calculus $S$ on $M$ [6], [10] and by using some ideas from [8] and [17], we find all the invariant symbolic calculi for $(G,\pi,M)$ provided that $S$ is injective.

More precisely, we associate to each invariant symbolic calculus $W$ for $(G,\pi,M)$ a *quantizer* $\Omega\colon M\to \mathrm{End}(\mathcal{H})$ via the relation

$$W(A)(x) = \mathrm{Tr}\left(A\Omega(x)\right).$$

Then we give an expression for $\Omega$ in terms of the eigenfunctions and eigenvalues of the so-called Berezin transform $B := SS^*$ (Proposition 4.5). Moreover, we prove that the invariant symbolic calculi are parametrized by a family of operators from the space of all Berezin symbols to $L^2(M,\mu)$ which intertwine the regular representations of $G$ on these spaces. As a consequence, we see that the problem of describing the invariant symbolic calculi is connected to that of decomposing the regular representation into irreducible components.

The preceding considerations apply in particular in the case when $G$ is a compact semisimple Lie group $G$ and $\pi$ is a unitary irreducible representation of $G$ with highest weight $\lambda$. In that case, $\pi$ is usually realized on a Hilbert space of holomorphic sections of a complex line bundle over a generalized flag manifold $G/H$ and can be also realized on a reproducing kernel Hilbert space of complex polynomials for which the Berezin calculus is injective, see [4], [10] and [32]. As an example, we treat in details the case where $G = SU(n+1)$, $H = S(U(1)\times U(n))$ and $\pi$ lies in the family of unitary irreducible representations of $G$ considered in [9].

## 2. Preliminaries

Here we introduce the notation and some generalities on reproducing kernel Hilbert spaces following essentially [5], [10] and [11].

Let $G$ be a compact Lie group and $M$ be a $G$-homogeneous space. Let $\mu$ be a $G$-invariant measure on $M$. Let $K$ be a continuous function on $M$ such that $K(x) > 0$ for each $x\in M$ and let $\tilde{\mu}$ be the measure on $M$ defined by $d\tilde{\mu}(x) = K(x)^{-1}d\mu(x)$.

Let $\mathcal{H}$ be a finite-dimensional space of continuous, square-integrable functions on $M$ with respect to $\tilde{\mu}$. Then $\mathcal{H}$ is a Hilbert space with respect to the $L^2$-norm relative to $\tilde{\mu}$ and, for each $x \in M$, there exists a unique function $e_x \in \mathcal{H}$ such that

$$f(x) = \langle f, e_x \rangle = \int_M f(y)\overline{e_x(y)}\, d\tilde{\mu}(y)$$

for every $f \in \mathcal{H}$ and $x \in M$. The function $k(x,y) := \overline{e_x(y)} = \langle e_y, e_x \rangle$ is then called the reproducing kernel of $\mathcal{H}$.

Consider now a continuous function $\alpha\colon G \times M \to \mathbb{C}^*$ such that

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 \cdot x)\alpha(g_2, x)$$

for each $g_1$, $g_2 \in G$ and $x \in M$. Then we get an action $\pi$ of $G$ on the space of continuous functions on $M$, according to the formula

$$\big(\pi(g)f\big)(x) = \alpha(g^{-1}, x)\, f(g^{-1} \cdot x)\,.$$

Assume moreover that $\pi(g)f \in \mathcal{H}$ for each $g \in G$ and $f \in \mathcal{H}$. Then $\pi$ is a representation of $G$ on $\mathcal{H}$.

**Proposition 2.1** ([5], [10]). *The representation $\pi$ is unitary if and only if $\alpha$ and $K$ are compatible in the sense that*

$$K(g \cdot x) = |\alpha(g,x)|^{-2} K(x)\,, \qquad g \in G,\, x \in M\,.$$

*In this case, we have*

$$\pi(g)e_x = \overline{\alpha(g,x)}\, e_{g\cdot x}\,, \qquad g \in G,\, x \in M\,,$$

*and*

$$k(g\cdot x, g \cdot y) = \alpha(g,x)^{-1}\overline{\alpha(g,y)}^{-1} k(x,y)\,, \qquad g \in G,\, x,\, y \in M\,.$$

*Moreover, there exists $c_\pi > 0$ such that $k(x,x) = c_\pi K(x)$ for each $x \in M$.*

In the rest of the note, $\pi$ is assumed to be unitary.

## 3. Berezin calculus

Here we review some general facts on Berezin quantization [6], [7], [10].

Let $A$ be an operator on $\mathcal{H}$. The Berezin covariant symbol of $A$ is the function defined on $M$ by

$$S(A)(x) = \frac{\langle A\, e_x\,,\, e_x \rangle}{\langle e_x\,,\, e_x\rangle}\,.$$

Moreover, the double Berezin symbol of $A$ is the function defined by

$$s(A)(x,y) = \frac{\langle A\, e_y\,,\, e_x \rangle}{\langle e_y\,,\, e_x\rangle}$$

for each $x$, $y \in M$ such that $\langle e_x, e_y \rangle \neq 0$.

The operator $A$ can be recovered from $s(A)$ by the following formula which is easy to verify, see e.g. [10], [14],

$$A\, f(x) = \int_M f(y)\, s(A)(x,y) \langle e_y, e_x \rangle\, K(y)^{-1}\, d\mu(y)\,.$$

Then we see that $s$ is injective. But in general $S$ is not injective, as showed by the following example. However, as mentionned in [6], p. 1118, in many cases of interest, $S$ is injective. This is the case when $M$ is a complex manifold and $\mathcal{H}$ consists of holomorphic functions on $M$ since the function $(x,y) \to \langle A\, e_y\,,\, e_x\rangle$, which is holomorphic in the variable $x$ and anti-holomorphic in the variable $y$, is then determined by its restriction to the diagonal of $M^2$ (see also Section 5).

**Example 3.1** (see [16])**.** We take $G = SO(3)$ and $M = \mathbb{S}^2$. Let $\pi_m$ be the $(2m+1)$-dimensional unitary irreducible representation of $SO(3)$ realized on the space $\mathcal{H}_m$ of all harmonic polynomials on $\mathbb{S}^2$. Observe that $\mathcal{H}_m$ is stable under complex conjugation $f \to \overline{f}$. Take $f,\, g \in \mathcal{H}_m$ linearily independant and consider the operator

$$A := \langle \cdot, \overline{f}\rangle_{\mathcal{H}_m} g - \langle \cdot, \overline{g}\rangle_{\mathcal{H}_m} f \neq 0\,.$$

Then, clearly, for each $x \in \mathbb{S}^2$, we have $Ae_x = f(x)g - g(x)f$ hence

$$\langle Ae_x, e_x\rangle_{\mathcal{H}_m} = f(x)g(x) - g(x)f(x) = 0\,.$$

This proves that $S$ is not injective.

Denote by $L(\mathcal{H})$ the space of all operators on $\mathcal{H}$ endowed with the Hilbert-Schmidt norm and by $\mathcal{S}$ the range of $S$, that is, the space of all symbols. In the following proposition, we collect some well-known properties of $S$ [6], [7].

**Proposition 3.2.**

(1) *For each $A \in L(\mathcal{H})$ we have*

$$\mathrm{Tr}(A) = c_\pi \int_M S(A)(x)\, d\mu(x)\,.$$

(2) *For each $A \in L(\mathcal{H})$, $g \in G$ and $x \in M$, we have*

$$S(\pi(g)^{-1}A\pi(g))(x) = S(A)(g\cdot x)\,.$$

(3) *Let $P_x$ be the orthogonal projection operator of $\mathcal{H}$ on the line generated by $e_x$. Then, for each $A \in L(\mathcal{H})$ and each $x \in M$, we have*

$$S(A)(x) = \mathrm{Tr}(AP_x)\,.$$

(4) *The adjoint $S^*$ of $S$ is the map from $\mathcal{S}$ to $L(\mathcal{H})$ given by*

$$S^*(F) = \int_M F(x)\, P_x\, d\mu(x)\,.$$

(5) *The map $B := SS^*$ is the operator on $\mathcal{S}$ given by*

$$B(F)(x) = \int_M F(y)\, \frac{|\langle e_x, e_y\rangle|^2}{\langle e_x, e_x\rangle\langle e_y, e_y\rangle}\, d\mu(y)\,.$$

We can fix the normalization of $\mu$ such that $\int_M d\mu(x) = 1$. Then, by the first assertion of the preceding proposition, we have $c_\pi = \dim(\mathcal{H})$.

Note that $B = SS^*$ is the so-called Berezin transform that has been intensively studied by many authors, especially for weighted Bergman spaces on bounded symmetric domains, see, in particular, [6], [7], [26], [27], [30] and [33].

Note also that one can interpret $S$ as a diagonal operator, see [27]. This goes as follows. Denote by $\mathcal{H}^- := \{\overline{f} : f \in \mathcal{H}\}$ the Hilbert space conjugate to $\mathcal{H}$. The norm on $\mathcal{H}^-$ is defined by $\|\overline{f}\|_{\mathcal{H}^-} = \|f\|$. Let $\overline{\pi}$ be the representation of $G$ on $\mathcal{H}^-$ defined by $\overline{\pi}(g)\overline{f} = \overline{\pi(g)f}$ for $f \in \mathcal{H}$. Then the representation $\pi^*$ of $G$ on $\mathcal{H}^*$ contragredient to $\pi$ is equivalent to $\overline{\pi}$, an intertwining operator between $\pi^*$ and $\overline{\pi}$ being $\overline{f} \to \langle \cdot, f\rangle$. Let us denote by $j$ the linear isomorphism from $\mathcal{H} \otimes \mathcal{H}^-$ onto $L(\mathcal{H})$ defined by

$$j\Big(\sum_i f_i \otimes \overline{g}_i\Big) = \sum_i \langle \cdot, g_i\rangle f_i.$$

Then one can easily verify that $S \circ j : \mathcal{H} \otimes \mathcal{H}^- \to \mathcal{S}$ is the diagonal operator

$$\sum_i f_i \otimes \overline{g}_i \to \sum_i f_i(x)\overline{g_i(x)}\langle e_x, e_x\rangle^{-1}.$$

## 4. Invariant symbolic calculus

First, we adapt Definition 1.1 to the context of Section 2.

**Definition 4.1.** An invariant symbolic calculus for the triple $(G, \pi, M)$ is a linear injective map $W$ from $L(\mathcal{H})$ to $L^2(M, \mu)$ such that

(1) $W(\mathrm{Id}_{\mathcal{H}}) = 1$;

(2) For each $A \in L(\mathcal{H})$, we have $W(A^*) = \overline{W(A)}$;

(3) For each $A \in L(\mathcal{H})$, $g \in G$ and $x \in M$, we have $W(\pi(g)\, A\, \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x)$.

Of course, the Berezin calculus $S$ is an example of such an invariant symbolic calculus, provided it is injective.

Now, let $W$ be an invariant symbolic calculus for $(G, \pi, M)$. For each $x \in M$ there exists a unique element $\Omega(x)$ of $L(\mathcal{H})$ such that

$$W(A)(x) = \mathrm{Tr}\,\big(A\Omega(x)\big) \tag{4.1}$$

for each $A \in L(\mathcal{H})$. The map $x \to \Omega(x)$ is called the *quantizer* associated with $W$. The properties of $W$ are reflected by similar properties of $\Omega$ and, clearly, we can construct $W$ from $\Omega$. By analogy to [20], Section 2.2, we can prove the following proposition.

**Proposition 4.2.** *Let $W$ be an invariant symbolic calculus for $(G, \pi, M)$ and let $\Omega$ the corresponding quantizer. Then $\Omega \neq 0$ and*

(1) *For each $x \in M$, we have $\mathrm{Tr}(\Omega(x)) = 1$.*

(2) *For each $x \in M$, we have $\Omega(x)^* = \Omega(x)$.*

(3) *For each $x \in M$ and $g \in G$, we have $\Omega(g \cdot x) = \pi(g)\Omega(x)\pi(g)^{-1}$.*

(4) *$L(\mathcal{H})$ is spanned by the $\Omega(x)$, $x \in M$.*

*Conversely, if a non-trivial square-integrable function $\Omega\colon M \to L(\mathcal{H})$ satisfies the preceding conditions, then the map $W$ defined by Equation 4.1 is an invariant symbolic calculus for $(G, \pi, M)$ with associated quantizer $\Omega$.*

In the rest of this note, we assume that $S$ is injective. Let $N = (\dim \mathcal{H})^2$. Since $SS^*$ is a self-adjoint operator of $\mathcal{S}$, there exists an orthonormal basis $(F_k)_{1\leq k\leq N}$ of $\mathcal{S}$ and a family $(\lambda_k)_{1\leq k\leq N}$ of positive real numbers such that $SS^*(F_k) = \lambda_k F_k$ for each $k = 1, 2, \ldots, N$. Moreover, since $SS^*$ commutes to conjugation, we may assume that the $F_k$ are real-valued. We can also assume that $F_1 = 1$ and $\lambda_1 = c_\pi^{-1}$. Indeed, by (5) of Proposition 3.2, we have, for each $x \in M$

$$\begin{aligned} B(1)(x) &= \langle e_x, e_x\rangle^{-1} \int_M |e_x(y)|^2 \langle e_y, e_y\rangle^{-1}\, d\mu(y) \\ &= c_\pi^{-1} \langle e_x, e_x\rangle^{-1} \|e_x\|^2 = c_\pi^{-1}\,. \end{aligned}$$

Now, we introduce a useful basis of $L(\mathcal{H})$, in the spirit of [8], Equation (3.9).

**Lemma 4.3.** *Let $D_k := \lambda_k^{-1/2} S^*(F_k)$ for $k = 1, 2, \ldots, N$. Then we have*

1. *For each $k = 1, 2, \ldots, N$, $S(D_k) = \lambda_k^{1/2} F_k$;*
2. *For each $k = 1, 2, \ldots, N$, $D_k^* = D_k$;*
3. *$D_1 = c_\pi^{-1/2}\,\mathrm{Id}_{\mathcal{H}}$;*
4. *For each $k, l = 1, 2, \ldots, N$, $\langle D_k, D_l\rangle = \delta_{kl}$, that is, $(D_k)$ is an orthonormal basis of $L(\mathcal{H})$ with respect to the Hilbert-Schmidt product;*
5. *For each $k = 1, 2, \ldots, N$, $\mathrm{Tr}(D_k) = c_\pi^{1/2} \delta_{1k}$.*

**Proof.** Assertion (1) is an immediate consequence of the definition of $(D_k)$ and Assertion (2) results from the fact that, for each symbol $F$ on $M$, one has $S^*(F)^* = S^*(\overline{F})$. To prove (3), note that, since we have for each $f \in \mathcal{H}$, $y \in M$,

$$P_y(f) = \frac{\langle f, e_y\rangle}{\langle e_y, e_y\rangle} e_y\,,$$

we can write, for each $f \in \mathcal{H}$ and $x \in M$,

$$\begin{aligned} (D_1 f)(x) &= c_\pi^{1/2} \int_M (P_y f)(x)\, d\mu(y) \\ &= c_\pi^{1/2} \int_M f(y)\overline{e_x(y)} c_\pi^{-1} K(y)^{-1}\, d\mu(y) \\ &= c_\pi^{-1/2} \langle f, e_x\rangle\,. \end{aligned}$$

Hence $D_1 = c_\pi^{-1/2}\,\mathrm{Id}_{\mathcal{H}}$.

To prove (4), note that for each $k, l$ we have

$$\begin{aligned} \langle D_k, D_l\rangle &= \lambda_k^{-1/2} \lambda_l^{-1/2} \langle S^*(F_k), S^*(F_l)\rangle = \lambda_k^{-1/2} \lambda_l^{-1/2} \langle SS^*(F_k), F_l\rangle \\ &= \lambda_k^{1/2} \lambda_l^{-1/2} \langle F_k, F_l\rangle = \delta_{kl}\,. \end{aligned}$$

In particular, for each $k$, we have $\langle D_1, D_k\rangle = \delta_{1k}$ hence $\mathrm{Tr}(D_k) = c_\pi^{1/2}\delta_{1k}$ and (5) is thus proved. □

Denote by $\rho$ the left-regular representation of $G$ on $L^2(M,\mu)$ defined by $(\rho(g)F)(x) = F(g^{-1}\cdot x)$. We also denote for $g\in G$ and $j,k = 1,2,\ldots,N$, $u_{jk}(g) := \langle \rho(g)F_j, F_k\rangle$.

**Lemma 4.4.**

(1) *For each $F\in\mathcal{S}$ and $g\in G$, we have $S^*(\rho(g)F) = \pi(g)S^*(F)\pi(g)^{-1}$.*

(2) *For each $g\in G$ and $j = 1,2,\ldots,N$, we have*

$$\pi(g)D_j\pi(g)^{-1} = \lambda_j^{-1/2}\sum_{k=1}^N \lambda_k^{1/2}u_{jk}(g)D_k\,.$$

**Proof.** The first assertion follows from (2) of Proposition 3.2 and implies that, for each $g\in G$ and $j = 1,2,\ldots,N$, we have

$$\begin{aligned}\pi(g)D_j\pi(g)^{-1} =& \lambda_j^{-1/2}\pi(g)S^*(F_j)\pi(g)^{-1}\\ =& \lambda_j^{-1/2}S^*(\rho(g)F_j)\\ =& \lambda_j^{-1/2}S^*\Big(\sum_{k=1}^N u_{jk}(g)F_k\Big)\\ =& \lambda_j^{-1/2}\sum_{k=1}^N \lambda_k^{1/2}u_{jk}(g)D_k\,.\end{aligned}$$

This proves the second assertion. □

**Proposition 4.5.** *Let $W$ be an invariant symbolic calculus for $(G,\pi,M)$ with associated quantizer $\Omega$. Then there exists an injective operator $C\colon\mathcal{S}\to L^2(M,\mu)$ which commutes to $\rho(g)$ for each $g\in G$ and to complex conjugation, such that $C(1) = 1$ and, for each $x\in M$, we have*

$$\Omega(x) = \sum_{k=1}^N \lambda_k^{1/2}C(F_k)(x)D_k\,.$$

*Conversely, for each operator $C$ as above, the preceding equation defines a quantizer for an invariant symbolic calculus for $(G,\pi,M)$.*

**Proof.** Assume that $W$ is an invariant symbolic calculus for $(G,\pi,M)$ and let $\Omega$ be the associated quantizer. Then there exists a family $(a_j)_{1\le j\le N}$ of functions on $M$ such that, for each $x\in M$, we have $\Omega(x) = \sum_{j=1}^N a_j(x)D_j$.

By (2) of Lemma 4.4, we see that the $G$-invariance of $\Omega$ (see Proposition 4.2) implies that for each $x\in M$, $g\in G$ and $k = 1,2,\ldots,N$, we have

$$a_k(g\cdot x) = \sum_{j=1}^N \lambda_j^{-1/2}\lambda_k^{1/2}u_{jk}(g)a_j(x)\,.$$

This is equivalent to the fact that the operator $C\colon \mathcal{S} \to L^2(M,\mu)$ defined by $C(F_k) = \lambda_k^{-1/2} a_k$, $k = 1, 2, \ldots, N$, commutes to $\rho(g)$ for each $g \in G$. Moreover, the property $\Omega(x)^* = \Omega(x)$ for each $x \in M$ implies that $a_k$ is real-valued for each $k = 1, 2, \ldots, N$. That shows that $C$ commutes to complex-conjugation. Also, the property $\mathrm{Tr}(\Omega(x)) = 1$ for each $x \in M$ gives $a_1(x) = c_\pi^{1/2}$ and, since $\lambda_1 = c_\pi^{-1}$, we get $C(1) = 1$. The rest of the proposition can be easily verified. $\square$

**Proposition 4.6.** *Let $C\colon \mathcal{S} \to L^2(M,\mu)$ be an operator as in the preceding proposition and let $W$ be the invariant symbol calculus $(G, \pi, M)$ with quantizer $\Omega$ defined by*

$$\Omega(x) = \sum_{k=1}^{N} \lambda_k^{1/2} C(F_k)(x) D_k \,.$$

*Then we have $W = CS$. In particular, the invariant symbol calculus corresponding to the case when $C$ is the inclusion $\mathcal{S} \to L^2(M,\mu)$ is $S$.*

**Proof.** For each $j = 1, 2, \ldots, N$ and $x \in M$, we can write

$$\begin{aligned} W(D_j)(x) &= \mathrm{Tr}(\Omega(x) D_j) \\ &= \sum_{k=1}^{N} \lambda_k^{1/2} C(F_k)(x)\, \mathrm{Tr}(D_j D_k) \\ &= \lambda_j^{1/2} C(F_j)(x) \\ &= C(S(D_j))(x) \end{aligned}$$

hence $W(D_j) = CS(D_j)$ for each $j = 1, 2, \ldots, N$ and we can conclude that $W = CS$. $\square$

We can easily identify the invariant symbolic calculi on $M$ which are also Stratonovich-Weyl correspondences.

**Proposition 4.7.** *Let $W$, $\Omega$ and $C$ as in Proposition 4.5.*

(1) *$W$ is a Stratonovich-Weyl correspondence if and only if the family $(\lambda_k^{1/2} C(F_k))_k$ is orthonormal;*

(2) *Let $C_0$ be the operator defined by $C_0(F_k) = \lambda_k^{-1/2} F_k$ for $k = 1, 2, \ldots, N$. Then the Stratonovich-Weyl correspondence $W_0$ corresponding to $C_0$ is the unitary part in the polar decomposition of $S$, that is, we have $W_0 = B^{-1/2} S$;*

(3) *Each Stratonovich-Weyl correspondence $W$ on $M$ can be written $W = UW_0$ where $U : \mathcal{S} \to L^2(M,\mu)$ is an isometric operator.*

**Proof.** Clearly, an invariant symbolic calculus $W$ on $M$ is a Stratonovich-Weyl correspondence if and only if $(W(D_k))_k$ is an orthonormal system of $L^2(M,\mu)$. But, for each $k = 1, 2, \ldots, N$, we have $W(D_k) = \lambda_k^{1/2} C(F_k)$, see the proof of Proposition 4.6. This implies Assertion (1).

Now, on the one hand, we have

$$W_0(D_k) = \lambda_k^{1/2} C_0(F_k) = F_k$$

for $k = 1, 2, \ldots, N$ and, on the other hand, by (1) of Lemma 4.3, we also have

$$(B^{-1/2}S)(D_k) = B^{-1/2}(\lambda_k^{1/2}F_k) = F_k$$

for $k = 1, 2, \ldots, N$. Hence we can conclude that $W_0 = B^{-1/2}S$.

To prove Assertion (3), consider a Stratonovich-Weyl correspondence $W$ and define an operator $U : \mathcal{S} \to L^2(M, \mu)$ by $U(F_k) = W(D_k)$ for $k = 1, 2, \ldots, N$. Since $(W(D_k))_k$ is orthonormal, $U$ is an isometry and we have $W(D_k) = U(F_k) = (UW_0)(D_k)$ for each $k = 1, 2, \ldots, N$, hence $W = UW_0$. $\square$

We see that the invariant symbolic calculi for $(G, \pi, M)$ are parametrized by the operators $C$ as in Proposition 4.5. In order to be a little more precise, let us introduce the decompositions of $\rho$ on $\mathcal{S}$ and $L^2(M, \mu)$ into irreducible unitary representations:

$$\mathcal{S} = \oplus_{j=1}^N V_i \,; \quad L^2(M, \mu) = \oplus_{j=1}^N W_j \,.$$

For each $i = 1, 2, \ldots, N$ and $j \geq 0$ define the operators $D_{ij} \colon \mathcal{S} \to L^2(M, \mu)$ as follows. If $V_i$ is not unitarily equivalent to $W_j$ then $D_{ij} = 0$ and if $V_i$ is unitarily equivalent to $W_j$ then $D_{ij}$ vanishes on $V_k$ for each $k \neq i$, maps $V_i$ to $W_j$ and induces a $G$-invariant isomorphism from $V_i$ onto $W_j$.

Let $C \colon \mathcal{S} \to L^2(M, \mu)$ be an operator as in Proposition 4.5. Then, by the Schur's Lemma, for each $i = 1, 2, \ldots, N$ and $j \geq 0$, the composition of $C$ with the inclusion $V_i \to L^2(M, \mu)$ and the projection $L^2(M, \mu) \to W_j$ is either 0 or an isomorphism $V_i \to W_j$ which is a multiple of $D_{ij}|_{V_i}$. Consequently, there exists a family $d_{ij}$ of complex numbers such that $C = \sum_{ij} d_{ij} D_{ij}$.

In the case when the decomposition of $L^2(M, \mu)$ is multiplicity-free (in that case the decomposition of $\mathcal{S}$ is also multiplicity-free), the situation is a little more simple. Indeed, for each $i = 1, 2, \ldots, N$, there exists a unique $j = j(i)$ such that $W_j$ is equivalent to $V_i$ and, denoting $d'_i := d_{ij(i)}$ and $D'_i = D_{ij(i)}$ we have $C = \sum_i d'_i D'_i$.

Note also that the decomposition of $\mathcal{S}$ is not always multiplicity-free. This can be seen by using the fact that the representation $\rho$ of $G$ in $\mathcal{S}$ is unitarily equivalent to $\pi \otimes \bar{\pi}$, see Section 3. Indeed, it is well-known that $\pi \otimes \bar{\pi}$ is not always multiplicity-free [28]. However, in many cases of interest, $L^2(M, \mu)$ is multiplicity-free, see [23].

## 5. The case of a compact semisimple Lie group

In this section, we consider the case where $G$ is a compact connected semisimple Lie group, $\pi$ a unitary irreducible representation of $G$ with highest weight $\Lambda$ and $M$ is the coadjoint orbit of $G$ associated with $\pi$ [10], [11], [12].

Fix a maximal torus $T$ of $G$ and let $\Delta$ be the root system of $G$ relative to $T$. Let $\Delta^+ \subset \Delta$ be a system of positive roots. Let $\mathfrak{t}$ be the Lie algebra of $T$. Let us denote by $\mathfrak{g}^c$ and $\mathfrak{t}^c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. Let $G^c$ and $T^c$ be the connected complex Lie groups whose Lie algebras are $\mathfrak{g}^c$ and $\mathfrak{t}^c$.

Let $\beta$ be the Killing form on $\mathfrak{g}^c$, that is, $\beta(X, Y) = \mathrm{Tr}(\operatorname{ad} X \operatorname{ad} Y)$ for $X$, $Y \in \mathfrak{g}^c$. For each $\lambda \in (\mathfrak{t}^c)^*$, we denote by $H_\lambda$ the element of $\mathfrak{t}^c$ satisfying $\beta(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{t}^c$. For $\lambda$, $\mu \in (\mathfrak{t}^c)^*$, let $(\lambda, \mu) := \beta(H_\lambda, H_\mu)$.

Fix $\lambda \in (\mathfrak{t}^c)^*$ a real-valued form on $i\mathfrak{t}$. Then $iH_\lambda$ lies in $\mathfrak{g}$. Let $T_1 \subset G$ be the torus generated by $\exp(iH_\lambda)$. We can apply to $T_1$ the construction of [31], Section

6.2. Let $H$ be the centralizer of $T_1$ in $G$. Then we have $T \subset H$ and the root system of $H$ relative to $T$ is $\Delta_1 = \{\alpha \in \Delta \,/\, (\lambda, \alpha) = 0\,\}$. Let $\mathfrak{h}$ be the Lie algebra of $H$, $\mathfrak{h}^c$ be the complexification of $\mathfrak{h}$ and $H^c$ be the connected subgroup of $G^c$ corresponding to $\mathfrak{h}^c$. Let $\mathfrak{g}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Let $\Delta_1^+ = \Delta^+ \cap \Delta_1$ and $\Phi = \Delta^+ \setminus \Delta_1^+$. We set $\mathfrak{n}^+ = \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \sum_{\alpha\in\Phi} \mathfrak{g}_{-\alpha}$. Then, by [31], 6.2.1, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras satisfying $[\mathfrak{h}^c\,,\,\mathfrak{n}^\pm] \subset \mathfrak{n}^\pm$ and we have the decompositions

$$\mathfrak{g}^c = \mathfrak{h}^c \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \qquad \mathfrak{h}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{g}_{-\alpha}\,.$$

We denote by $N^+$ and $N^-$ the analytic subgroups of $G^c$ with Lie algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively.

Now, we consider the generalized flag manifold $M := G/H$. A complex structure on $M$ is defined by the diffeomorphism $M = G/H \simeq G^c/H^cN^-$ [31], 6.2.11. We denote by $\tau\colon G^c \to M \simeq G^c/H^cN^-$ the natural projection. Recall that (1) each $g$ in a dense open subset of $G^c$ has a unique Gauss decomposition $g = n^+h\,n^-$ where $n^+ \in N^+$, $h \in H^c$ and $n^- \in N^-$ and (2) the map $\sigma\colon Z \to \tau(\exp Z)$ is a holomorphic diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of $M$ (see [21], Chap. VIII). Then the natural action of $G^c$ on $M \simeq G^c/H^cN^-$ induces an action (defined almost everywhere) of $G^c$ on $\mathfrak{n}^+$. We denote by $g \cdot Z$ the action of $g \in G^c$ on $Z \in \mathfrak{n}^+$. Following [25], we introduce the projections $\kappa\colon N^+H^cN^- \to H^c$ and $\zeta\colon N^+H^cN^- \to N^+$. Then, for $g \in G^c$ and $Z \in \mathfrak{n}^+$ we have $g \cdot Z = \log\zeta(g\exp Z)$.

We set $(X+iY)^* = -X+iY$ for $X$, $Y \in \mathfrak{g}$ and we denote by $g \to g^*$ the involutive anti-automorphism of $G^c$ which is obtained by exponentiating $X+iY \to (X+iY)^*$ to $G^c$. Also, let $\theta$ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ and let $\tilde{\theta}$ be the automorphism of $G^c$ for which $d\tilde{\theta} = \theta$. Then we have $\theta(X) = -X^*$ for $X \in \mathfrak{g}^c$ and $\tilde{\theta}(g) = (g^*)^{-1}$ for $g \in G^c$.

Let us assume that $\lambda$ is integral and dominant, that is, $\frac{2\lambda(H_\alpha)}{\alpha(H_\alpha)}$ is a nonnegative integer for each $\alpha \in \Delta^+$. Let $\chi_0$ be the unique character on $H$ such that $\lambda = d\chi_0|_{\mathfrak{t}}$ and let $\chi_\lambda$ be the unique extension of $\chi_0$ to $H^cN^-$. There exists a unique (up to equivalence) unitary irreducible representation $\pi_\lambda$ of $G$ with highest weight $\lambda$. This representation is usually realized in the space of the holomorphic sections of the holomorphic line bundle $L_\lambda = G^c \times_\chi \mathbb{C}$. Here we use the realization of $\pi_\lambda$ which was obtained in [10] by trivializing $L_\lambda$ by means of the section $Z \in \mathfrak{n}^+ \to [\exp Z, 1]$.

Let $\chi_\Lambda$ be the character of $H^c$ corresponding to $\Lambda = \sum_{\alpha\in\Phi} \alpha$, that is, $\chi_\Lambda(h) = \mathrm{Det}_{\mathfrak{n}^+} \mathrm{Ad}(h)$ for each $h \in H^c$. Then the $G$-invariant measure on $\mathfrak{n}^+$ is

$$d\mu(Z) = c_0\chi_\Lambda(k(Z))\,d\mu_L(Z)$$

where $k(Z) := \kappa(\exp Z^* \exp Z)$, $d\mu_L(Z)$ is a Lebesgue measure on $\mathfrak{n}^+$ and, according to Section 3, the constant $c_0$ is defined by

$$c_0^{-1} = \int_{\mathfrak{n}^+} \chi_\Lambda(k(Z))\,d\mu_L(Z)\,.$$

The representation space of $\pi_\lambda$ is then the finite-dimensional Hilbert space $\mathcal{H}_\lambda$ consisting of complex polynomials $f$ satisfying

$$\|f\|_\lambda^2 := \int_{\mathfrak{n}^+} |f(Z)|^2\, \chi_\lambda(k(Z))\, d\mu(Z) < +\infty\,.$$

We denote by $\langle \cdot\,,\,\cdot\rangle_\lambda$ the Hilbert product on $\mathcal{H}_\lambda$ and by $\|\cdot\|_\lambda$ the corresponding Hilbert norm.

Moreover, the representation $\pi_\lambda$ acts on $\mathcal{H}_\lambda$ as

$$(\pi_\lambda(g)f)(Z) = \chi_\lambda(\exp(-Z)g\exp(g^{-1}\cdot Z))f(g^{-1}\cdot Z)\,.$$

**Proposition 5.1** ([10]). *With the notation as in Section 2, we have*

(1) *For each $g \in G$ and $Z \in \mathfrak{n}^+$, we have $\alpha(g,Z) = \chi_\lambda(\exp(-Z)g^{-1}$ $\exp(g\cdot Z)) = \chi_\lambda(\kappa(g\exp Z))^{-1}$;*

(2) *For each $Z \in \mathfrak{n}^+$, we have $K(Z) = \chi_\lambda(k(Z))^{-1}$;*

(3) *For each $Z, W \in \mathfrak{n}^+$, we have $e_Z(W) = k(W,Z) = c_{\pi_\lambda}\chi_\lambda(\kappa(\exp Z^*$ $\exp W))^{-1}$ where $c_{\pi_\lambda} = \dim(\mathcal{H}_\lambda)$.*

Then we can define the covariant Berezin symbol $S_\lambda(A)$ of each operator $A$ on $\mathcal{H}_\lambda$ as in Section 3. The kernel of $A$ is the function $k_A(Z,W) = \langle Ae_W, e_Z\rangle_\lambda$ which is holomorphic in the variable $Z$ and anti-holomorphic in the variable $W$. Then $k_A$ is determined by its restriction to the diagonal. Consequently, the map $S_\lambda$ is here injective and the results of Section 4 work in this case.

In [10], we proved the following result.

**Proposition 5.2.** *The map $\psi_\lambda\colon \mathfrak{n}^+ \to \mathfrak{g}$ defined by*

$$\psi_\lambda(Z) := \mathrm{Ad}\big(\tilde{\theta}(\exp W)\,\zeta(\exp Z^*\exp Z)\big)\,(-iH_\lambda)$$

*is a diffeomorphism from $\mathfrak{n}^+$ onto a dense open subset of the orbit $\mathcal{O}_\lambda$ of $-iH_\lambda \in \mathfrak{g}$ for the adjoint action of $G$. Moreover, for each $X \in \mathfrak{g}^c$ and $Z \in \mathfrak{n}^+$, we have*

$$S_\lambda(d\pi_\lambda(X))(Z) = i\beta(\psi_\lambda(Z), X)$$

*and, for each $g \in G$ and $Z \in \mathfrak{n}^+$, we have $\psi_\lambda(g\cdot Z) = \mathrm{Ad}(g)\psi_\lambda(Z)$.*

This proposition allows us to transfer $S_\lambda$ as well each invariant symbolic calculus on $\mathfrak{n}^+$ to $\mathcal{O}_\lambda$. Then the results of Section 4 also give a description of the invariant symbolic calculi for $(G, \pi_\lambda, \mathcal{O}_\lambda)$.

Note that the computations of $(F_k)$, $(\lambda_k)$ and $(D_k)$, with the notation of Section 4, are difficult in general. However, in the particular case when $G/K$ is a (compact) Hermitian symmetric space, a formula for the $\lambda_k$ in terms of the Gamma function is given in [33].

## 6. Example

Here we take $G = SU(n+1)$ and use the notation of Section 5. Each $g \in G$ can be written as a block matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with matrices $a(1\times 1)$, $b(1\times n)$, $c(n\times 1)$ and $d(n\times n)$.

We take $T_1$ to be the torus of $G$ consisting of the matrices

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{is}I_n \end{pmatrix} \qquad t, s \in \mathbb{R}, \qquad e^{it}e^{ins} = 1\,.$$

Then $T_1$ is contained in the maximal torus $T$ of $G$ of all matrices

$$\mathrm{Diag}(e^{it_1}, e^{it_2}, \ldots, e^{it_{n+1}}) \qquad t_1, t_2, \ldots, t_{n+1} \in \mathbb{R}\,, \qquad \prod_{k=1}^{n+1} e^{it_k} = 1$$

and $H$ consists of the matrices

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \qquad |a| = 1,\ d \in U(n), \qquad a\,\mathrm{Det}(d) = 1\,.$$

Note that $G^c = SL(n+1, \mathbb{C})$. Let $\mathfrak{t}$ the Lie algebra of $T$. Then we have

$$\mathfrak{t}^c = \left\{ X = \mathrm{Diag}(x_1, x_2, \ldots, x_{n+1}) \ :\ x_k \in \mathbb{C}, \quad \sum_{k=1}^{n+1} x_k = 0 \right\}\,.$$

The set $\Delta$ of roots of $\mathfrak{t}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le n+1$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{t}^c$ as above. Here we take $\Delta^+$ to be $\lambda_j - \lambda_i$ for $1 \le i < j \le n+1$. Consequently, we have

$$N^+ = \left\{ \begin{pmatrix} 1 & 0 \\ z^t & I_n \end{pmatrix} \ :\ z \in \mathbb{C}^n) \right\}, \ N^- = \left\{ \begin{pmatrix} 1 & y \\ 0 & I_n \end{pmatrix} \ :\ y \in \mathbb{C}^n \right\}$$

and we can identify $\mathfrak{n}^+$ to $\mathbb{C}^n$ via

$$z \longrightarrow \begin{pmatrix} 1 & 0 \\ z^t & I_n \end{pmatrix}\,.$$

Here the subscript $t$ denotes transposition.

Note also that

$$H^c = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \ :\ a \in \mathbb{C},\ d \in M_n(\mathbb{C}), \quad a\,\mathrm{Det}(d) = 1 \right\}\,.$$

Then we can easily verify that the Gauss decomposition of a matrix $g \in G^c$ is given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}c & I_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - a^{-1}cb \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & I_n \end{pmatrix}$$

if $a \ne 0$. In particular, we see that $G \subset N^+H^cN^-$ and that the action of $G^c$ on $\mathfrak{n}^+ \simeq \mathbb{C}^n$ is given by

$$g \cdot z = (a + bz^t)^{-1}(c^t + zd^t)\,, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

Now, we fix a positive integer $m$ and consider the character $\chi$ of $H$ defined by

$$\chi \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = a^{-m}\,.$$

Then the corresponding unitary irreducible representation $\pi := \pi_\chi$ of $G$ has highest weight $-m\lambda_1$. Moreover, in this case, the reproducing kernel of $\mathcal{H}$ is

$$k(w, z) = e_z(w) = c_\pi(1 + \bar{z}w^t)^m\,.$$

Moreover, since $\Lambda = -(n+1)\lambda_1$ here, the $G$-invariant measure $\mu$ on $\mathfrak{n}^+ \simeq \mathbb{C}^n$ is given by

$$d\mu(z) = c(1+\bar{z}z^t)^{-(n+1)}dxdy$$

with the notation $z_k = x_k + iy_k$, $k = 1,2,\dots,n$ and $dx = dx_1dx_2\dots dx_n$, $dy = dy_1dy_2\dots dy_n$. The constant $c$ is then given by

$$c^{-1} = \int_{\mathbb{C}^n}(1+\bar{z}z^t)^{-(n+1)}dxdy = \frac{\pi^n}{n!}\,.$$

Thus we see that $\mathcal{H}$ is the space of all polynomials on $\mathbb{C}^n$ of degree $\leq m$ endowed this the Hilbert product

$$\langle f_1, f_2\rangle = \frac{n!}{\pi^n}\int_{\mathbb{C}^n} f_1(z)\overline{f}_2(z)(1+\bar{z}z^t)^{-(m+n+1)}\,dx\,dy\,.$$

This implies, in particular, that $c_\pi = \dim(\mathcal{H}) = \binom{m+n}{m}$.

On the other hand, since we have

$$\alpha(g^{-1},z) = (a+bz^t)^m\,,\quad z\in\mathbb{C}^n,\ g^{-1} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G\,,$$

$\pi$ is given by

$$(\pi(g)f)(z) = (a+bz^t)^m f(g^{-1}\cdot z),\quad g^{-1} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\,.$$

Let $\beta_0$ be the bilinear form on $\mathfrak{g}$ defined by $\beta_0(X,Y) = \frac{1}{n+1}\operatorname{Tr}(XY)$. Let $X_0 = im\begin{pmatrix} n & 0\\ 0 & -I_n\end{pmatrix}\in\mathfrak{g}$ and $\xi_0 = \beta_0(X_0,\cdot)\in\mathfrak{g}^*$. Then we have $d\chi = i\xi_0$ on $\mathfrak{t}$ and $\pi$ is associated with the coadjoint orbit $O(\xi_0)$ of $\xi_0$. Moreover, if we introduce the section $z\to g_z$ for the action of $G$ on $\mathbb{C}^n$ defined by

$$g_z := \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} 1 & \bar{z}\\ -z^t & b(z)\end{pmatrix},\quad b(z) := \sqrt{1+|z|^2}I_n - (1+\sqrt{1+|z|^2})^{-1}z^t\bar{z}$$

then the map $\Psi : z\to \operatorname{Ad}^*(g_z)\xi_0$ is a diffeomorphism from $\mathbb{C}^n$ onto a dense open set of $O(\xi_0)$ such that for each $z\in\mathbb{C}^n$, $X\in\mathfrak{g}$, we have $S(d\pi(X))(z) = i\langle\Psi(z),X\rangle$ and, for each $z\in\mathbb{C}^n$, $g\in G$, we have $\Psi(g\cdot z) = \operatorname{Ad}^*(g)\Psi(z)$ [9]. More precisely, for each $z\in\mathbb{C}^n$, we have $\Psi(z) = \beta_0(X_z,\cdot)$ where

$$X_z := \frac{im}{1+|z|^2}\begin{pmatrix} n-|z|^2 & (n+1)\bar{z}\\ (n+1)z^t & (n+1)z^t\bar{z} - (1+|z\ |^2)I_n\end{pmatrix}\,.$$

We focus now on the case $m=1$ in which the explicit calculation of the Berezin transform on $\mathcal{S}$ is possible (the computation of $B$ for general $m$ is a difficult task). We need the following computational lemmas.

**Lemma 6.1.** (1) *For each real number $a\geq 0$, the volume of*

$$\left\{x = (x_1,x_2,\dots,x_n)\in[0,+\infty[^n\ :\ \sum_{i=1}^n x_i\leq a\right\}$$

*is $a^n/n!$.*

(2) *For each integer $p > n$, we have*

$$\int_{[0,+\infty[^n} \frac{1}{(1+\sum_{i=1}^n x_i)^p}\, dx = \frac{(p-n-1)!}{(p-1)!}\,.$$

(3) *For each $j, k = 1, 2, \ldots, n$, $j \neq k$, we have*

$$\int_{[0,+\infty[^n} \frac{x_j}{(1+\sum_{i=1}^n x_i)^{n+3}}\, dx = \frac{1}{(n+2)!}\,;$$

$$\int_{[0,+\infty[^n} \frac{x_j^2}{(1+\sum_{i=1}^n x_i)^{n+3}}\, dx = \frac{2}{(n+2)!}\,;$$

$$\int_{[0,+\infty[^n} \frac{x_j x_k}{(1+\sum_{i=1}^n x_i)^{n+3}}\, dx = \frac{1}{(n+2)!}\,.$$

**Proof.** (1) can be proved by induction on $n$. (2) can be deduced from (1) by making first the change of variables defined by $y_1 = \sum_{i=1}^n x_i$, $y_k = x_k$ for $k > 1$ and then the change $t = \frac{y_1}{1+y_1}$. (3) can be proved similarly. Details are left to the reader. □

**Lemma 6.2.** *For each $j, k = 1, 2, \ldots, n$, $j \neq k$, we have*

$$\int_{\mathbb{C}^n} \frac{1}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = \frac{2}{(n+2)!}\pi^n\,;$$

$$\int_{\mathbb{C}^n} \frac{z_j}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = \int_{\mathbb{C}^n} \frac{\bar{z_j}}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = 0\,;$$

$$\int_{\mathbb{C}^n} \frac{z_j \bar{z_k}}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = 0\,;$$

$$\int_{\mathbb{C}^n} \frac{|z_j|^2}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = \frac{1}{(n+2)!}\pi^n\,;$$

$$\int_{\mathbb{C}^n} \frac{|z_j z_k|^2}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = \frac{1}{(n+2)!}\pi^n\,;$$

$$\int_{\mathbb{C}^n} \frac{|z_j|^4}{(1+\sum_{i=1}^n |z_i|^2)^{n+3}}\, dx\, dy = \frac{2}{(n+2)!}\pi^n\,.$$

**Proof.** Passing to polar coordinates $z_j = r_j e^{i\theta_j}$, $j = 1, 2, \ldots, n$, we reduce the calculations of these integrals to those of the preceding lemma. □

The functions $1, z_1, z_2, \ldots, z_n$ constitute a basis of $\mathcal{H}$. For $0 \leq i,\ j \leq n$, we write $A_{ij}$ for the operator $\mathcal{H}$ whose matrix in this basis has $ij$-th entry 1 and all of the other entries 0. Then $\mathcal{S}$ is here spanned by

$$f_{00}(z) := S(A_{00})(z) = \frac{1}{1+\sum_{i=1}^n |z_i|^2}\,; \quad f_{0j}(z) := S(A_{0j})(z) = \frac{\bar{z_j}}{1+\sum_{i=1}^n |z_i|^2}\,;$$

$$f_{i0}(z) := S(A_{i0})(z) = \frac{z_i}{1+\sum_{i=1}^n |z_i|^2}\,; \quad f_{ij}(z) := S(A_{ij})(z) = \frac{z_i \bar{z_j}}{1+\sum_{i=1}^n |z_i|^2}$$

for $i, j = 1, 2, \ldots, n$. We have the following result.

**Proposition 6.3.** (1) *For $i, j = 1, 2, \ldots, n$, we have*

$$B(f_{00}) = \frac{1}{n+2}(1 + f_{00}); \quad B(f_{0j}) = \frac{1}{n+2} f_{0j}; \quad B(f_{i0}) = \frac{1}{n+2} f_{i0};$$
$$B(f_{ii}) = \frac{1}{n+2}(1 + f_{ii}); \quad B(f_{ij}) = \frac{1}{n+2} f_{ij},\, i \neq j.$$

(2) *The functions $F_{00} = 1$, $F_{0j} = \sqrt{(n+1)(n+2)} f_{0j}$, $F_{i0} = \sqrt{(n+1)(n+2)} f_{i0}$, $F_{ij} = \sqrt{(n+1)(n+2)} f_{ij}$ $(i \neq j)$ and $F_{ii} = \sqrt{\frac{1}{2}(n+1)(n+2)}(f_{00} - f_{ii})$ for $i, j = 1, 2, \ldots, n$ form an orthonormal basis of $\mathcal{S}$ consisting of eigenfunctions of $B$, the corresponding eigenvalues being $\frac{1}{n+1}$ for $F_{00}$ and $\frac{1}{n+2}$ for $F_{ij}$ with $(i, j) \neq (0, 0)$.*

(3) *The corresponding operators $D_{ij}$ of $\mathcal{H}$ are given by $D_{00} = \frac{1}{\sqrt{n+1}} \mathrm{Id}_{\mathcal{H}}$, $D_{ij} = \sqrt{n+1} A_{ij}$ and $D_{ii} = \sqrt{\frac{1}{2}(n+1)}(A_{00} - A_{ii})$ for $i, j = 1, 2, \ldots, n$, $i \neq j$.*

**Proof.** Taking Lemma 6.2 into account, (1) immediately follows from the integral formula for $B$ (see (5) of Proposition 3.2) which is here

$$B(F)(z) = \frac{(n+1)!}{\pi^n} \int_{\mathbb{C}^n} F(w) \frac{(1 + \bar{z}w)(1 + \bar{w}z)}{(1 + |z|^2)(1 + |w|^2)} (1 + |w|^2)^{-(n+1)}\, dx\, dy$$

and (2) from the computations of the $\|f_{ij}\|$ for $i, j = 0, 1, \ldots, n$. Finally, to prove (3), we have just to use (1) and (3) of Lemma 4.3. $\square$

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