ARCHIVUM MATHEMATICUM (BRNO)
Tomus 55 (2019), 195–201

ON INSTANCES OF FOX'S INTEGRAL EQUATION CONNECTION TO THE RIEMANN ZETA FUNCTION

Alexander E. Patkowski

Abstract. We consider some applications of the singular integral equation of the second kind of Fox. Some new solutions to Fox's integral equation are discussed in relation to number theory.

1. Introduction

The Fredholm singular integral equation of the second kind has the form

$$\Delta(x) = f(x) + \int_0^\infty k(x,t)\Delta(t)\,dt\,, \tag{1.1}$$

where $0 < x < \infty$. Integral equations of this form are known to have many applications, particularly in boundary value problems. Fox [2] studied the special case $k(x,t) = k(xt)$, and noted that a simple method to solve (1.1) would be to employ Mellin transforms. The general (formal) solution is given by

$$\Delta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\bar{f}(s) + \bar{k}(s)\bar{f}(1-s)}{1-\bar{k}(s)\bar{k}(1-s)} x^{-s}\,ds\,, \tag{1.2}$$

where the bar signifies the Mellin transform of a suitable function f

$$\bar{f}(s) := \int_0^\infty t^{s-1} f(t)\,dt\,. \tag{1.3}$$

Here we must assume that each of the Mellin transforms in (1.2) exist. The kernels $k(x,t) = \dot{k}(xt) := \sin(xt)$, and $k(x,t) = \ddot{k}(xt) := \cos(xt)$, correspond to the theory of Fourier sine and cosine transforms respectively [4]. In fact, the reader who is familiar with such transforms might easily recognize solutions to (1.1) from tables [2]. Indeed, the Fourier sine transform of $(e^{2\pi t}-1)^{-1}$, is given by $w > 0$

$$\int_0^\infty \frac{\sin(wt)\,dt}{e^{2\pi t}-1} = \frac{1}{2}\Big(\frac{1}{e^w-1} + \frac{1}{2} - \frac{1}{w}\Big)\,. \tag{1.4}$$

2010 *Mathematics Subject Classification*: primary 42A38; secondary 11L20.

Key words and phrases: Fourier integrals, Fox's integral equation, Riemann prime counting function.

Received October 26, 2018, revised April 2019. Editor G. Teschl.

DOI: 10.5817/AM2019-3-195

Hence, a formal solution to the singular Fredholm integral equation of the second kind

$$\Delta(x) = f(x) + \int_0^\infty \sin(xt)\Delta(2\pi t)\,dt\,, \tag{1.5}$$

where

$$f(x) = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{x}\Big)$$

is

$$\Delta(x) = \frac{1}{2}\frac{1}{e^x - 1}\,. \tag{1.6}$$

Instances of this sort appear to be ubiquitous. Furthermore, solutions to the case $f(x) = 0$ in (1.1), which gives the first-order homogeneous equation, are essentially Fourier pairs when we choose the kernels $\dot{k}(xt)$, $\ddot{k}(xt)$.

From the above observations we may make the following statement. Let $\zeta(s) = \sum_{n\geq 1} n^{-s}$ denote the Riemann zeta function [6].

Proposition 1.1. *Put $\dot{\gamma}(x) := \frac{\bar{h}(1)}{x} - \frac{h(0+)}{2}$, where $\bar{h}$ is the Mellin transform of h, and h is bounded at the origin. If $\Delta(x) = \sum_{n\geq 1} h(nx)$, then the non-homogeneous second-order integral equation*

$$\Delta(x) = \dot{\gamma}(x) + \int_0^\infty k(xt)\Delta(2\pi t)\,dt\,, \tag{1.7}$$

may be transformed into the functional equation

$$\bar{h}(s)\zeta(s) = (2\pi)^{s-1}\bar{k}(s)\zeta(1-s)\bar{h}(1-s)\,. \tag{1.8}$$

Furthermore, if $k(x) = 2\sin(x)$ then

$$\bar{h}(s)\zeta(s) = 2(2\pi)^{s-1}\Gamma(s)\sin\Big(\frac{\pi}{2}s\Big)\zeta(1-s)\bar{h}(1-s)\,. \tag{1.9}$$

Proof. If we assume $\dot{\gamma}(x)$ is as defined in the proposition, and the solution may be written as the series $\Delta(x) = \sum_{n\geq 1} h(nx)$, then we may write (for $-\epsilon < c < 0$, $\epsilon > 0$)

$$\sum_{n\geq 1} h(nx) - \frac{\bar{h}(1)}{x} + \frac{h(0+)}{2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{h}(s)\zeta(s)x^{-s}\,ds\,. \tag{1.10}$$

by [4, p.118, eq.(4.1.3)]. Taking Mellin transforms of both sides of (1.10) over the appropriate region gives us

$$\int_0^\infty t^{s-1}\Big(\sum_{n\geq 1} h(nx) - \frac{\bar{h}(1)}{x} + \frac{h(0+)}{2}\Big)\,dt = \bar{h}(s)\zeta(s)\,, \tag{1.11}$$

for $-\epsilon < \Re(s) < 0$. Therefore, after rearranging (1.7), taking Mellin transforms and invoking the series for $\Delta(x)$, we obtain the proposition. In the case of (1.9), we use the known Mellin transform

$$\int_0^\infty t^{s-1}\sin(xt)\,dt = \frac{\Gamma(s)\sin(\frac{\pi}{2}s)}{x^s}\,, \tag{1.12}$$

where $-1 < \Re(s) < 1$. □

Note that putting $h(x) = e^{-x}$ in Proposition 1.1 implies (1.4) may be transformed into the functional equation for the Riemann zeta function [6]:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\Big(\frac{\pi}{2}s\Big)\zeta(1-s)\,. \tag{1.13}$$

Appropriate modifications may be made to (1.7) in the proposition to work with different kernels in the functional equation (1.8). This idea would of course be used in studying other Dirichlet series.

2. A slightly modified Fox integral equation

Here we consider our main object of study and re-write the Fox equation as

$$\pi\Delta(ax) = -f(x) + \int_0^\infty k(xt)\Delta(t)\,dt\,, \tag{2.1}$$

$a \in \mathbb{R}$, and put $k(x) = \sin(x)$. In this case taking Mellin transforms gives us

$$\pi a^{-s}\bar{\Delta}(s) = -\bar{f}(s) + \Gamma(s)\sin(\frac{\pi}{2}s)\bar{\Delta}(1-s)\,. \tag{2.2}$$

Replacing s by $1-s$ and applying some standard computations toward the formal solution (1.2), we have that

$$\bar{\Delta}(s) = \frac{-\pi\bar{f}(s)}{\pi^2 a^{-s} - a^{1-s}\frac{\pi}{2}} - \frac{a^{1-s}}{\pi^2 a^{-s} - a^{1-s}\frac{\pi}{2}}\bar{k}(s)\bar{f}(1-s)\,. \tag{2.3}$$

Hence,

$$\Delta(x) = \frac{-\pi f(x/a)}{\pi^2 - a\frac{\pi}{2}} - \frac{a}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(xt)f(t)\,dt\,. \tag{2.4}$$

Inserting (2.4) into (2.1) we find

$$\begin{aligned}
\pi\Delta(ax) &= -f(x) + \int_0^\infty k(xt)\Delta(t)\,dt \\
&= -f(x) + \int_0^\infty \sin(xt)\Big(\frac{-\pi f(t/a)}{\pi^2 - a\frac{\pi}{2}} - \frac{a}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(yt)f(y)dy\Big)\,dt \\
&= -f(x) - \frac{\pi}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(xt)f(t/a)\,dt \\
&\quad - \frac{a}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty\int_0^\infty \sin(xt)\sin(yt)f(y)dy\,dt \\
&= -f(x) - \frac{\pi}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(xt)f(t/a)\,dt - \frac{a\frac{\pi}{2}}{\pi^2 - a\frac{\pi}{2}}f(x) \\
&= -f(x) - \frac{\pi a}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(axt)f(t)\,dt - \Big(\frac{\pi^2}{\pi^2 - a\frac{\pi}{2}} - 1\Big)f(x) \\
&= \frac{-\pi a}{\pi^2 - a\frac{\pi}{2}}\int_0^\infty \sin(axt)f(t)\,dt - \frac{\pi^2}{\pi^2 - a\frac{\pi}{2}}f(x)\,.
\end{aligned}$$

Now replacing x by x/a, and dividing both sides by π in our resulting equation gives (2.4). Therefore (2.4) is a formal solution to (2.1). However, it should be noted that the solution has a singularity at $a = 2\pi$, and so a different treatment appears to be warranted in this case. As it turns out (2.1) when $a = 2\pi$ has some relevance to number theory.

It is easily shown through the calculus of residues applied to the function

$$-\frac{(2\pi t)^{-x}\Gamma(x)}{x\zeta(1-x)}$$

that, valid for $t > 0$,

$$R(e^{-2\pi t}) = \frac{1}{\pi}\sum_{n\geq 1}\frac{(-1)^{n-1}t^{-2n-1}}{(2n+1)\zeta(2n+1)} + \frac{1}{2}\sum_{\rho}\frac{t^{-\rho}}{\rho\cos(\pi\rho/2)\zeta'(\rho)}\,, \tag{2.5}$$

where $\mu(n)$ is the Möbius function. The sum over ρ is taken over all the non-real zeros of $\zeta(s)$, with the added assumption that they are all simple. Here $R(x)$ is Riemann's approximation to the prime counting function (which counts the number of prime numbers $\leq x$), typically written in the form

$$R(x) = \sum_{n\geq 1}\frac{\mu(n)}{n}L(x^{1/n})\,,$$

where $L(x)$ is the logarithmic integral

$$L(x) := \int_{\mu'}^{x}\frac{dt}{\ln(t)}\,.$$

Here μ' is Soldner's constant. Note also that $L'(e^{-2\pi x/n}) = x^{-1}e^{-2\pi x/n}$. We write (for $x > 1$)

$$\begin{aligned}\int_0^\infty \sin(xt)\sum_{n\geq 1}\frac{\mu(n)}{n}e^{-t/n}dt &= \sum_{n\geq 1}\frac{\mu(n)}{n}\frac{x}{x^2+1/n^2}\\ &= \frac{1}{x}\sum_{n\geq 1}\frac{\mu(n)}{n}\sum_{k\geq 0}(xn)^{-2k}(-1)^k\\ &= \frac{1}{x}\sum_{n\geq 0}\frac{(-1)^n x^{-2n}}{\zeta(2n+1)}\\ &= -x\frac{d}{dx}\Big(\frac{\pi}{2}\sum_{\rho}\frac{x^{-\rho}}{\rho\cos(\pi\rho/2)\zeta'(\rho)} - \pi R(e^{-2\pi x})\Big)\,.\end{aligned}$$

Here the expansion of $1/(1+(nx)^{-2})$ has been employed which is valid for $nx > 1$. As $n \geq 1$ this implies $x > 1$. Further, we have used the fact that $1/\zeta(1) = 0$. We therefore have the following solution.

Theorem 2.1. *Let $x > 1$, and assume that the zeros of $\zeta(s)$ are simple. A solution to the singular Fredholm integral equation of the second kind (2.1) when $a = 2\pi$,* *and*

$$f(x) = \frac{\pi}{2}\sum_{\rho} \frac{x^{-\rho}}{\cos(\pi\rho/2)\zeta'(\rho)}\,, \tag{2.6}$$

is given by

$$\Delta(x) = \sum_{n\geq 1} \frac{\mu(n)}{n} e^{-x/n}\,. \tag{2.7}$$

3. Comments

The Fredholm alternative theorem [7, p.23, Theorem 1.3.4] implies that, if the homogeneous integral equation

$$\pi\Delta(2\pi x) = \int_0^\infty \sin(xt)\Delta(t)\,dt \tag{3.1}$$

has nontrivial solutions (i.e. $\Delta(x)\neq 0$), then (2.1) with $a = 2\pi$ has infinitely many solutions or no solution at all. From Titchmarsh [5, p.23, eq. (2.7.1)–(2.7.3)] we have that (for $0 < \Re(s) < 1$)

$$\zeta(s)\Gamma(s) = \int_0^\infty t^{s-1}\Big(\frac{1}{e^t - 1} - \frac{1}{t}\Big)\,dt\,, \tag{3.2}$$

and for $x > 0$

$$\frac{1}{e^{2\pi x} - 1} - \frac{1}{2\pi x} = \frac{1}{\pi}\int_0^\infty \sin(xt)\Big(\frac{1}{e^t - 1} - \frac{1}{t}\Big)\,dt\,. \tag{3.3}$$

We may simply use (3.3) directly. (Alternatively, we may take the Mellin transform of (3.1) over the region $0 < \Re(s) < 1$, invoke (3.2), and then find the desired $\Delta(x)$ using the functional equation (1.13).) Therefore, in conjunction with our solution in Theorem 2.1, the Fredholm alternative theorem tells us there are infinitely many solutions of (2.1) when $a = 2\pi$.

4. An associated integral

Here we offer an interesting integral involving the series in (2.7). Define the sine integral as $S(x) := \int_0^x \frac{\sin(t)}{t}\,dt$.

Theorem 4.1. *For a positive real number a,*

$$\int_0^\infty \frac{\{t\}}{t}\Big(\sum_{n\geq 1} \frac{\mu(n)}{n} e^{-t/(na)}\Big)\,dt = -\frac{1}{2} + \frac{1}{\pi}\int_0^\infty S(\frac{x}{2\pi a})e^{-x}\,dx\,. \tag{4.1}$$

Proof. From Ivic [3, eq.(13)], we have for $0 < c < 1$,

$$\int_0^\infty \frac{\{t\}}{t} e^{-t/a}\,dt = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s} a^s\Gamma(s)\,ds\,, \tag{4.2}$$

where $\{t\}$ denotes the fractional part of t. We move the line of integration of the integral on the right side of (4.2) to the left $\Re(s) = b$, $-1 < b < 0$, and compute the residue from the pole of order two at $s=0$. We use $\zeta(1)^{-1}=0$, $\frac{d}{ds}\zeta(s)^{-1}|_{s=0}=-1$, and $\Gamma'(1)=-\gamma$ to get

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s)}{s}a^s\Gamma(s)\,ds = \frac{1}{2}\log(a)-\frac{1}{2}\gamma+\frac{1}{2}\log(2\pi)-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta(s)}{s}a^s\Gamma(s)\,ds\,. \tag{4.3}$$

Using uniform convergence of $\zeta(v)^{-1}=\sum_{n\ge1}\mu(n)/n^v$ for $\Re(v)\ge1$, we have

$$\int_0^\infty\frac{\{t\}}{t}\Big(\sum_{n\ge1}\frac{\mu(n)}{n}e^{-t/(na)}\Big)\,dt=-\frac{1}{2}-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta(s)}{s\zeta(1-s)}a^s\Gamma(s)\,ds\,, \tag{4.4}$$

for $-1<b<0$. This is justified through use of the known formulas $\sum_{n\ge1}\mu(n)n^{-1}=0$ and $\sum_{n\ge1}\mu(n)n^{-1}\log(n)=-1$. Using (1.13) we write the right side of (4.4) as

$$-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta(s)}{s\zeta(1-s)}a^s\Gamma(s)\,ds = -\frac{1}{2\pi^2 i}\int_{b-i\infty}^{b+i\infty}\frac{\Gamma(1-s)\Gamma(s)\sin(\frac{\pi}{2}s)}{s}(2\pi a)^s\,ds\,. \tag{4.5}$$

If $S(x)$ denotes the sine integral, then we have by Parseval's theorem [4, p.83]

$$\frac{1}{\pi}\int_0^\infty S(\frac{x}{2\pi a})e^{-x}x^{s-1}\,dx = -\frac{1}{2\pi^2 i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\Gamma(s-r)\Gamma(r)\sin(\frac{\pi}{2}r)}{r}(2\pi a)^r\,dr\,, \tag{4.6}$$

where c_1 is a real number confined to the strip of holomorphy when both $0>c_1>-1$ and $(s-c_1)>0$. We put $s=1$ in (4.6) and note that we have set $-1<c_1<0$. Comparison of our resulting integrals gives the result. □

If we instead work with the integral

$$-\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\zeta(s)}{s}z^s\,ds=\{z\}-\frac{1}{2}\,, \tag{4.7}$$

valid for $-1<c'<0$, we may deduce a slightly different form of Theorem 4.1.

Theorem 4.2. *For a positive real number a,*

$$\int_0^\infty\frac{\{t\}-\frac{1}{2}}{t}\Big(\sum_{n\ge1}\frac{\mu(n)}{n}(e^{-t/(na)}-1)\Big)\,dt=\frac{1}{\pi}\int_0^\infty S(\frac{x}{2\pi a})e^{-x}\,dx\,. \tag{4.8}$$

We further note that the right hand side of (4.8) may be evaluated by the known result [1, p.177, eq.(18)]

$$\int_0^\infty e^{-at}S(t)\,dt=\frac{1}{a}\tan^{-1}\Big(\frac{1}{a}\Big)\,.$$

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1390 Bumps River Rd.,
Centerville, MA 02632, USA
E-mail: alexpatk@hotmail.com