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NON-SPLIT ALMOST COMPLEX AND NON-SPLIT RIEMANNIAN SUPERMANIFOLDS

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Abstract. Non-split almost complex supermanifolds and non-split Riemannian supermanifolds are studied. The first obstacle for a splitting is parametrized by group orbits on an infinite dimensional vector space. For almost complex structures, the existence of a splitting is equivalent to the existence of local coordinates in which the almost complex structure can be represented by a purely numerical matrix, i.e. containing no Grassmann variables. For Riemannian metrics, terms up to degree 2 are allowed in such a local matrix representation, in order to preserve non-degeneracy. It is further shown that non-split structures appear in the almost complex case as deformations of a split reduction and in the Riemannian case as the deformation of an underlying metric. In contrast to non-split deformations of complex supermanifolds, these deformations can be restricted by cut-off functions to local deformations. A class of examples of nowhere split structures constructed from almost complex manifolds of dimension 6 and higher, is provided for both cases.

Even almost complex structures and Riemannian metrics define global tensor fields on real supermanifolds. Denoting a real supermanifold by $\mathcal{M} = (M, \mathcal{C}^\infty_{\mathcal{M}})$ and the global super vector fields on $\mathcal{M}$ by $\mathcal{V}_{\mathcal{M}}$, the tensors lie in $\operatorname{End}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$, resp. $\operatorname{Hom}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}^*_{\mathcal{M}})_{\bar{0}}$. We fix a Batchelor model $\mathcal{M} \to (M, \Gamma^\infty_{\Lambda E^*})$ with vector bundle $E \to M$ (see [1]). Then the $\mathbb{Z}$-degree zero part J_R of an even almost complex structure $J \in \operatorname{End}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ is again an almost complex structure on $\mathcal{M}$. This raises the question, whether there is a Batchelor model, such that J equals its reduction J_R. In the case of a positive answer we call the tensor split. The analogous question can be formulated for even Riemannian metrics where the reduction g_R of a metric $g \in \operatorname{Hom}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}^*_{\mathcal{M}})_{\bar{0}}$ is given by the Riemannian metric $g_R = g_0 + g_2$.

For complex structures being integrable almost complex structures, existence of a splitting was studied in [6], [13] and [16]: there exist non-split complex supermanifolds, all of them being deformations of split complex supermanifolds. The parameter spaces of deformations are given by orbits of the automorphism group of the associated Batchelor bundle on a certain non-abelian first cohomology. Here the existence of local complex coordinates makes the splitting problem a problem of *global* cohomology. The splitting question for even symplectic supermanifolds was

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answered in [14] by identifying the symplectic supermanifold with an underlying symplectic manifold and a Batchelor bundle with metric and connection. It is shown that all terms of degree higher than 2 in a symplectic form can be erased by the choice of a Batchelor model. Hence all symplectic supermanifolds are split in the above sense. Results on the splitting problem for homogeneous complex supermanifolds are proved in [18], while the splitting problem for supermoduli spaces of super Riemannian surfaces is considered in [5].

In this paper the existence of a splitting for even almost complex structures as well as even Riemannian metrics is studied. It is shown that all almost complex structures appear as deformations of split structures and all Riemannian metrics appear as deformations of underlying metrics. In both cases but in contrast to the complex case mentioned above, these deformations can be restricted by smooth cut-off functions to *local* deformations. For almost complex structures the splitting problem stated above can be expressed as: what is the obstacle for having local coordinates near any point such that the almost complex structure is represented by a purely numerical matrix. In the Riemannian case (similar to the symplectic case in [14]), the reduction is asked to be a purely numerical matrix on $\mathcal{V}_{\mathcal{M},-1}^{\otimes 2}$ and to have matrix entries of degree less or equal to 2 on the three remaining blocks of $(\mathcal{V}_{\mathcal{M},0} \oplus \mathcal{V}_{\mathcal{M},-1})^{\otimes 2}$. The first obstacle for a splitting is described for both problems. It appears in degrees of nilpotency $2k, \ldots, 4k-2$ for a $k \in \mathbb{N}$ and is well-defined as soon as all candidates for obstacles in the lower even degrees $2l$, $1 \leq l < k$ vanish. For the complex case involving global cohomologies, a similar behavior of the obstacles appears (see [5] and [13]). Finally explicit examples of non-split almost complex structures, resp. Riemannian metrics are given. The results and the applied methods are summarized in the following.

Contents. In the first section an almost complex structure is decomposed via the finite log series into its reduction (the degree zero term) and its degree increasing term. With respect to these components the lowest degree obstacle for isomorphy of almost complex supermanifolds is deduced. For a fixed reduction, these obstacles are parametrized by group orbits on a quotient of tensor spaces. The group is a quotient of the transformations that are almost holomorphic with respect to the reduction up to a certain degree.

The second section deals with Riemannian metrics in a similar way producing results analogous to those in the almost complex case. Here the isometries of the reduction play the role of the almost holomorphic transformations. However the more complicated action of the automorphism group of the supermanifold on a metric and the fact that the reduction has no pure degree, require an adjustment of the techniques.

Finally the third section contains a class of non-split examples for almost complex structures and Riemannian metrics. These are constructed on the supermanifold of differential forms on an arbitrary almost complex manifold of dimension higher than 4. In the almost complex and in the Riemannian case, the constructed non-split tensors are nowhere split, i.e. at no point of the manifold the matrix elements of the respective tensors satisfy the respective properties mentioned above.

Introductory references. For an introduction to the theory of supermanifolds see e.g. [3], [4], [7], [9], and [17]. For the relation of the differing approaches to supermanifolds and for the associated vector bundle for the construction of a Batchelor model see [1] and [2].

1. Non-split almost complex supermanifolds

Let $(\mathcal{M}, J)$ be an almost complex supermanifold (see e.g. [12]) with sheaf of superfunctions $\mathcal{C}^\infty_{\mathcal{M}}$. Denote the $\mathcal{C}^\infty_{\mathcal{M}}(M)$-module of global superderivations of $\mathcal{C}^\infty_{\mathcal{M}}$ by $\mathcal{V}_{\mathcal{M}} = \mathcal{V}_{\mathcal{M},\bar{0}} \oplus \mathcal{V}_{\mathcal{M},\bar{1}}$. Furthermore fix a Batchelor model $\mathcal{M} \to (M, \Gamma^\infty_{\Lambda E^*})$ yielding $\mathbb{Z}$-gradings (denoted by lower indeces) and filtrations (denoted by upper indeces in brackets) on $\mathcal{C}^\infty_{\mathcal{M}}$, $\mathcal{V}_{\mathcal{M}}$ and $\mathrm{End}(\mathcal{V}_{\mathcal{M}})$, the last denoting $\mathcal{C}^\infty_{\mathcal{M}}(M)$-linear maps. The even automorphism of $\mathcal{C}^\infty_{\mathcal{M}}(M)$-modules $J \in \mathrm{End}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ can be uniquely decomposed into $J = J_R(\mathrm{Id} + J_N)$ with invertible $J_R = J_0$ and nilpotent J_N. The finite exp and log series yield a unique representation $\mathrm{Id} + J_N = \exp(Y)$ with $Y \in \mathrm{End}^{(2)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$.

Lemma 1.1. *The tensor J is an almost complex structure if and only if J_R is an almost complex structure and $YJ_R + J_RY = 0$.*

Proof. From $J^2 = -\mathrm{Id}$ we obtain $J_R^2 = -\mathrm{Id}$ and $\exp(Y)J_R\exp(Y) = J_R$. For reasons of degree $Y_2J_R + J_RY_2 = 0$. Assume that $Y_{2k}J_R + J_RY_{2k} = 0$ holds for all $k < n$. Set $Y_{[2k]} := \sum_{j=1}^k Y_{2j}$. We have $\exp(Y) \equiv \exp(Y_{[2n-2]}) + Y_{2n}$ up to terms of degree $> 2n$. Hence $\exp(Y)J_R\exp(Y) \equiv \exp(Y_{[2n-2]})\exp(-Y_{[2n-2]})J_R + Y_{2n}J_R + J_RY_{2n}$ up to terms of degree $> 2n$. This completes the induction. The converse implication follows directly. □

We call J_R the *reduction* of J, deforming J_R by $t \mapsto J_R\exp(tY)$. In particular J_R yields an almost complex structure on M and an almost complex structure on the vector bundle $E \to M$. Hence even and odd dimension of $\mathcal{M}$ are even. Further topological conditions on M and E for the existence of an almost complex structure can be obtained from [10] and e.g. [15]. Adapted to our considerations the almost complex supermanifold $(\mathcal{M}, J)$ is split if there is a Batchelor model, such that the almost complex structure J has nilpotent component $Y = 0$. Note that this problem is completely local since Lemma 1.1 allows cutting off the nilpotent Y in $J = J_R\exp(Y)$.

Let $\Phi = (\varphi, \varphi^*)$ be an automorphism of the supermanifold $\mathcal{M}$. The global even isomorphism of superalgebras $\varphi^* \in \mathrm{Aut}\big(\mathcal{C}^\infty_{\mathcal{M}}(M)\big)_{\bar{0}}$ over φ is decomposable into $\varphi^* = \exp(\zeta)\varphi_0^*$ with $\zeta \in \mathcal{V}^{(2)}_{\mathcal{M},\bar{0}}$ and φ_0^* preserving the $\mathbb{Z}$-degree induced by the Batchelor model (see e.g. [13]). Denote by $\mathrm{Aut}(E^*)$ the bundle automorphisms over arbitrary diffeomorphisms of M, then φ_0^* is induced by an element $\varphi_0 \in \mathrm{Aut}(E^*)$ over φ. The automorphism φ^* transforms J into $\varphi^*.J$ given by $(\varphi^*.J)(\chi) := \varphi^*(J((\varphi^*)^{-1}\chi\varphi^*))(\varphi^*)^{-1}$. Denoting $\mathrm{ad}(\zeta) := [\zeta, \cdot]$, assuming $\zeta \in \mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}}$ and applying $\varphi^* \equiv (\mathrm{Id} + \zeta)\varphi_0^*$ up to terms in $\mathcal{V}^{(4k)}_{\mathcal{M},\bar{0}}$, we have $\varphi^*.J \equiv \varphi_0^*.J + [\mathrm{ad}(\zeta), \varphi_0^*.J]$ up to terms in $\mathrm{End}^{(4k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$. Comparing both sides with respect to the degree yields:

Proposition 1.2. *The almost complex supermanifolds* $(\mathcal{M}, J)$ *and* $(\mathcal{M}, J')$ *with structures* $J = J_R \exp(Y)$*,* $J' = J'_R \exp(Y')$*,* Y*,* $Y' \in \operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ *are isomorphic up to error terms in* $\operatorname{End}^{(4k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ *via an automorphism* φ^* *with* $\varphi^*(\varphi_0^*)^{-1} \in \exp(\mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}})$ *if and only if there exist* $\varphi_0 \in \operatorname{Aut}(E^*)$ *and* $\zeta \in \mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}}$ *such that* $J'_R = \varphi_0^*.J_R$ *and:*

$$Y'_{2j} = \varphi_0^*.Y_{2j} - \operatorname{ad}(\zeta_{2j}) - J'_R \operatorname{ad}(\zeta_{2j}) J'_R\,, \quad k \le j < 2k\,.$$

From now on we fix the reduction J_R and hence assume that for an automorphism ψ^* of $\mathcal{M}$, the map ψ_0^* is pseudo-holomorphic with respect to J_R, denoted $\psi_0^* \in \operatorname{Hol}(\mathcal{M}, J_R)$. Let $\operatorname{Hol}(\mathcal{M}, J_R, 2k)$ be the automorphisms $\psi^* = \exp(\xi)\psi_0^*$ of $\mathcal{M}$ such that $J_R \equiv \psi^*.J_R$ up to terms in $\operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$. Note that $\psi^* \in \operatorname{Hol}(\mathcal{M}, J_R, 2k)$ includes $\psi_0^* \in \operatorname{Hol}(\mathcal{M}, J_R)$ and that $\exp(\mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}}) \subset \operatorname{Hol}(\mathcal{M}, J_R, 2k)$ is a normal subgroup.

Define on the endomorphisms of real vector spaces $\operatorname{End}_{\mathbb{R}}(\mathcal{V}_{\mathcal{M}})$ the $\mathcal{C}^\infty_{\mathcal{M}}(M)$-linear $\mathbb{Z}$-degree preserving map:

$$F_{J_R}\colon \operatorname{End}_{\mathbb{R}}(\mathcal{V}_{\mathcal{M}}) \to \operatorname{End}_{\mathbb{R}}(\mathcal{V}_{\mathcal{M}})\,, \quad F_{J_R}(\gamma) := \gamma + J_R \gamma J_R\,.$$

The set $F_{J_R}\big(\operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})\big)$ is by Lemma 1.1 exactly the nilpotent parts Y of almost complex structures $J = J_R \exp(Y)$ deforming J_R in degree $2k$ and higher. Note further that $F_{J_R}(\operatorname{ad}(\mathcal{V}_{\mathcal{M}})) \subset \operatorname{End}(\mathcal{V}_{\mathcal{M}})$ and more precisely $F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}})\big) \subset \operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$.

Definition 1.3. Let the upper index $2k \in 2\mathbb{N}$ in curly brackets denote the sum of terms of $\mathbb{Z}$-degree $2k$ up to $4k-2$. For $J = J_R \exp(Y)$, $Y \in \operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ we call the class $[Y^{\{2k\}}]$ in the quotient of vector spaces $F_{J_R}(\operatorname{End}^{\{2k\}}(\mathcal{V}_{\mathcal{M}})_{\bar{0}})/F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{\{2k\}}_{\mathcal{M},\bar{0}})\big)$ the $2k$-th split obstruction class of J.

The $\operatorname{Hol}(\mathcal{M}, J_R, 2k)$-action on $F_{J_R}\big(\operatorname{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)$ is given up to terms in $\operatorname{End}^{(4k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ by $(\varphi^*, Y) \mapsto J_R(J_R - \varphi^*.J_R) + \varphi^*.Y$. Since $\varphi^*.F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{\{2k\}}_{\mathcal{M},\bar{0}})\big) \subset F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{\{2k\}}_{\mathcal{M},\bar{0}})\big)$, it is well-defined on $F_{J_R}\big(\operatorname{End}^{\{2k\}}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)/F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{\{2k\}}_{\mathcal{M},\bar{0}})\big)$. By Proposition 1.2 it induces an action of $P\operatorname{Hol}(\mathcal{M}, J_R, 2k) := \operatorname{Hol}(\mathcal{M}, J_R, 2k)/\exp(\mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}})$ on $F_{J_R}\big(\operatorname{End}^{\{2k\}}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)/F_{J_R}\big(\operatorname{ad}(\mathcal{V}^{\{2k\}}_{\mathcal{M},\bar{0}})\big)$. It follows that for an almost complex supermanifold that is split up to terms of degree $2k$ and higher, the $2k$-th split obstruction class is well-defined up to the $P\operatorname{Hol}(\mathcal{M}, J_R, 2k)$-action. Note that for a given almost complex structure $J = J_R \exp(Y)$ the obstructions can be checked starting with $j = 1$ iteratively: if $Y_{2j} = \operatorname{ad}(\zeta_{2j}) + J_R \operatorname{ad}(\zeta_{2j}) J_R$ can be solved for a $\zeta_{2j} \in \mathcal{V}_{\mathcal{M},2j}$ then there is an automorphism of the supermanifold $\mathcal{M}$ such that $J = J_R \exp(Y')$ with $Y' \in \operatorname{End}^{(2(j+1))}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$. In the non-split case this procedure ends with a well-defined $2k$ and associated orbit of $2k$-th split obstruction classes. We note as a special case:

Proposition 1.4. *Let* $(\mathcal{M}, J_R)$ *be a split almost complex supermanifold of odd dimension* $2(2m+r)$*,* $m \ge 0$*,* $r \in \{0, 1\}$*. The almost complex supermanifolds* $(\mathcal{M}, J)$

with reduction J_R that are split up to terms of degree $(2m+r)+1$ and higher, correspond bijectively to the $P\,\mathrm{Hol}(\mathcal{M}, J_R, 2(m+1))$-orbits on $F_{J_R}\big(\mathrm{End}^{(2(m+1))}(\mathcal{V}_\mathcal{M})_{\bar 0}\big)/F_{J_R}\big(\mathrm{ad}(\mathcal{V}^{(2(m+1))}_{\mathcal{M},\bar 0})\big)$.

As a technical tool for application in Section 3, we note an identification for the quotient appearing in the split obstruction classes. Denote by $\mathcal{E}^1_\mathcal{M} = \mathcal{E}^1_{\mathcal{M},\bar 0} \oplus \mathcal{E}^1_{\mathcal{M},\bar 1}$ the global super-1-forms on $\mathcal{M}$ and by $\mathrm{d}_\mathcal{M}$ the de Rham operator on the algebra $\mathcal{E}_\mathcal{M}$ of superforms. Observe $\mathrm{End}(\mathcal{V}_\mathcal{M}) = \mathcal{V}_\mathcal{M} \otimes_{\mathcal{C}^\infty_\mathcal{M}(M)} \mathcal{E}^1_\mathcal{M}$. Further for homogeneous components of $\chi \otimes \mathrm{d}_\mathcal{M} f \in \mathcal{V}_\mathcal{M} \otimes_{\mathcal{C}^\infty_\mathcal{M}(M)} \mathcal{E}^1_\mathcal{M}$ we have the decomposition:

$$\chi \otimes \mathrm{d}_\mathcal{M} f = (-1)^{|f||\chi|}\big(f \cdot \mathrm{ad}(\chi) - \mathrm{ad}(f\chi)\big) \tag{1}$$

We can follow:

Proposition 1.5. *For all k, the map*

$$\Theta_{J_R}\colon F_{J_R}\big(\mathcal{V}_\mathcal{M} \otimes_{\mathcal{C}^\infty_\mathcal{M}(M)} \mathcal{E}^1_\mathcal{M}\big)^{(2k)}_{\bar 0} \longrightarrow F_{J_R}\big(\mathrm{End}^{(2k)}(\mathcal{V}_\mathcal{M})_{\bar 0}\big)/F_{J_R}\big(\mathrm{ad}(\mathcal{V}^{(2k)}_{\mathcal{M},\bar 0})\big)$$

locally for homogeneous arguments defined by

$$F_{J_R}(\chi \otimes \mathrm{d}_\mathcal{M} f) \longmapsto (-1)^{|f|\,|\chi|} f \cdot F_{J_R}\big(\mathrm{ad}(\chi)\big) + F_{J_R}\big(\mathrm{ad}(\mathcal{V}^{(2k)}_{\mathcal{M},\bar 0})\big)$$

is a well-defined, surjective morphism of $\mathbb{Z}$-filtered super vector spaces. For any element $\psi^* \in \mathrm{Hol}(\mathcal{M}, J_R, 2k)$ and $[\psi^*] \in P\,\mathrm{Hol}(\mathcal{M}, J_R, 2k)$ we have $\Theta_{J_R}(\psi^*.Z) = [\psi^*].\big(\Theta_{J_R}(Z)\big)$.

2. Non-Split Riemannian supermanifolds

Let $(\mathcal{M}, g)$ be a Riemannian supermanifold with even non-degenerate supersymmetric form $g \in \mathrm{Hom}(\mathcal{V}_\mathcal{M} \otimes_{\mathcal{C}^\infty_\mathcal{M}(M)} \mathcal{V}_\mathcal{M}, \mathcal{C}^\infty_\mathcal{M}(M))_{\bar 0}$. Here we will mostly regard g as an isomorphism of $\mathcal{C}^\infty_\mathcal{M}$-modules $g \in \mathrm{Hom}(\mathcal{V}_\mathcal{M}, \mathcal{V}^*_\mathcal{M})_{\bar 0}$ with $g(X)(Y) = (-1)^{|X|\,|Y|} g(Y)(X)$ for homogeneous arguments. The context will fix which point of view is used. For a given Batchelor model $\mathcal{M} \to (M, \Gamma^\infty_{\Lambda E^*})$ decompose $g = g_R(\mathrm{Id} + g_N)$ with invertible $g_R = g_0 + g_2$ and nilpotent $g_N \in \mathrm{End}^{(2)}(\mathcal{V}_\mathcal{M})_{\bar 0}$ such that $g_0 g_N \in \mathrm{Hom}^{(4)}(\mathcal{V}_\mathcal{M}, \mathcal{V}^*_\mathcal{M})_{\bar 0}$. With the finite log and exp series we write $g = g_R \exp(W)$ with $W \in \mathrm{End}^{(2)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar 0}$, where $\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar 0}$ denotes those $W \in \mathrm{End}^{(2k)}(\mathcal{V}_\mathcal{M})_{\bar 0}$ such that $g_0 W \in \mathrm{Hom}^{(2k+2)}(\mathcal{V}_\mathcal{M}, \mathcal{V}^*_\mathcal{M})_{\bar 0}$.

Lemma 2.1. *If the tensor $g = g_R \exp(W)$, $W \in \mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar 0}$ is a Riemannian metric then g_R is a Riemannian metric and $g_R\big(W(\cdot), \cdot\big) \equiv g_R\big(\cdot, W(\cdot)\big)$ up to terms of degree $4k+2$ and higher.*

Proof. Due to supersymmetry $g_R\big(\exp(W)(\cdot), \cdot\big) = g_R\big(\cdot, \exp(W)(\cdot)\big)$. The approximation $\exp(W) \equiv 1 + W$ holds up to terms of degree $4k$ with error term $\frac{1}{2} W^2_{2k}$ in degree $4k$. Since $W \in \mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar 0}$ we have $g_R W^2_{2k} \in \mathrm{Hom}^{(4k+2)}(\mathcal{V}_\mathcal{M}, \mathcal{V}^*_\mathcal{M})_{\bar 0}$. □

We call g_R the *reduction* of g. Here the metric g appears as a deformation of the underlying Riemannian metric g_0 on M via $t \mapsto (g_0 + t \cdot g_2)\exp(\sum_{j=1}^\infty t^j W_{2j})$. Note that g_R also yields a non-degenerate alternating form on the bundle E. So in contrast to the non-graded case there is a true condition for the existence of a

Riemannian metric: the existence of a nowhere vanishing section of $E \wedge E \to M$. In particular the odd dimension of $\mathcal{M}$ has to be even. A Riemannian supermanifold $(\mathcal{M}, g)$ is split, if there is a Batchelor model, such that the Riemannian metric g has nilpotent component $W = 0$. Again the appearing deformations are essentially local via cutting off $g_R \exp(W)$ by $(g_0 + f \cdot g_2)\exp(\sum_{j=1}^{\infty} f^j W_{2j})$ with cut-off function f.

As before let $\Phi = (\varphi, \varphi^*)$, $\varphi^* = \exp(\zeta)\varphi_0^*$ be an automorphism of the supermanifold $\mathcal{M}$. We obtain $\varphi^*.g$ given by $(\varphi^*.g)(\chi, \chi') = \varphi^*\big(g((\varphi^*)^{-1}\chi\varphi^*, (\varphi^*)^{-1}\chi'\varphi^*)\big)$. Assuming $\zeta \in \mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}}$ this yields:

$$\varphi^*.g \equiv \varphi_0^*.g - (\varphi_0^*.g)(\mathrm{ad}(\zeta) \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{ad}(\zeta)) + \zeta(\varphi_0^*.g) \tag{2}$$

in $\mathrm{Hom}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$ up to terms in $\mathrm{Hom}^{(4k)}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$. Note that for the term $\zeta(\varphi_0^*.g)$, the metric is regarded as an element in $\mathrm{Hom}(\mathcal{V}_{\mathcal{M}} \otimes_{\mathcal{C}^\infty_{\mathcal{M}}(M)} \mathcal{V}_{\mathcal{M}}, \mathcal{C}^\infty_{\mathcal{M}}(M))_{\bar{0}}$. Define $\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}}$ to be the elements in $\zeta \in \mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}}$ satisfying $g_0\big(\,\mathrm{ad}(\zeta)\otimes\mathrm{Id} + \mathrm{Id}\otimes ad(\zeta)\big) + \zeta g_0 \in \mathrm{Hom}^{(2k+2)}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$. Comparing the terms in (2) with respect to the degree yields:

Proposition 2.2. *The Riemannian supermanifolds* $(\mathcal{M}, g)$ *and* $(\mathcal{M}, g')$ *with Riemannian metrics* $g = g_R \exp(W)$, $g' = g'_R \exp(W')$, $W \in \mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$, $W' \in \mathrm{End}^{(2k)}_{g'_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ *are isomorphic up to error terms in* $\mathrm{Hom}^{(4k)}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$ *via an automorphism* φ^* *with* $\varphi^*(\varphi_0^*)^{-1} \in \exp(\mathcal{V}^{(2k)}_{\mathcal{M},g'_0,\bar{0}})$ *if and only if there exist* $\varphi_0 \in \mathrm{Aut}(E^*)$ *and* $\zeta \in \mathcal{V}^{(2k)}_{\mathcal{M},g'_0,\bar{0}}$ *such that* $g'_R = \varphi_0^*.g_R$ *and:*

$$W'_{2j} = \varphi_0^*.W_{2j} - \mathrm{ad}(\zeta_{2j}) - \big((g'_R)^{-1}(\mathrm{ad}^*(\zeta) - \zeta)g'_R\big)_{2j}\,, \quad k \le j < 2k\,.$$

Here $\varphi_0^*.W$ *is defined by* $(\varphi_0^*.W)(\chi) := \varphi_0^*\big(W((\varphi_0^*)^{-1}\chi\varphi_0^*)\big)(\varphi_0^*)^{-1}$, *the homomorphism* $\mathrm{ad}^*\colon \mathcal{V}_{\mathcal{M}} \to \mathrm{End}_{\mathbb{R}}\big(\mathrm{End}_{\mathbb{R}}(\mathcal{V}_{\mathcal{M}}, \mathcal{C}^\infty_{\mathcal{M}}(M))\big)$ *denotes the representation dual to* ad, *and finally* $\big((\zeta g'_R)(X)\big)(Y) = \zeta\big(g'_R(X,Y)\big)$.

Fix g_R from now on and denote by $\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$ the automorphisms $\psi^* = \exp(\xi)\psi_0^*$ of $\mathcal{M}$ such that $g_R \equiv \psi^*.g_R$ up to a term $S := g_R - \psi^*.g_R \in \mathrm{Hom}^{(2k+2)}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$. Note that this forces $g_0 g_R^{-1} S \in \mathrm{Hom}^{(2k+2)}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}_{\mathcal{M}}^*)_{\bar{0}}$. Further $\exp(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}}) \subset \mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$ is a normal subgroup.

Parallel to the analysis of the almost complex structures we define the maps

$$\begin{aligned} F_{g_R} &: \mathrm{End}(\mathcal{V}_{\mathcal{M}}) \to \mathrm{End}(\mathcal{V}_{\mathcal{M}})\,, && F_{g_R}(\gamma) := \gamma + g_R^{-1}\gamma^* g_R \\ G_{g_R} &: \quad \mathcal{V}_{\mathcal{M}} \quad \to \mathrm{End}(\mathcal{V}_{\mathcal{M}})\,, && G_{g_R}(\zeta) := \mathrm{ad}(\zeta) + g_R^{-1}\big(\,\mathrm{ad}^*(\zeta) - \zeta\big)g_R \end{aligned}$$

denoting by γ^* the induced element in $\mathrm{End}(\mathcal{V}_{\mathcal{M}}^*)$ and ad^* as above. By Lemma 2.1 the elements in $F_{g_R}(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}})$ are up to degree $\ge 4k+2$ the appearing Ws in Riemannian metrics $g = g_R \exp(W)$ that are split up to degree $\ge 2k$. Further $G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})$ lies in $F_{g_R}\big(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)$.

Definition 2.3. For $g = g_R \exp(W)$ with $W \in \mathrm{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ we call the class $[W^{\{2k\}}]$ in the quotient of vector spaces $F_{g_R}(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}})^{\{2k\}}/G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})^{\{2k\}}$ the $2k$-th split obstruction class of g.

The $\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$-action on $F_{g_R}\big(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)$ is given up to terms in $\mathrm{End}^{(4k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ by $(\psi^*, W) \mapsto g_R^{-1}(\psi^*.g_R - g_R) + \psi^*.W$. We have $\psi^*.G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}}) \subset G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})$ by direct calculation. Analog to the almost complex case using Proposition 2.2, the action of $\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$ induces a $P\,\mathrm{Iso}(\mathcal{M}, g_R, 2k+2) := \mathrm{Iso}(\mathcal{M}, g_R, 2k+2)/\exp(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})$-action on the quotient $F_{g_R}(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}})^{\{2k\}}/G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})^{\{2k\}}$. Hence the $2k$-th split obstruction class is well-defined up to the $P\,\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$-action for a Riemannian supermanifold that is split up to terms of degree $2k+2$ and higher. We have in particular analogously to the almost complex case:

Proposition 2.4. *Let $(\mathcal{M}, g_R)$ be a split Riemannian supermanifold of odd dimension $2(2m+r)$, $m \geq 0$, $r \in \{0,1\}$. The Riemannian supermanifolds $(\mathcal{M}, g)$ with reduction g_R that are split up to terms of degree $(2m+r)+3$ and higher, correspond bijectively to the $\mathrm{Iso}\big(\mathcal{M}, g_R, 2(m+2)\big)$-orbits on $F_{g_R}\big(\mathrm{End}^{(2(m+1))}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)/G_{g_R}(\mathcal{V}^{(2(m+1))}_{\mathcal{M},g_0,\bar{0}})$.*

In an analogy to Proposition 1.5 and for later use in Section 3, it follows:

Proposition 2.5. *The map*

$$\Theta_{g_R}\colon F_{g_R}\big(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big) \longrightarrow F_{g_R}\big(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}\big)/G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})$$

locally defined by

$$F_{g_R}(\chi \otimes \mathrm{d}_{\mathcal{M}} f) \longmapsto (-1)^{|f|\,|\chi|} f \cdot G_{g_R}(\chi) + G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}})$$

is a well-defined surjective morphism of $\mathbb{Z}$-filtered vector spaces. For any element ψ^* in $\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$ and $[\psi^*] \in P\,\mathrm{Iso}(\mathcal{M}, g_R, 2k+2)$ we have $\Theta_{g_R}(\psi^*.Z) = [\psi^*].\big(\Theta_{g_R}(Z)\big)$.

Proof. Apply F_{g_R} to (1) and add $f\chi - f\chi$ in the bracket. This yields a well-defined map $F_{g_R}(\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}) \to F_{g_R}(\mathrm{End}^{(2k)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}})/G_{g_R}(\mathcal{V}^{(2k)}_{\mathcal{M},\bar{0}})$. Since $\chi \otimes \mathrm{d}_{\mathcal{M}} f$ is in $\mathrm{End}^{(2k)}_{g_0}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$, its degree $2k$ term is of the form $\sum \tilde{f}_i \frac{\partial}{\partial \xi_i} \otimes d_{\mathcal{M}} \hat{f}_i$ for an odd coordinate system (ξ_i). This forces $f\chi \in \mathcal{V}^{(2k)}_{\mathcal{M},g_0,\bar{0}}$ by direct calculation. □

3. Examples of global nowhere split structures

Here explicit examples of non-split almost complex structures, resp. non-split Riemannian metrics are given. The constructed tensors are nowhere split.

Let (M, J_M) be an almost complex manifold of dimension $2n$ and let $\mathcal{M}$ be the supermanifold defined by differential forms, i.e. $\mathcal{C}^\infty_{\mathcal{M}} = \mathcal{E}_M$. The vector fields in $\mathcal{V}_M$ act on $\mathcal{C}^\infty_{\mathcal{M}}$ by Lie derivation. Let further $\pi\colon \mathcal{V}_{\mathcal{M}} \to \mathcal{V}_{\mathcal{M}}$ be the odd $\mathcal{C}^\infty_{\mathcal{M}}$-linear operator well-defined by $\pi^2 = \mathrm{Id}$ and $\pi(\chi)(\omega) := \iota_\chi \omega$ for $\chi \in \mathcal{V}_M \subset \mathcal{V}_{\mathcal{M},\bar{0}}$ and $\omega \in \mathcal{C}^\infty_{\mathcal{M}}$.

By [11, Prop. 4.1] there exist non-degenerate 2-forms $\eta \in \mathcal{C}^\infty_{\mathcal{M}}$ compatible with J_M. We fix one and denote by g' the J_M-invariant Riemannian metric $\eta(\cdot, J_M(\cdot))$ on M. Furthermore we embed

$$\mathrm{End}(\mathcal{V}_M) \cong (\mathcal{V}_M \otimes_{\mathcal{C}^\infty_M(M)} \mathcal{E}^1_M) \hookrightarrow \mathcal{V}_{\mathcal{M}} \quad \text{by} \quad \chi \otimes \alpha \mapsto \xi_{\chi\otimes\alpha} := \alpha \cdot \chi$$

and obtain $\xi_{\mathrm{Id}}, \xi_{J_M} \in \mathcal{V}_{\mathcal{M},1}$ and $\pi(\xi_{Id}), \pi(\xi_{J_M}) \in \mathcal{V}_{\mathcal{M},0}$.

We define on $\mathcal{M}$ by $\mathcal{C}^\infty_{\mathcal{M}}$-linear continuation to $\mathrm{End}(\mathcal{V}_{\mathcal{M}})$, resp. $\mathrm{Hom}(\mathcal{V}_{\mathcal{M}}, \mathcal{V}^*_{\mathcal{M}})$:

(i) the split almost complex structure J_R by $J_M \oplus (\pi \circ J_M \circ \pi) \in \mathrm{End}_{\mathcal{C}^\infty_M}(\mathcal{V}_M \oplus \pi(\mathcal{V}_M))$;

(ii) the split Riemannian metric g_R by $g' + \big(\eta \circ (\pi \otimes \pi)\big) \in \mathrm{Hom}_{\mathcal{C}^\infty_M}(\mathcal{V}_M^{\otimes 2} \oplus \pi(\mathcal{V}_M)^{\otimes 2}, \mathcal{C}^\infty_M)$ and $g_R\big(\mathcal{V}_M \otimes \pi(\mathcal{V}_M)\big) = 0$.

Note the following technical lemma for later application:

Lemma 3.1. *Let f, $g \in \mathrm{End}(\mathcal{V}_M)$, $\omega \in \mathcal{C}^\infty_{\mathcal{M},2}$. Then:*

(a) $\pi(\xi_f)(\omega) = \frac{1}{2}\big(\omega(f(\cdot), \cdot) + \omega(\cdot, f(\cdot))\big)$;

(b) $[\pi(\xi_f), \pi(\xi_g)] = -\pi(\xi_{[f,g]})$;

(c) $J_R(\xi_{J_M}) = -\xi_{\mathrm{Id}}$ *and* $J_R\big(\pi(\xi_{J_M})\big) = -\pi(\xi_{Id})$.

Further fix in the almost complex, resp. Riemannian case the tensors:

(i) $J = J_R \exp(\eta \cdot Y_\eta)$ with $Y_\eta \in \mathrm{End}^{(2)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ by $Y_\eta = F_{J_R}(\pi(\xi_{J_M}) \otimes \mathrm{d}_{\mathcal{M}}\eta)$;

(ii) $g = g_R \exp(\eta \cdot W_\eta)$ with $W_\eta \in \mathrm{End}^{(2)}(\mathcal{V}_{\mathcal{M}})_{\bar{0}}$ by $W_\eta = F_{g_R}(\pi(\xi_{J_M}) \otimes d_{\mathcal{M}}\eta)$.

We prove:

Lemma 3.2. *Assume that $n > 1$. The endomorphisms Y_η and W_η are nowhere vanishing. In particular $\big(Y_\eta(\pi(\xi_{J_M}))\big)(\eta) = \eta^2 \equiv \big(W_\eta(\pi(\xi_{J_M}))\big)(\eta)$ up to terms of degree 6 and higher.*

Proof. With Lemma 3.1 (c) follows $Y_\eta = \pi(\xi_{J_M}) \otimes \mathrm{d}_{\mathcal{M}}\eta - \pi(\xi_{\mathrm{Id}}) \otimes \big((\mathrm{d}_{\mathcal{M}}\eta) \circ J_R\big)$. Applying $\big(Y_\eta(\pi(\xi_{J_M}))\big)(\eta)$ we obtain $\big(\pi(\xi_{J_M})(\eta)\big)^2 + \big(\pi(\xi_{Id})(\eta)\big)^2$. By Lemma 3.1 (a) we have $\pi(\xi_{J_M})(\eta) = 0$ since η is compatible with J_M, and $\big(Y_\eta(\pi(\xi_{J_M}))\big)(\eta) = \eta^2$. So Y_η is nowhere vanishing.

For the second statement note $W_\eta = \pi(\xi_{J_M}) \otimes d_{\mathcal{M}}\eta + g_R^{-1}(d_{\mathcal{M}}\eta) \cdot g_R\big(\pi(\xi_{J_M})\big)$. Further $\big(W_\eta(\pi(\xi_{J_M}))\big)(\eta) = \big(\pi(\xi_{J_M})(\eta)\big)^2 + \big(g_R^{-1}(d_{\mathcal{M}}\eta)\big)(\eta) \cdot \eta(\xi_{J_M}, \xi_{J_M})$. Due to the compatibility of η and J_M, we have $\eta(\xi_{J_M}, \xi_{J_M}) = \eta$ and as before, $\pi(\xi_{J_M})(\eta) = 0$. Further a calculation yields $\big(g_R^{-1}(d_{\mathcal{M}}\eta)\big)(\eta) \equiv g_R\big(g_R^{-1}(d_{\mathcal{M}}\eta), g_R^{-1}(d_{\mathcal{M}}\eta)\big) \equiv \eta$ up to terms of degree 4 and higher. Hence $\big(W_\eta(\pi(\xi_{J_M}))\big)(\eta) \equiv \eta^2$ up to terms of degree 6 and higher. So W_η is nowhere vanishing. $\square$

Finally it follows:

Theorem 3.3. *Assume that $n > 2$. The almost complex structure $J = J_R \exp(\eta \cdot Y_\eta)$ and the Riemannian metric $g = g_R \exp(\eta \cdot W_\eta)$ on $\mathcal{M}$ are nowhere split.*

Proof. For $\psi^* = \exp(\xi)\psi_0^* \in \mathrm{Hol}(\mathcal{M}, J_R, 4)$ we obtain $[\mathrm{ad}(\xi_2), J_R] = 0$. With the identity $\exp(\xi_2).J_R = \exp\big(\mathrm{ad}(\xi_2)\big) J_R \exp\big(\mathrm{ad}(\xi_2)\big) = \exp([\mathrm{ad}(\xi_2), \cdot])(J_R) = J_R$ it follows that ψ^* maps $Y \in F_{J_R}\big(\mathrm{End}^{\{4\}}(\mathcal{V}_\mathcal{M})_{\bar{0}}\big)$ to $F_{J_R}\big(\mathrm{ad}(\xi_4)\big) + \psi_0^*.Y$ up to terms of degree ≥ 6. Hence $F_{J_R}\big(\mathrm{ad}(\mathcal{V}^{\{4\}}_{\mathcal{M},\bar{0}})\big)$ is in a $P\,\mathrm{Hol}(\mathcal{M}, J_R, 4)$-orbit up to terms of degree ≥ 6. Using Proposition 1.5 and $\eta Y_\eta = F_{J_R}\big(\eta\pi(\xi_{J_M}) \otimes \mathrm{d}_\mathcal{M}\eta\big)$ it is sufficient to check that $\eta F_{J_R}\big(\mathrm{ad}(\eta\pi(\xi_{J_M}))\big)$ does not vanish in degree 4. Following (1), $F_{J_R}\big(\mathrm{ad}(\eta\pi(\xi_{J_M}))\big) = \eta \cdot F_{J_R}\big(\mathrm{ad}(\pi(\xi_{J_M}))\big) - Y_\eta$. Now Lemma 3.1 (b) and (c) yield that $F_{J_R}\big(\mathrm{ad}(\pi(\xi_{J_M}))\big)\big(\pi(\xi_{J_M})\big) = 0$. Using this we obtain $F_{J_R}\big(\mathrm{ad}(\eta\pi(\xi_{J_M}))\big)\big(\pi(\xi_{J_M})\big)(\eta) = -Y_\eta\big(\pi(\xi_{J_M})\big)(\eta)$ which is η^2 following Lemma 3.2. This proves the first statement.

We have by direct calculation $\eta W_\eta \in \mathrm{End}^{(4)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar{0}}$. Let $\psi^* = \exp(\xi)\psi_0^* \in \mathrm{Iso}(\mathcal{M}, g_R, 6)$, then the degree 4 term of $\psi^*.g$ vanishes while the degree 6 term is $\psi_0^*.(g_2 W_4 + g_0 W_6) + (\psi^*.g_R)_6$. Further $(1 + \xi_2)\psi_0^*$ preserves g_R and so does $\exp(\xi_2)\psi_0^*$. The term $\big(\exp(\xi_4 + \xi_6).g_R\big)_6$ equals $\big(g_R(\mathrm{ad}(\xi_4 + \xi_6) \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{ad}(\xi_4 + \xi_6)) - (\xi_4 + \xi_6) g_R\big)_6$. So ψ^* maps $W \in \mathrm{End}^{(4)}_{g_0}(\mathcal{V}_\mathcal{M})_{\bar{0}}$ to $G_{g_R}(\xi_4) + \psi_0^*(W_4)$ up to terms of degree ≥ 6. Since ψ^* preserves the vanishing degree four term of g_R, it follows that $\xi_4 \in \mathcal{V}^{\{4\}}_{\mathcal{M},g_0,\bar{0}}$. Hence $G_{g_R}(\mathcal{V}^{\{4\}}_{\mathcal{M},g_0,\bar{0}})$ is in a $P\,\mathrm{Iso}(\mathcal{M}, g_R, 6)$-orbit up to terms of degree ≥ 6. In analogy to the almost complex case and following Proposition 2.5 it is sufficient to show that $\eta G_{g_R}(\eta\pi(\xi_{J_M}))$ is nowhere vanishing. We have the identity $G_{g_R}\big(\eta\pi(\xi_{J_M})\big) = \eta \cdot G_{g_R}\big(\pi(\xi_{J_M})\big) - W_\eta$. Note that for $\alpha \in \mathcal{V}^*_\mathcal{M}$ it follows $(g_R^{-1}(\alpha))(\eta) = \alpha(g_R^{-1}(d_\mathcal{M}\eta))$ and $g_R^{-1}(d_\mathcal{M}\eta) \equiv \pi(\xi_{\mathrm{Id}})$ up to terms of degree two and higher. Using these details, Lemma 3.1 (b), and the definition of g_R, one obtains $G_{g_R}\big(\pi(\xi_{J_M})\big)\big(\pi(\xi_{J_M})\big)(\eta) \equiv -\pi(\xi_{J_M})\big(\eta(\pi(\xi_{\mathrm{Id}}), \pi(\xi_{J_M}))\big)$ up to terms of degree four and higher. A direct calculation using the graded Leibniz rule and J_M-invariance of η shows that $G_{g_R}\big(\pi(\xi_{J_M})\big)\big(\pi(\xi_{J_M})\big)(\eta)$ vanishes up to terms of degree four and higher. Hence $G_{g_R}\big(\eta\pi(\xi_{J_M})\big)\big(\pi(\xi_{J_M})\big)(\eta) \equiv -W_\eta\big(\pi(\xi_{J_M})\big)(\eta)$ up to terms of degree 6 and higher. Following Lemma 3.2, the degree 4 term of this expression is η^2. This proves the second statement. $\square$

Following [8], connections on supermanifolds are in direct correspondence to splittings. The present results in particular show that e.g. the Levi-Civita connection of a metric does not in general yield a splitting that reduces the metric in the above sense.

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