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# SOME ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF LIPSCHITZ ALGEBRAS AND THEIR SECOND DUALS

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Abstract. Let $(X,d)$ be a metric space and $\alpha > 0$. We study homological properties and different types of amenability of Lipschitz algebras $\operatorname{Lip}_\alpha X$ and their second duals. Precisely, we first provide some basic properties of Lipschitz algebras, which are important for metric geometry to know how metric properties are reflected in simple properties of Lipschitz functions. Then we show that all of these properties are equivalent to either uniform discreteness or finiteness of $X$. Finally, some results concerning the character space and Arens regularity of Lipschitz algebras are provided.

## 1. Introduction

Let $(X,d)$ be a metric space and $B(X)$ (resp. $C_b(X)$) indicates the Banach space, consisting of all bounded (resp. continuous and bounded) complex valued functions on $X$, endowed with the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \quad (f \in C_b(X))\,.$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then $\operatorname{Lip}_\alpha X$ is the subspace of $C_b(X)$, consisting of all functions $f$ such that

$$p_\alpha(f) := \sup\Big\{\frac{|f(x)-f(y)|}{d(x,y)^\alpha} : x,y \in X,\ \ x \neq y\Big\} < \infty\,. \tag{1.1}$$

It is known that $\operatorname{Lip}_\alpha X$, endowed with the norm $\|\cdot\|_\alpha$, given by

$$\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty\,,$$

and pointwise product is a unital commutative Banach algebra, called Lipschitz algebra. Moreover, following [19], $\operatorname{lip}_\alpha X$ is the subalgebra of $\operatorname{Lip}_\alpha X$, consisting of all functions $f \in \operatorname{Lip}_\alpha X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\dfrac{|f(x)-f(y)|}{d(x,y)^\alpha} < \varepsilon$, whenever $0 < d(x,y) < \delta$. Lipschitz algebras were first considered by Sherbert [19] and then Johnson [11, 10], for $0 < \alpha \leq 1$. But in the recent works, these algebras were introduced and studied for any $\alpha > 0$. However,

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in the earlier works, the condition of (1.1), for $\alpha > 1$, has not been known as a Lipschitz condition, but a Hölder condition of exponent $\alpha$.

There are valuable works related to some notions of amenability of Lipschitz algebras. Let $\mathcal{A}$ be a Banach algebra and $\Delta(\mathcal{A})$ be the character space of $\mathcal{A}$, consisting of all nonzero multiplicative functionals on $\mathcal{A}$. Gourdeau [7] proved that if $\mathcal{A}$ is amenable, then $\Delta(\mathcal{A})$ is uniformly discrete with respect to norm topology, induced by $\mathcal{A}^*$; see also Bade, Curtis, and Dales [3] and Zhang [20]. Moreover, Hu, Monfared, and Traynor investigated character amenability of Lipschitz algebras [9]. They showed that if $X$ is an infinite compact metric space and $0 < \alpha < 1$, then $\operatorname{Lip}_\alpha X$ is not character amenable. Recently, $C-$character amenability of $\operatorname{Lip}_\alpha X$ ($\alpha > 0$), was studied by Dashti, Nasr Isfahani and Soltani [5, Theorem 3.1]. In fact as a generalization of [9], they showed that for $\alpha > 0$ and any locally compact metric space $X$, $\operatorname{Lip}_\alpha X$ is $C-$character amenable, for some $C > 0$, if and only if $X$ is $\varepsilon$-uniformly discrete, for some $\varepsilon > 0$. Furthermore, in a recent work [1], the first and third authors joint with Azizi, fully investigated the structure of Lipschitz algebras and their arbitrary intersections. Also a necessary and sufficient condition for amenability and character amenability of Lipschitz algebras was provided.

In the present paper, we first study some useful properties of Lipschitz algebras. Then we continue our study on the more general notions of amenability and also homological properties of Lipschitz algebras and their second duals. We study approximate amenability, pseudo-amenability, approximate contractibility, pseudo-contractibility and (approximate) biprojectivity and biflatness of these algebras and present some necessary and sufficient conditions. In fact, we show that all of these properties are equivalent to uniform discreteness or finiteness of $X$. Moreover, we study weak amenability of Lipschitz algebras and as a main result we show that weak amenability of $\operatorname{Lip}_\alpha X$ implies the discreteness of $X$, whenever $(X, d)$ is any metric space and $0 < \alpha \leq 1$. This result is in fact a generalization of [4, Corollary 4.4.33], that in which $X$ is a compact metric space. Finally, some results related to character space, semisimplicity and Arens regularity of Lipschitz algebras and their second duals are provided.

## 2. Basic results

Let $X$ be a metric space and $\alpha > 0$. It is easily verified that if $\mathbb{R}$ is endowed with usual Euclidean metric and $\alpha > 1$, then $\operatorname{Lip}_\alpha \mathbb{R}$ is just $\operatorname{Cons}(\mathbb{R})$, the space consisting of all constant functions on $\mathbb{R}$. However, constant functions are not the only functions which belong to Lipschitz algebras, for the general metric spaces. For instance, take $X$ to be the subset $[0, 1] \cup \{2\}$ of $\mathbb{R}$, endowed with the induced Euclidean metric. For example consider the function $f$, defined as

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 2 & x = 2\,. \end{cases}$$

Then

$$p_2(f) = \sup_{x \in [0,1]} \frac{1}{(x-2)^2} = 1\,,$$

which implies that $f \in \operatorname{Lip}_2 X$, whereas $f$ is not a constant function. This fact encourages us to consider Lipschitz algebras for the case where $\alpha > 1$.

In this section, we present some special and useful properties of Lipschitz algebras, which will be used in the next sections.

A subalgebra $\mathcal{A}$ of $C_b(X)$ is called weakly separating the points of $X$ (or briefly weakly separating) if for all $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Moreover, $\mathcal{A}$ is called strongly separating the points of $X$ (or briefly strongly separating) if for all $x$, $y \in X$ with $x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Also recall that $\delta_x$ represents the Dirac function at $x$ ($x \in X$), belonging to $(\operatorname{Lip}_\alpha X)^*$, defined by $\delta_x(f) = f(x)$, for all $f \in \operatorname{Lip}_\alpha X$. We commence this section with the following result. Note that this result is well known, for the case where $\alpha = 1$; see [18, Lemma 3.1].

**Proposition 2.1.** *Let $(X, d)$ be a metric space, and $\alpha > 0$. Then the following statements are equivalent:*

(i) $\operatorname{Lip}_\alpha X$ *is strongly separating the points of* $X$.

(ii) *The functionals* $\delta_x$ $(x \in X)$ *are distinct in* $(\operatorname{Lip}_\alpha X)^*$.

(iii) *The set* $\{\delta_x : \ x \in X\}$ *is linearly independent in* $(\operatorname{Lip}_\alpha X)^*$.

**Proof.** (i)⇒(iii) First, note that for all $x$, $y \in X$ with $x \neq y$, there is $f \in \operatorname{Lip}_\alpha X$ such that $f(x) = 0$ and $f(y) \neq 0$. Indeed, by the hypothesis there is $g \in \operatorname{Lip}_\alpha X$ such that $g(x) \neq g(y)$. Then the function $f$ defined as $f(t) = g(t) - g(x)$ is the desired function. Now suppose that $x_1, x_2, \ldots, x_n \in X$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ such that $\alpha_1 \delta_{x_1}(f) + \alpha_2 \delta_{x_2}(f) + \cdots + \alpha_n \delta_{x_n}(f) = 0$. We show that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0\,.$$

For all $x_k$ $(k = 1, \ldots, n)$, there are the functions $f_k \in \operatorname{Lip}_\alpha X$ such that

$$f_k(x_1) \neq 0 \quad \text{and} \quad f_k(x_k) = 0 \quad (k = 2, \ldots, n)\,.$$

Let $f = f_1 f_2 \ldots f_n$. Then $f \in \operatorname{Lip}_\alpha X$ such that $f(x_1) \neq 0$ and $f(x_i) = 0$, for all $i = 2, \ldots, n$. Since

$$\alpha_1 \delta_{x_1}(f) + \alpha_2 \delta_{x_2}(f) + \cdots + \alpha_n \delta_{x_n}(f) = 0\,,$$

it follows that

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n) = 0\,,$$

which implies $\alpha_1 f(x_1) = 0$ and so $\alpha_1 = 0$. Similarly one can obtain that

$$\alpha_2 = \cdots = \alpha_n = 0\,.$$

Consequently the set $\{\delta_x : \ x \in X\}$ is linearly independent in $(\operatorname{Lip}_\alpha X)^*$.

(iii)⇒(ii) It is clear.

(ii)⇒(i) Suppose that $x, y \in X$ with $x \neq y$. By (ii) we have $\delta_x \neq \delta_y$. Consequently, there exists $f \in \operatorname{Lip}_\alpha X$ such that $\delta_x(f) \neq \delta_y(f)$, which implies $f(x) \neq f(y)$. Therefore $\operatorname{Lip}_\alpha X$ is strongly separating the points of $X$. □

**Corollary 2.2.** *Let $(X, d)$ be a metric space and $\alpha > 0$. Then the following statements are equivalent:*

(i) $\operatorname{Lip}_\alpha X$ *is finite dimensional and strongly separating the points of* $X$*.*

(ii) $X$ *is finite.*

**Proof.** (i)$\Rightarrow$(ii) By Proposition 2.1, $\{\delta_x : x \in X\}$ is a linearly independent subset of the finite dimensional space $(\operatorname{Lip}_\alpha X)^*$. Thus $\{\delta_x : x \in X\}$ is finite, which implies the finiteness of $X$.

(ii)$\Rightarrow$(i) It is obvious. □

**Remark 2.3.** It is clear that every strongly separating Banach algebra $\mathcal{A}$ is also weakly separating. In fact, suppose that $x \in X$. Take $y \in X$ with $x \neq y$. So there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Define $g(z) := f(z) - f(y)$ $(z \in X)$. Thus $g \in \mathcal{A}$ and $g(x) = f(x) - f(y) \neq 0$. The converse of the above statement is not necessarily valid. For example, take $X$ to be the real line $\mathbb{R}$, endowed with the usual Euclidean metric. It is not hard to see that $\operatorname{Lip}_2 \mathbb{R} = \operatorname{Cons}(\mathbb{R})$. It follows that $\operatorname{Lip}_2 \mathbb{R}$ is weakly separating, but not strongly separating. Note that for an arbitrary metric space $(X, d)$ and $\alpha > 0$, since $\operatorname{Lip}_\alpha X$ and $\operatorname{lip}_\alpha X$ contain the space of constant functions on $X$, denoted by $\operatorname{Cons}(X)$, thus both are weakly separating the points of $X$.

**Examples 2.4.** The following examples show that the assumptions, given in Proposition 2.1 and Corollary 2.2 are necessary.

(1) Take $X$ to be $\mathbb{R}$, with the usual Euclidean metric and $\alpha > 1$. Then

$$\operatorname{Lip}_\alpha X = \operatorname{Cons}(X)\,,$$

which is finite dimensional, whereas $X$ is not finite. Note that $\operatorname{Cons}(X)$ is not strongly separating.

(2) Let $X = \mathbb{R}$ with the usual discrete metric, defined as

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y\,. \end{cases}$$

Then $\operatorname{Lip}_\alpha \mathbb{R} = B(\mathbb{R})$, for all $\alpha > 0$. Thus in this case, all the Lipschitz spaces $\operatorname{Lip}_\alpha X$ are strongly separating the points of $X$, but no one is finite dimensional. Moreover $X$ is not finite. This example shows that in Corollary 2.2, the condition of being finite dimensional for $\operatorname{Lip}_\alpha X$ can not be removed.

Recall that $X$ is called $\varepsilon$-uniformly discrete, for some $\varepsilon > 0$, if $d(x, y) \geq \varepsilon$, for all $x$, $y \in X$ with $x \neq y$. We say that $X$ is uniformly discrete, if it is $\varepsilon$-uniformly discrete, for some $\varepsilon > 0$.

The following lemma is useful in its own right.

**Lemma 2.5.** *Let* $(X, d)$ *be a metric space and* $0 < \alpha$*,* $\beta \leq 1$*. Then* $\operatorname{Lip}_\alpha X = \operatorname{Lip}_\beta X$ *if and only if either* $(X, d)$ *is uniformly discrete or* $\alpha = \beta$*.*

**Proof.** Suppose that $\operatorname{Lip}_\alpha X = \operatorname{Lip}_\beta X$ and $\alpha < \beta$. By [11, page 1] and [1, Corollary 2.4] we have

$$\operatorname{Lip}_\beta X \subseteq \operatorname{lip}_\alpha X \subseteq \operatorname{Lip}_\alpha X\,.$$

It follows that $\operatorname{Lip}_\alpha X = \operatorname{lip}_\alpha X$ and so $X$ is uniformly discrete, by [11, Lemma 2.5]. Conversely, suppose that $X$ is uniformly discrete. Therefore again by [11, Lemma 2.5], we have $\operatorname{Lip}_\alpha X = \operatorname{Lip}_\beta X = B(X)$. □

**Proposition 2.6.** *Let $(X, d)$ be a metric space and $\alpha$, $\beta > 0$. Then $\operatorname{Lip}_\alpha X$ is a Banach $\operatorname{Lip}_\beta X-$bimodule if and only if $(X, d)$ is uniformly discrete or $\alpha \leq \beta$.*

**Proof.** If $(X, d)$ is uniformly discrete, then $\operatorname{Lip}_\alpha X = \operatorname{Lip}_\beta X = B(X)$, by [1, Proposition 2.1]. It follows that $\operatorname{Lip}_\alpha X$ is a Banach $\operatorname{Lip}_\beta X-$bimodule. Now suppose that $\alpha \leq \beta$. Then by [1, Corollary 2.4], $\operatorname{Lip}_\beta X \subseteq \operatorname{Lip}_\alpha X$ and so

$$\operatorname{Lip}_\alpha X\ \operatorname{Lip}_\beta X \subseteq \operatorname{Lip}_\alpha X\ \operatorname{Lip}_\alpha X = \operatorname{Lip}_\alpha X .$$

It follows that $\operatorname{Lip}_\alpha X$ is a Banach $\operatorname{Lip}_\beta X-$bimodule. For the converse, suppose that $\operatorname{Lip}_\alpha X$ is a Banach $\operatorname{Lip}_\beta X-$bimodule and $\alpha > \beta$. Then by [1, Corollary 2.4], $\operatorname{Lip}_\alpha X \subseteq \operatorname{Lip}_\beta X$. Moreover, since $\operatorname{Lip}_\alpha X\ \operatorname{Lip}_\beta X \subseteq \operatorname{Lip}_\alpha X$, and $1 \in \operatorname{Lip}_\alpha X$, we obtain $\operatorname{Lip}_\beta X \subseteq \operatorname{Lip}_\alpha X$. It follows that $\operatorname{Lip}_\beta X = \operatorname{Lip}_\alpha X$. Now Lemma 2.5 implies that $(X, d)$ is uniformly discrete. This completes the proof. □

## 3. Homological properties and Character space of Lipschitz algebras

In this section, we study some various notions of amenability and homological properties of Lipschitz algebras. We first provide some preparations.

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. A derivation is a bounded linear map $D\colon \mathcal{A} \to X$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$ $(a, b \in \mathcal{A})$. For $x \in X$, the map $ad_x\colon \mathcal{A} \to X$ defined as $ad_x(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$ is clearly a derivation on $\mathcal{A}$, called an inner derivation. A derivation $D$ is called approximately inner if there is a net $(x_\alpha)$ in $X$ such that $D(a) = \lim_\alpha ad_{x_\alpha}(a)$, for all $a \in \mathcal{A}$.

The groundwork for amenability of Banach algebras was laid by Johnson in [10]. In fact a Banach algebra $\mathcal{A}$ is called contractible (or super amenable) if for each Banach $\mathcal{A}$-bimodule $X$, every continuous derivation $D\colon \mathcal{A} \to X$ is inner. Also $\mathcal{A}$ is called (approximately) amenable if, for each Banach $A$-bimodule $X$, every continuous derivation $D\colon \mathcal{A} \to X^*$ is (approximately) inner. Note that $X^*$ is the dual space of $X$. Moreover $\mathcal{A}$ is called weakly amenable if every derivation $D\colon \mathcal{A} \to \mathcal{A}^*$ is inner.

To confirm or rule out whether or not a given Banach algebra is amenable, it is often difficult to use the above definitions of amenability and contractibility. Following [6], $\mathcal{A}$ is amenable (resp. pseudo-amenable) if and only if it has a bounded (resp. not necessarily bounded) approximate diagonal, i.e., a net $(m_\lambda)$ in the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\|am_\lambda - m_\lambda a\|_{\mathcal{A}\widehat{\otimes}\mathcal{A}} \to_\lambda 0$ and $\|a\pi_{\mathcal{A}}(m_\lambda) - a\|_{\mathcal{A}} \to_\lambda 0$, for each $a \in \mathcal{A}$. Here and in the sequel, $\pi_{\mathcal{A}}$ always denotes the product morphism from $\mathcal{A}\widehat{\otimes}\mathcal{A}$ into $\mathcal{A}$, specified by $\pi_{\mathcal{A}}(a \otimes b) = ab$. Similarly, $\mathcal{A}$ is contractible (resp. pseudo-contractible) if and only if it has a diagonal (resp. a central approximate diagonal), i.e., an element $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ for which $am = ma$ and $\pi_{\mathcal{A}}(m)a = a$, (resp. an approximate diagonal $(m_\lambda)$, satisfying $am_\lambda = m_\lambda a$) for all $a \in \mathcal{A}$ and all $m_\lambda$. We also refer to [4], for a full information about projective tensor product of Banach algebras.

Let $\phi \in \Delta(\mathcal{A})$. The concept of $\phi$-amenability for Banach algebras was introduced by Kaniuth et al. [13]. Precisely, $\mathcal{A}$ is called $\phi$-amenable if there exists a bounded linear functional $m$ on $\mathcal{A}^*$ satisfying

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = m(f)\phi(a)\,,$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$, for all $b \in \mathcal{A}$. Any such $m$ is called a $\phi$-mean. Moreover, for some $C > 0$, $\mathcal{A}$ is called $C - \phi$-amenable if there exists a $\phi$-mean bounded by $C$; see Hu, Monfared, and Traynor [9]. The notion of (right) character amenability was introduced and studied by Monfared [15]. In fact, character amenability of $\mathcal{A}$ is equivalent to $\mathcal{A}$ being $\varphi$-amenable, for all $\varphi \in \Delta(\mathcal{A})$, and $\mathcal{A}$ having a bounded right approximate identity. The concept of $C-$character amenability is defined similarly; see [9] for more details in this field.

We also recall the definitions and basic relationships of the standard homological properties. We refer to [8], as a standard reference in this field. Following this reference, we say that $\mathcal{A}$ is biprojective if there is a bounded $\mathcal{A}$-bimodule map $\xi\colon \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \xi = id_{\mathcal{A}}$. Also $\mathcal{A}$ is called biflat if there is a bounded $\mathcal{A}$-bimodule map $\theta : (\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}^*$ such that $\theta \circ {\pi_{\mathcal{A}}}^* = \mathrm{id}_{\mathcal{A}^*}$. We also remind from [20] that $\mathcal{A}$ is approximately biprojective if there exists a net $(\xi_\lambda)$ of bounded $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\|\pi_{\mathcal{A}} \circ \xi_\lambda(a) - a\|_{\mathcal{A}} \to_\lambda 0$, for each $a \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is called approximately biflat if there exists a net $\theta_\delta : (\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}^*$, $(\delta \in \Delta)$, of bounded $\mathcal{A}$-bimodule morphisms such that

$$W^*OT - \lim_{\delta}\ \theta_\delta \circ {\pi_{\mathcal{A}}}^* = \mathrm{id}_{\mathcal{A}^*}\,,$$

where $(W^*OT)$ is the weak$^*$ operator topology on $B(\mathcal{A}^*)$. Recall that the weak$^*$ operator topology $(W^*OT)$ on $B(\mathcal{A}^*)$ is the locally convex topology, determined by the seminorms $\{p_{a,g} : a \in \mathcal{A}, g \in \mathcal{A}^*\}$, where $p_{a,g}(T) = |\langle g, T(a)\rangle|$, for all $a \in \mathcal{A}$ and $g \in \mathcal{A}^*$; see [17].

We also recall the first and second Arens products $\Box$ and $\Diamond$ on $\mathcal{A}^{**}$, as follows. Let $a$, $b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $F$, $G \in \mathcal{A}^{**}$. Then the functionals $f \cdot a$ and $a \cdot f$ and also $F \cdot f$ and $f \cdot F$ are defined as

$$f \cdot a(b) = f(ab) \quad \text{and} \quad a \cdot f(b) = f(ba) \quad (b \in \mathcal{A})$$

and

$$f \cdot F(a) = F(a \cdot f) \quad \text{and} \quad F \cdot f(a) = F(f \cdot a) \quad (a \in \mathcal{A})\,.$$

Finally

$$F \Box G(g) = F(G \cdot g) \quad \text{and} \quad F \Diamond G(g) = G(g \cdot F) \quad (g \in \mathcal{A}^*)\,.$$

Then by [4, Theorem 2.6.15], $\mathcal{A}^{**}$ is a Banach algebra under both Arens products $\Box$ and $\Diamond$, containing $\mathcal{A}$ as a closed subalgebra. Moreover by [4, Definition 2.6.16], $\mathcal{A}$ is said to be Arens regular, if these two multiplications coincide.

3.1. **Amenability, biprojectivity and biflatness.** Let $(X, d)$ be a locally compact metric space. Since $B(X)$ and $C_b(X)$ and so their second duals are unital, then by [16, Section 4, Exercises 4.1.1, 4.3.1], $B(X)$ (resp. $B(X)$, $C_b(X)^{**}$ and $C_b(X)$) is biprojective if and only if $B(X)$ (resp. $B(X)$, $C_b(X)^{**}$ and $C_b(X)$) is contractible. Also for the unital Banach algebras, the concept of contractibility and pseudo-contractibility are equivalent; see [6, Theorem 2.4]. Furthermore, for all Banach algebras, pseudo-contractibility and approximate biprojectivity are equivalent. Moreover, since these Banach algebras are commutative, $B(X)^{**}$ (resp. $B(X)$, $C_b(X)^{**}$ and $C_b(X)$) is contractible if and only if $B(X)$ (resp. $B(X)$, $C_b(X)^{**}$ and $C_b(X)$) is finite dimensional; see [16, Corollary 4.1.3]. In this situation, since the set $\{\delta_x : x \in X\}$ is strongly separating the points of $B(X)$, thus the same arguments similar to that, given in the proof of Proposition 2.1 and Corollary 2.2 imply that $X$ is finite. These observations provide the following proposition.

**Proposition 3.1.** *Let $X$ be a locally compact metric space. Then the following statements are equivalent.*

(i) $B(X)^{**}$ *is biprojective (resp. approximate biprojective).*

(ii) $B(X)^{**}$ *is contractible (resp. pseudo-contractible).*

(ii) $C_b(X)^{**}$ *is biprojective (resp. approximate biprojective).*

(iv) $C_b(X)^{**}$ *is contractible (resp. pseudo-contractible).*

(v) $B(X)$ *is biprojective (resp. approximate biprojective).*

(vi) $B(X)$ *is contractible (resp. pseudo-contractible).*

(vii) $C_b(X)$ *is biprojective (resp. approximate biprojective).*

(viii) $C_b(X)$ *is contractible (resp. pseudo-contractible).*

(ix) $X$ *is finite.*

Now we investigate Proposition 3.1 for Lipschitz algebras. Since $\operatorname{Lip}_\alpha X$ and so $(\operatorname{Lip}_\alpha X)^{**}$ are unital, by [16, Corollary 4.1.3], $(\operatorname{Lip}_\alpha X)^{**}$ (resp. $\operatorname{Lip}_\alpha X$) is biprojective if and only if $(\operatorname{Lip}_\alpha X)^{**}$ (resp. $\operatorname{Lip}_\alpha X$) is contractible. Moreover, since these Banach algebras are commutative, then by [16, Section 4, Exercises 4.1.1, 4.3.1], $(\operatorname{Lip}_\alpha X)^{**}$ (resp. $\operatorname{Lip}_\alpha X$) is contractible if and only if $(\operatorname{Lip}_\alpha X)^{**}$ (resp. $\operatorname{Lip}_\alpha X$) is finite dimensional. Now we obtain the following result from Proposition 2.1 and Corollary 2.2.

**Theorem 3.2.** *Let $(X, d)$ be a metric space and $\alpha > 0$. Then the following statements are equivalent.*

(i) $(\operatorname{Lip}_\alpha X)^{**}$ *is contractible.*

(ii) $(\operatorname{Lip}_\alpha X)^{**}$ *is pseudo-contractible.*

(iii) $(\operatorname{Lip}_\alpha X)^{**}$ *is biprojective.*

(iv) $(\operatorname{Lip}_\alpha X)^{**}$ *is approximately biprojective.*

(v) $\operatorname{Lip}_\alpha X$ *is contractible.*

(vi) $\operatorname{Lip}_\alpha X$ *is pseudo-contractible.*

(vii) $\mathrm{Lip}_\alpha X$ *is biprojective.*

(viii) $\mathrm{Lip}_\alpha X$ *is approximately biprojective.*

*Moreover, if* $\mathrm{Lip}_\alpha X$ *is strongly separating, then all of the above statements are equivalent to the finiteness of* $X$*.*

Note that by [16], a Banach algebra is amenable if and only if it is biflat and it has a bounded approximate identity. It follows that for the unital Banach algebras, the concept of amenability and biflatness are equivalent. Also by [6, Theorem 3.1], the concept of approximate amenability and approximate contractibility of Banach algebras are equivalent. Moreover, Propositions 3.2 and 3.8 of [6] mention that for the unital Banach algebras, pseudo-amenability and approximate amenability are equivalent. In addition, for the unital Banach algebras, the concept of pseudo-amenability and character amenability are equivalent; see [2, Theorem 1.1]. Furthermore, by [17, Theorem 2.4], every unital approximately biflat Banach algebra is pseudo-amenable and so character amenable. We also in [1], provided a necessary and sufficient condition for $C-$character amenability of Lipschitz algebras, for the case where $\mathrm{Lip}_\alpha X$ is strongly separating the points of $X$. In fact, we proved that $\mathrm{Lip}_\alpha X$ is amenable if and only if it is $C$-character amenable, for some $C > 0$, if and only if $(X, d)$ is $\varepsilon$-uniformly discrete, for some $\varepsilon > 0$. Thus the following result is immediately obtained.

**Theorem 3.3.** *Let* $(X, d)$ *be a metric space and* $\alpha > 0$ *such that* $\mathrm{Lip}_\alpha X$ *is strongly separating the points of* $X$*. Then the following statements are equivalent;*

(i) $\mathrm{Lip}_\alpha X$ *is amenable.*

(ii) $\mathrm{Lip}_\alpha X$ *is character amenable.*

(iii) $\mathrm{Lip}_\alpha X$ *is pseudo-amenable.*

(iv) $\mathrm{Lip}_\alpha X$ *is approximately amenable.*

(v) $\mathrm{Lip}_\alpha X$ *is approximately contractible.*

(vi) $\mathrm{Lip}_\alpha X$ *is approximately biflat.*

(vii) $\mathrm{Lip}_\alpha X$ *is biflat.*

(viii) $X$ *is* $\varepsilon-$*uniformly discrete, for some* $\varepsilon > 0$.

3.2. **Weak amenability of Lipschitz algebras.** Let $\mathcal{A}$ be a commutative Banach algebra and $\phi \in \Delta(A)$. A non-zero linear functional $d_\phi$ on $\mathcal{A}$ is called a point derivation at $\phi$ if

$$d_\phi(ab) = \phi(a)d_\phi(b) + \phi(b)d_\phi(a) \quad (a, b \in \mathcal{A}).$$

By [4, Theorem 2.8.63], $\mathcal{A}$ is weakly amenable if and only if there are no non zero point derivation on $\mathcal{A}$; i.e. $d_\phi = 0$, for all $\phi \in \Delta(A)$.

In this section, we study weak amenability of Lipschitz algebras. By the preceding explanations, we should investigate point derivations on $\mathrm{Lip}_\alpha X$. The next elementary lemma will be used in the further results.

**Lemma 3.4.** *Let* $(X, d)$ *be a metric space and* $\alpha > 0$*. Then the characteristic function* $\chi_{\{x\}}$ *at* $x \in X$ *belongs to* $\mathrm{Lip}_\alpha X$ *if and only if* $x$ *is an isolated point of* $X$*.*

**Proof.** First suppose that $x$ is an isolated point of $X$. Thus $\inf_{t\neq x} d(x,t) > 0$ and so

$$p_\alpha(\chi_{\{x\}}) = \sup_{s\neq t} \frac{|\chi_{\{x\}}(t) - \chi_{\{x\}}(s)|}{d(s,t)^\alpha} = \sup_{x\neq t} \frac{1}{d(x,t)^\alpha} = \frac{1}{\inf_{t\neq x} d(x,t)^\alpha} < \infty\,,$$

which implies $\chi_{\{x\}} \in \operatorname{Lip}_\alpha X$. The converse is obvious. □

The following corollary is immediately obtained from Lemma 3.4.

**Corollary 3.5.** *Let $(X,d)$ be a metric space and $\alpha > 0$. Then $X$ is discrete if and only if all the characteristic functions $\chi_{\{x\}}$ ($x \in X$) belong to $\operatorname{Lip}_\alpha X$.*

The following result has been proved in [19, Proposition 9.2].

**Proposition 3.6.** *Let $(X,d)$ be a metric space and $0 < \alpha \le 1$. If $x$ is a cluster point of $X$, then there is a non-zero point derivation at $\delta_x$ on $\operatorname{Lip}_\alpha X$.*

**Corollary 3.7.** *Let $(X,d)$ be a metric space and $0 < \alpha \le 1$. If $\operatorname{Lip}_\alpha X$ is weakly amenable, then $X$ is discrete.*

**Corollary 3.8.** *Let $(X,d)$ be a compact metric space and $0 < \alpha \leqslant 1$. Then $\operatorname{Lip}_\alpha X$ is weakly amenable if and only if $X$ is finite.*

**Proof.** If $\operatorname{Lip}_\alpha X$ is weakly amenable, then by Corollary 3.7, $X$ is discrete and since $X$ is compact, it follows that $X$ is finite. Conversely, if $X$ is finite then $\operatorname{Lip}_\alpha X = C_b(X)$, and so it is amenable and consequently weakly amenable. □

**Remark 3.9.**

(i) Note that Corollary 3.7 is not necessarily valid, whenever $\alpha > 1$. For example consider $\mathbb{R}$ endowed with the usual Euclidean metric. It is easily verified that $\operatorname{Lip}_\alpha \mathbb{R} = \operatorname{Cons}(\mathbb{R})$, which is homeomorphic to $\mathbb{C}$. Thus $\operatorname{Lip}_\alpha \mathbb{R}$ is amenable and so weakly amenable, whereas $\mathbb{R}$ is not discrete.

(ii) By [6, Corollary 3.7], any pseudo-amenable commutative Banach algebra is weakly amenable. Now Theorem 3.3 implies that if $X$ is uniformly discrete, then $\operatorname{Lip}_\alpha X$ is always weakly amenable. In fact, in this case all the spaces $\operatorname{Lip}_\alpha X$, $C_b(X)$ and $B(X)$ coincide.

Regarding to the converse of Corollary 3.7, the following partial useful results are obtained. Note that these results have been proved for the case where $0 < \alpha \le 1$; see [19, Proposition 8.5].

**Proposition 3.10.** *Let $(X,d)$ be a metric space and $\alpha > 0$. If $a \in X$ is an isolated point of $X$, then every point derivation at $\delta_a$ on $\operatorname{Lip}_\alpha X$ is zero.*

**Proof.** Suppose that $d_{\delta_a} : \operatorname{Lip}_\alpha X \to \mathbb{C}$ is a point derivation at $\delta_a$. Then for all $f$, $g \in \operatorname{Lip}_\alpha X$

$$d_{\delta_a}(fg) = d_{\delta_a}(f)\delta_a(g) + d_{\delta_a}(g)\delta_a(f) = d_{\delta_a}(f)g(a) + d_{\delta_a}(g)f(a)\,.$$

In particular,

$$d_{\delta_a}(f\chi_{\{a\}}) = d_{\delta_a}(f)\delta_a(\chi_{\{a\}}) + d_{\delta_a}(\chi_{\{a\}})\delta_a(f) \quad (f \in \operatorname{Lip}_\alpha X)\,.$$

It follows that $f(a)d_{\delta_a}(\chi_{\{a\}}) = d_{\delta_a}(f) + d_{\delta_a}(\chi_{\{a\}})f(a)$, which implies $d_{\delta_a}(f) = 0$. Therefore $d_{\delta_a} = 0$. □

**Corollary 3.11.** *Let $(X, d)$ be a discrete metric space and $\alpha > 0$. Then every point derivation at $\delta_x$ $(x \in X)$ on $\operatorname{Lip}_\alpha X$ is zero*

**Remark 3.12.** In [3] and also [4], there are valuable results about amenability and weak amenability of $\operatorname{lip}_\alpha X$, for the case where $(X, d)$ is a compact metric space and $0 < \alpha \leq 1$. For example by [3, Theorem 3.9], $\operatorname{lip}_\alpha X$ is not amenable, for any infinite, compact metric space X and $0 < \alpha < 1$. Also by [3, Theorem 3.10], for each compact metric space $X$ and $0 < \alpha < \frac{1}{2}$, $\operatorname{lip}_\alpha X$ is weakly amenable. Also for the unit circle $\mathbb{T}$ with the usual metric, $\operatorname{lip}_\alpha \mathbb{T}$ is weakly amenable if and only if $0 < \alpha \leq \frac{1}{2}$; see [3, Theorem 3.14].

3.3. **Character space of Lipschitz algebras.** Let $\mathcal{A}$ be a commutative Banach algebra. We endow $\Delta(\mathcal{A})$ with the weakest topology with respect to which all the functions

$$\Delta(\mathcal{A}) \to \mathbb{C}\,, \quad \varphi \mapsto \varphi(x) \quad (x \in \mathcal{A})$$

are continuous. A neighborhood basis at $\varphi_0 \in \Delta(\mathcal{A})$ is then given by the collection of sets

$$U(\varphi_0, x_1, \ldots, x_n, \varepsilon) = \{\varphi \in \Delta(\mathcal{A}) : \ |\varphi(x_i) - \varphi_0(x_i)| < \varepsilon, \ 1 \leq i \leq n\}\,,$$

where $\varepsilon > 0$, $n \in \mathbb{N}$, and $x_1, \ldots, x_n$ are arbitrary elements of $\mathcal{A}$. This topology on $\Delta(\mathcal{A})$ is called the Gelfand topology. The Gelfand topology obviously coincides with the relative $w^*$-topology of $\mathcal{A}^*$ on $\Delta(\mathcal{A})$.

In this section, the character space of Lipschitz algebras will be introduced. It requires some preliminaries.

Following [14], an algebra of functions, defined on a set $X$ is called inverse-closed if for every function in the algebra satisfying $|f(x)| \geqslant \varepsilon > 0$, for all $x \in X$ and some $\varepsilon > 0$, the inverse $\frac{1}{f}$ is also in the algebra. In [19, Proposition 1.7], it has been proved that $\operatorname{Lip}_1 X$ and $\operatorname{lip}_1 X$ are both inverse-closed. By using similar arguments, we generalize this result for each $\alpha > 0$.

**Lemma 3.13.** *Let $(X, d)$ be a metric space and $\alpha > 0$. Then $\operatorname{Lip}_\alpha X$ and $\operatorname{lip}_\alpha X$ are inverse-closed.*

**Proof.** Suppose that $f \in \operatorname{Lip}_\alpha X$ and there is $\varepsilon > 0$ such that for all $x \in X$, $|f(x)| \geqslant \varepsilon$. Then for all $x, y \in X$ with $x \neq y$

$$\frac{\left|\frac{1}{f}(x) - \frac{1}{f}(y)\right|}{d(x,y)^\alpha} = \frac{|f(x) - f(y)|}{|f(x)||f(y)|d(x,y)^\alpha} \leqslant \frac{1}{\varepsilon^2} \cdot \frac{|f(x) - f(y)|}{d(x,y)^\alpha}\,. \tag{3.1}$$

Consequently

$$p_\alpha(\frac{1}{f}) \leq \frac{1}{\varepsilon^2}\, p_\alpha(f) < \infty\,,$$

which implies that $\dfrac{1}{f} \in \operatorname{Lip}_\alpha X$. It follows that $\operatorname{Lip}_\alpha X$ is inverse-closed. Moreover by (3.1), if $f \in \operatorname{lip}_\alpha X$, then

$$\frac{\left|\frac{1}{f}(x) - \frac{1}{f}(y)\right|}{d(x,y)^\alpha} \to 0\,,$$

whenever $d(x,y) \to 0$. It follows that $\dfrac{1}{f} \in \operatorname{lip}_\alpha X$. Therefore $\operatorname{lip}_\alpha X$ is also inverse-closed. □

By the lemma and corollary given in [14, page 55], if $\mathcal{A}$ is a weakly separating, self-adjoint and inverse-closed algebra of continuous complex-valued functions on a compact space $X$, then $\Delta(\mathcal{A}) = \delta_X$, where

$$\delta_X = \{\delta_x : \ x \in X\}\,.$$

Moreover in the case where $X$ is not necessarily compact then $\delta_X$ is dense in $\Delta(\mathcal{A})$.

Now let $L_\alpha(X)$ (resp. $l_\alpha(X)$) be the closure of the set $\{\delta_x : \ x \in X\}$ in the weak*-topology of $(\operatorname{Lip}_\alpha X)^*$ (resp. $(\operatorname{lip}_\alpha X)^*$). Thus the following result is obtained, by the above discussion. Note that this result is known for the case where $0 < \alpha \leq 1$; see [19].

**Theorem 3.14.** *Let $(X,d)$ be a metric space and $\alpha > 0$. Then*

$$\Delta(\operatorname{Lip}_\alpha X) = L_\alpha(X) \quad \textit{and} \quad \Delta(\operatorname{lip}_\alpha X) = l_\alpha(X)\,. \tag{3.2}$$

*Moreover if $(X,d)$ is non-empty and compact and $0 < \alpha \leq 1$, then*

$$\Delta(\operatorname{Lip}_\alpha X) = \Delta(\operatorname{lip}_\alpha X) = \Delta((\operatorname{lip}_\alpha X)^{**}) = \delta_X\,. \tag{3.3}$$

**Proof.** By the explanations given at the beginning of the second section, $\operatorname{Lip}_\alpha X$ and $\operatorname{lip}_\alpha X$ are always weakly separating. Also all the Lipschitz algebras are clearly self adjoint. Moreover $\operatorname{Lip}_\alpha X$ and $\operatorname{lip}_\alpha X$ are inverse-closed by Lemma 3.13. Now [14, Corollary, page 55] implies that $\delta_X$ is dense in $\Delta(\operatorname{Lip}_\alpha X)$ and $\Delta(\operatorname{lip}_\alpha X)$, in the respective Gelfand topology. Thus the equation of (3.2) is obtained. In the case where $X$ is compact, then

$$\Delta(\operatorname{Lip}_\alpha X) = \Delta(\operatorname{lip}_\alpha X) = \delta_X\,, \tag{3.4}$$

by [14, Lemma, page 55]. Moreover by [3, Theorem 3.8], for the non-empty compact metric space $(X,d)$ and $0 < \alpha \leq 1$, the space $(\operatorname{lip}_\alpha X)^{**}$ is isometrically isomorphic to $\operatorname{Lip}_\alpha X$. Now [12, Lemma 2.2.12] implies

$$\Delta((\operatorname{lip}_\alpha X)^{**}) = \Delta(\operatorname{Lip}_\alpha X) = \delta_X\,.$$

Therefore the equations, given in (3.3) are satisfied. □

**Remark 3.15.** Let $(X,d)$ be a metric space and $\alpha > 0$. We show that for each $f \in \Delta(\operatorname{Lip}_\alpha X)$, $\hat{f} \in \Delta((\operatorname{Lip}_\alpha X)^{**})$, where $\hat{f}$ is the corresponding element of $f$ in $(\operatorname{Lip}_\alpha X)^{***}$, defined by $\hat{f}(F) = F(f)$ $(F \in (\operatorname{Lip}_\alpha X)^{**})$. To that end, take $F$, $G \in (\operatorname{Lip}_\alpha X)^{**}$. By the Goldstein Theorem, there are the nets $(a_\alpha)$ and $(b_\beta)$ in

$\mathrm{Lip}_\alpha X$, converging respectively to $F$ and $G$, in the weak*-topology of $(\mathrm{Lip}_\alpha X)^{**}$. Thus

$$\begin{aligned}\hat{f}(F\square G) = (F\square G)(f) &= \lim_\alpha \lim_\beta (f(a_\alpha b_\beta)) \\ &= \lim_\alpha \lim_\beta (f(a_\alpha) f(b_\beta)) \\ &= F(f)G(f) = \hat{f}(F)\hat{f}(G)\end{aligned}$$

and so $\hat{f} \in \Delta((\mathrm{Lip}_\alpha X)^{**})$. It follows that $\hat{\delta_x} \in \Delta((\mathrm{Lip}_\alpha X)^{**})$, for all $x \in X$. Consequently

$$\overline{\{\hat{\delta_x} : x \in X)\}}^{w^*} \subseteq \overline{\{\hat{f} : f \in \Delta(\mathrm{Lip}_\alpha X)\}}^{w^*} \subseteq \Delta((\mathrm{Lip}_\alpha X)^{**})\,. \tag{3.5}$$

But it is unclear to us if the equality holds. In fact, the following natural question arises:

**Question 3.16.** For which metric space $(X, d)$ and $\alpha > 0$, the equality is achieved in the inclusions (3.5)?

3.4. **Arens regularity and semisimplicity of Lipschitz algebras.** Let $\mathcal{A}$ be a commutative Banach algebra. Then the radical of $\mathcal{A}$, denoted by $\mathrm{rad}(\mathcal{A})$, is defined by

$$\mathrm{rad}(\mathcal{A}) = \bigcap_{\varphi\in\Delta(\mathcal{A})} \ker(\varphi)\,.$$

Clearly, $\mathrm{rad}(\mathcal{A})$ is a closed ideal of $\mathcal{A}$. Then $\mathcal{A}$ is called semisimple if $\mathrm{rad}(\mathcal{A}) = \{0\}$. We refer to [12], for more information. We conclude this work with the next theorem, which contains some results related to Arens regularity and semisimplicity of the Lipschitz algebras and their second duals.

**Theorem 3.17.** *Let $(X, d)$ be a metric space and $\alpha > 0$. Then the following statements hold;*

(i) $\mathrm{Lip}_\alpha X$ *and* $\mathrm{lip}_\alpha X$ *are semisimple.*

(ii) *If $X$ is compact and $0 < \alpha \le 1$, then* $\mathrm{Lip}_\alpha X$, $\mathrm{lip}_\alpha X$ *and also* $(\mathrm{lip}_\alpha X)^{**}$ *are Arens regular.*

(iii) *If $X$ is compact and $0 < \alpha \le 1$, then* $(\mathrm{lip}_\alpha X)^{**}$ *is semisimple.*

**Proof.** (i) Since

$$\mathrm{rad}(\mathrm{Lip}_\alpha X) = \bigcap_{f\in\Delta(\mathrm{Lip}_\alpha X)} \ker(f) \subseteq \bigcap_{x\in X} \ker(\delta_x) = \{0\}\,,$$

it follows that $rad(\mathrm{Lip}_\alpha X) = \{0\}$ and so $\mathrm{Lip}_\alpha X$ is semisimple. Similarly $\mathrm{lip}_\alpha X$ is semisimple.

(ii) By [3, Theorem 3.8], $\mathrm{Lip}_\alpha X$ and $(\mathrm{lip}_\alpha X)^{**}$ are isomorphic and Arens regular. Also by [4, Corollary 2.6.18], the closed subalgebras of any Arens regular Banach algebra are again Arens regular. It follows that $\mathrm{lip}_\alpha X$ is also Arens regular.

(iii) It is immediately obtained by [3, Theorem 3.8] and part (i). $\square$

**Remark 3.18.** Although for any non-empty compact metric space $(X, d)$ and $0 < \alpha \leq 1$, all the Lipschitz algebras $\operatorname{Lip}_\alpha X$ are semisimple, but their second duals are not necessarily semisimple. For example take $X$ to be the subset $[0, 1]$ of $\mathbb{R}$, with the induced Euclidean metric. Then by [4, Proposition 3.6], $(\operatorname{Lip}_\alpha[0, 1])^{**}$ is not semisimple.

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