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RECOGNIZABILITY OF FINITE GROUPS BY SUZUKI GROUP

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Abstract. Let G be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. In this paper, we will show that $G \cong Sz(q)$ if and only if $\mathcal{S}(G) \cong \mathcal{S}(Sz(q))$, where $q = 2^{2m+1} \geq 8$.

1. Introduction

Let G be a finite group and $x \in G$. The order of x is denoted by $o(x)$. The set of all element orders of G is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. It is clear that $\pi_e(G)$ is determined by the subset $\mu(G)$ of maximal element orders with respect to divisibility. We set $m_i = m_i(G) = |\{g \in G | o(g) = i\}|$.

The *main supergraph* $\mathcal{S}(G)$ is the graph whose vertices are the group elements and two elements x and y are connected if either $o(x) \mid o(y)$ or $o(y) \mid o(x)$. We also denote the subgraph of $\mathcal{S}(G)$ with the identity removed by $\mathcal{S}^*(G)$ [4]. We write $x \sim y$ when two vertices x and y are adjacent.

For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G | x^d = 1\}$. We say that the groups G_1 and G_2 are of *the same order type* if $|G_1(d)| = |G_2(d)|$, for all $d \in \mathbb{N}$. By the definition of the main supergraph, it is clear that if G_1 and G_2 are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. The example $G_1 = Z_4 \times Z_4$ and $G_2 = Z_4 \times Z_2 \times Z_2$ shows that the converse statement is not true in general.

In 1987, J.G. Thompson [9, Problem 12.37] posed the following Problem:
Thompson's Problem. Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

In [5], the set of m_i (also known as nse) was used to prove no solvable group has the same order type as $Sz(2^{2m+1})$, where $2^{2m+1} - 1$ is a prime power. In this paper we use instead the supergraph to remove the requirement that $2^{2m+1} - 1$ is a prime power. As two groups having the same order type implies their supergraphs coincide, if a solvable group is uniquely determined by $\mathcal{S}(G)$, then Thompson's conjecture holds for G. In [7, 8, 10], the authors proved that alternating groups of degree p, $p+1$ and $p+2$, the symmetric groups of degree p, the small Ree groups ${}^2G_2(3^{2n+1})$ and $\mathrm{PSL}_2(q)$, where q is a prime power are uniquely determined by

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their main supergraph. Furthermore, in [6], it was also proven for sporadic simple groups and for $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(p)$, where p is a prime. Our main theorem is as follows:

Main Theorem. Let $\mathcal{S}(G) \cong \mathcal{S}(Sz(q))$, where $q = 2^{2m+1} \geq 8$. Then $G \cong Sz(q)$.

We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Throughout this paper, we denote by ϕ the Euler's totient function.

2. Preliminary results

In this section, we present some preliminary results which will turn out to be useful in what follows. First, we quote some known results about Frobenius group and 2-Frobenius group, which are useful in the sequel.

Lemma 2.1 ([2]). *Let G be a 2-Frobenius group of even order, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*
(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$, $(|G/K|, |K/H|) = 1$ and $G/K \lesssim \mathrm{Aut}(K/H)$.

Lemma 2.2 ([2]). *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$.*

Lemma 2.3 ([12]). *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*
(a) *G is a Frobenius group or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Lemma 2.4 ([3]). *The Suzuki groups are only non-abelian simple groups of order prime to* 3.

Lemma 2.5 ([1, 11]). *Let $S = Sz(q)$ with $q = 2^{2m+1} \geq 8$, $m \geq 1$. Then $m_2(S) = (q-1)(q^2+1)$, $m_4(S) = (q^2 - q)(q^2 + 1)$ and $m_{2^r} = 0$ for $r \geq 3$.*

3. Proof of the Main Theorem

Now we are ready to prove the main theorem of this paper.

Proof of the main theorem. It is well known that $Sz(q)$ has no elements of order $2r$ with r an odd number (see [11]). Therefore, by Lemma 2.5, $\mathcal{S}^*(Sz(q))$ has a complete component consisting of the $(q^2+1)(q^2-1)$ elements whose order is a power of 2. Let K_1 denote this component in $\mathcal{S}^*(G)$. Since K_1 is complete, it follows that K_1 contains only elements of prime power order for a fixed prime p. Moreover, as 2 divides the number of elements who order is a power of p (by pairing g and g^{-1}) but does not divide $(q^2+1)(q^2-1)$, it follows that $p = 2$. Therefore, 2 is an isolated vertex in the prime graph $\Gamma(G)$.

If G is a Frobenius group with complement H, then by Lemma 2.2 $|H| = q^2$ or $(q^2+1)(q-1)$. However $|H| \mid |G|/|H| - 1$ which gives a contradiction. While, if G is a 2-Frobenius group with series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ as in Lemma 2.1, then $(q^2+1)(q-1) = |K/H| \mid |H| - 1 = 2^t - 1$ for some t which is a contradiction. Thus by Lemma 2.3 G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with $K/H \cong Sz(q^{'})$ for some $q^{'} < q$ as $3 \nmid |G|$ by Lemma 2.4. Furthermore, Lemma 2.3 implies H and G/K are 2-groups and therefore $(q^2+1)(q-1) \mid (q^{'2}+1)(q^{'}-1)$ showing that $q^{'} = q$, $K = G$ and $H = 1$. In particular, $G \cong Sz(q)$. □

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