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# HOMOGENEOUS RANDERS SPACES ADMITTING JUST TWO HOMOGENEOUS GEODESICS

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Abstract. The existence of a homogeneous geodesic in homogeneous Finsler manifolds was investigated and positively answered in previous papers. It is conjectured that this result can be improved, namely that any homogeneous Finsler manifold admits at least two homogenous geodesics. Examples of homogeneous Randers manifolds admitting just two homogeneous geodesics are presented.

## 1. Introduction

The existence of at least one homogeneous geodesics in arbitrary homogeneous Riemannian manifold was proved by O. Kowalski and J. Szenthe in [7]. In the papers [6] and [9], it was proved that this result is optimal, namely, examples of homogeneous Riemannian metrics on solvable Lie groups were constructed which admit just one homogeneous geodesic through any point. Generalization of the above existence result to Finsler geometry was proved in the series of papers [12] by Z. Yan and S. Deng for Randers metrics, [4] by the author for odd-dimensional Finsler metrics, [5] by the author for Berwald or reversible Finsler metrics, [13] by Z. Yan and L. Huang in general. However, due to the nonreversibility of general Finsler metrics, it is conjectured that the result and its proofs in the nonreversible situation are not optimal.

**Conjecture 1.** *An arbitrary homogeneous Finsler manifold admits at least two homogeneous geodesics through arbitrary point.*

In comparison with Riemannian geometry, the situation is rather delicate. In the context of Finsler geometry, the trajectory of the unique homogeneous geodesic in a Riemannian manifold should be regarded as two geodesics with initial vectors $X$ and $-X$. For a general homogeneous Finsler manifold, the initial vectors of the two homogeneous geodesics may be non-opposite. For the moment, it is not clear how to adapt existing proofs in mentioned papers to prove the above conjecture in full generality.

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In the present paper, examples of invariant Randers metrics which admit just two homogeneous geodesics are constructed. The initial vectors of these geodesics are $X+Y$ and $-X+Y$, for certain vectors $X, Y \in T_pM$. For the construction, Randers metrics which are modifications of Riemannian metrics of examples from [6] and [9] are used. Further, it is demonstrated with an example that general Randers metrics whose underlying Riemannian metric admits just two homogeneous geodesics (with initial vectors $X$ and $-X$) may admit more than two homogeneous geodesics.

## 2. Basic settings

A *Minkowski norm* on the vector space $\mathbb{V}$ is a nonnegative function $F\colon \mathbb{V} \to \mathbb{R}$ which is smooth on $\mathbb{V}\setminus\{0\}$, positively homogeneous ($F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$) and whose Hessian $g_{ij} = (\frac{1}{2}F^2)_{y^i y^j}$ is positively definite on $\mathbb{V}\setminus\{0\}$. Here $(y^i)$ are the components of a vector $y \in \mathbb{V}$ with respect to a fixed basis $B$ of $\mathbb{V}$ and putting $y^i$ to a subscript means the partial derivative. Then the pair $(\mathbb{V}, F)$ is called the *Minkowski space*. The tensor $g_y$ with components $g_{ij}(y)$ is the *fundamental tensor*. A Finsler metric on the smooth manifold $M$ is a function $F$ on $TM$ which is smooth on $TM \setminus \{0\}$ and whose restriction to any tangent space $T_xM$ is a Minkowski norm. Then the pair $(M, F)$ is called the *Finsler manifold*. On a Finsler manifold, functions $g_{ij}$ depend smoothly on $x \in M$ and on $o \neq y \in T_xM$.

Special Minkowski norms are the Randers norms. They are determined by a symmetric positively definite bilinear form $\alpha$ and a vector $V$ such that $\alpha(V,V) < 1$, or, equivalently, its $\alpha$-equivalent 1-form $\beta$ related with $V$ by the formula

$$\beta(U) = \alpha(V, U) \qquad \forall U \in \mathbb{V}.$$

The Randers norm $F$ is defined by the formula

$$F(U) = \sqrt{\alpha(U,U)} + \beta(U) \qquad \forall U \in \mathbb{V}. \tag{1}$$

If a Finsler metric $F$ on $M$ restricted to any tangent space $T_pM$ is a Randers norm, it is called a *Randers metric*. Obviously, the Randers metric $F$ is determined by a Riemannian metric $\alpha$ and a smooth 1-form $\beta$ and formula (1) holds on each tangent space $T_pM$. We remark that, in the literature, the letter $\alpha$ is sometimes used for the norm induced by the 2-form $\alpha$ and then formula (1) above is without the square root. We choose the notation above because for $\beta = 0$, $F$ is the Riemannian norm and components $g_{ij}$ of the fundamental tensor are just the components of the Riemannian metric $\alpha$.

Let $M$ be a Finsler manifold $(M, F)$. If there is a connected Lie group $G$ which acts transitively on $M$ as a group of isometries, then $M$ is called a *homogeneous manifold*. We remark that a homogeneous manifold $(M, F)$ may admit more presentations as a homogeneous space in the form $G/H$, corresponding to various transitive isometry groups. The following theorem gives the relation between the isometry group of a Randers manifold and the isometry group of the corresponding underlying Riemannian manifold. We shall use this theorem later.

**Theorem 2** ([2]). *Let $(M, F)$ be a Randers manifold with the Finsler function $F = \sqrt{\alpha} + \beta$. Then the group of isometries of $(M, F)$ is a closed subgroup of the group of isometries of the Riemannian manifold $(M, \alpha)$.*

Homogeneous manifold $M$ can be naturally identified with the *homogeneous space* $G/H$, where $H$ is the isotropy group of the origin $p \in M$. A homogeneous Finsler space $(G/H, F)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of $H$ on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m}+\mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space $T_pM$ via the projection $\pi\colon G \to G/H = M$. Using this natural identification, from the Minkovski norm and its fundamental tensor on $T_pM$, we obtain the $\mathrm{Ad}(H)$-invariant Minkowski norm and the $\mathrm{Ad}(H)$-invariant fundamental tensor on $\mathfrak{m}$ and we denote these again by $F$ and $g$. In particular, for the invariant Randers metrics, we shall use the following theorem.

**Theorem 3** ([2]). *Let $(G/H, \alpha)$ be a Riemannian homogeneous space with the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Then there is a one-to-one correspondence between $G$-invariant Randers metrics on $G/H$ whose underlying Riemannian metric is $\alpha$ and the set*

$$\mathcal{V} = \{V \in \mathfrak{m} \mid \alpha(V) < 1, \mathrm{Ad}(H)(V) = V\}\,.$$

In our examples which follow, the algebra $\mathfrak{h}$ is always trivial. In such a case, any vector $V \in \mathfrak{g}$ gives rise to a $G$-invariant Randers metric.

We further recall that the *slit tangent bundle* $TM_0$ is defined as $TM_0 = TM \setminus \{0\}$. Using the restriction of the natural projection $\pi\colon TM \to M$ to $TM_0$, we naturally construct the pullback vector bundle $\pi^*TM$ over $TM_0$. The *Chern connection* is the unique linear connection on the vector bundle $\pi^*TM$ which is torsion free and almost $g$-compatible, see some monograph, for example [1] by D. Bao, S.-S. Chern and Z. Shen or [2] by S. Deng for details. Using the Chern connection, the derivative along a curve $\gamma(t)$ can be defined. A regular smooth curve $\gamma$ with tangent vector field $T$ is a *geodesic* if $D_T(\frac{T}{F(T)}) = 0$. In particular, a geodesic of constant speed satisfies $D_TT = 0$.

A geodesic $\gamma(s)$ through the point $p$ is *homogeneous* if it is an orbit of a one-parameter group of isometries. More explicitly, if there exists a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector $X$ is called a *geodesic vector*. Geodesic vectors are characterized by the following *geodesic lemma*, proved in Riemannian geometry by O. Kowalski and L. Vanhecke in [8] and generalized to Finsler geometry by D. Latifi in [10].

**Lemma 4** ([10]). *Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. A nonzero vector $Y \in \mathfrak{g}$ is geodesic if and only if it holds*

$$g_{Y_\mathfrak{m}}(Y_\mathfrak{m}, [Y, U]_\mathfrak{m}) = 0 \qquad \forall U \in \mathfrak{m}\,,$$

*where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.*

In the special situation of a Randers space, the characterization of geodesic vectors using the Riemannian metric $\alpha$ and the 1-form $\beta$ is given in the following statement. See the paper [3] by the author for a direct proof or [11] by Z. Yan and S. Deng for a proof using the navigation data.

**Lemma 5** ([3]). *Let $F = \sqrt{\alpha} + \beta$ be a homogeneous Randers metric on $G/H$, let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be a reductive decomposition and $V \in \mathfrak{m}$ be the vector $\alpha$-equivalent with $\beta$. The vector $Y = X + \xi(X) \in \mathfrak{g}$, where $X \in \mathfrak{m}$ and $\xi(X) \in \mathfrak{h}$, is geodesic if and only if*

$$\alpha\big(X + \sqrt{\alpha(X,X)} \cdot V, [X + \xi(X), U]_{\mathfrak{m}}\big) = 0 \qquad \forall U \in \mathfrak{m}\,. \tag{2}$$

We shall use formula (2) in the case with trivial algebra $\mathfrak{h}$. In such a case, we obviously have $\xi(X) = 0$.

## 3. An $n$-dimensional example

We shall consider the series of examples in arbitrary dimension, constructed with Riemannian metrics in [9]. We shall modify these metrics and obtain invariant Randers metrics. We identify a special family of these Randers metrics, which have desired properties. Let us start with the Lie algebra $\mathfrak{n}$ with the orthonormal basis $B = \{E_1, \dots, E_{n+1}\}$ and generated by the Lie brackets

$$\begin{aligned} [E_i, E_j] &= 0\,, \qquad \forall i, j, \leq n\,, \\ [E_{n+1}, E_i] &= a_i E_i + E_{i+1}\,, \quad \forall i < n\,, \\ [E_{n+1}, E_n] &= a_n E_n\,, \end{aligned}$$

for arbitrary nonzero parameters $a_1, \dots, a_n \in \mathbb{R}$. The corresponding Lie group $N$ is solvable and it is endowed with the invariant Riemannian metric induced by the scalar product determined by the given orthonormal basis. It was proved in [9] that, for generic choice of the parameters $a_1$, $a_2$, the group $N$ acting on itself by left translations is the maximal group of isometries. Using Theorem 3, with respect to the isometry group $N$, it follows that an arbitrary vector $V \in \mathfrak{n}$ such that $\|V\| < 1$ gives rise to the Randers norm $F$ on $\mathfrak{n}$ given by the formula (1) and consequently to an invariant Randers metric $F$ on $N$. Using Theorem 2, we see that the group $N$ acting on itself by left translations is the maximal isometry group of the Finsler manifold $(N, F)$. For the simplicity, we shall consider only Randers metrics generated by the vector $V = kE_1,\ 0 < k < 1$, which are suitable for our purposes.

**Theorem 6.** *Let $(N, F)$ be the $n$-dimensional homogeneous Randers space constructed above, with parameters $a_i$ such that $\min\{a_i\}_{i=1}^n > n$, with a Randers metric $F$ determined by a vector $V = kE_1$ and such that $ka_1 < 1$. Then $(N, F)$ admits just two homogeneous geodesics through the origin $e \in N$. The initial vectors of these geodesics are $Z_1 = E_{n+1} - \frac{k}{\sqrt{1-k^2}}E_1$ and $Z_2 = -E_{n+1} - \frac{k}{\sqrt{1-k^2}}E_1$.*

**Proof.** We express an arbitrary vector $X \in \mathfrak{n}$ with respect to the basis $B$ as $X = x_1E_1 + \cdots + x_{n+1}E_{n+1}$ and we write down the Lie brackets

$$\begin{aligned}
[X, E_i] &= x_{n+1}(a_iE_i + E_{i+1})\,, \qquad 1 \le i < n\,,\\
[X, E_n] &= x_{n+1}a_nE_n\,,\\
[X, E_{n+1}] &= -x_1a_1E_1 - \sum_{i=2}^{n}(x_{i-1} + x_ia_i)E_i\,.
\end{aligned}$$

We now substitute for the vector $U$ in formula (2), step by step, all vectors from the orthonormal basis $B$. According to Lemma 5, the vector $X$ is geodesic if it satisfies the system of equations

$$\begin{aligned}
x_{n+1}\big(a_1(x_1 + k\|X\|) + x_2\big) &= 0\,,\\
x_{n+1}\big(a_ix_i + x_{i+1}\big) &= 0\,, \qquad 1 < i < n\,,\\
x_{n+1}a_nx_n &= 0\,,\\
a_1x_1(x_1 + k\|X\|) + \sum_{i=2}^{n}(x_{i-1} + x_ia_i)x_i &= 0\,.
\end{aligned} \tag{3}$$

First, let $x_{n+1} \neq 0$ in the system of equations (3). Because these equations are homogeneous, we can assume, without the loss of generality, $x_{n+1} = \pm 1$. From the equations number $n, \dots, 1$ in the system (3), we obtain immediately the conditions $x_n = x_{n-1} = \cdots = x_2 = 0$ and $x_1 + k\|X\| = 0$. It follows that $x_1 = \frac{-k}{\sqrt{1-k^2}}$, for both choices $x_{n+1} = \pm 1$, and we obtain just the solutions in the statement.

Second, let $x_{n+1} = 0$. The first $n$ equations in the system (3) are satisfied and the last equation is $p(x_i) = 0$ for the homogeneous polynomial

$$p(x_i) = a_1x_1^2 + a_1x_1k\|X\| + \sum_{i=2}^{n} x_{i-1}x_i + \sum_{i=2}^{n} a_ix_i^2\,.$$

Without the loss of generality, we can assume $\|X\| = \sum_{i=1}^{n} x_i^2 = 1$. We now use the estimate $|x_{i-1}x_i| < 1$ for each term in the first sum and also $|a_1x_1k| < 1$, using the assumption. Using other assumptions, we obtain the estimate for the whole polynomial $p(x_i)$ from below

$$p(x_i) > \min\{a_i\} \cdot \sum_{i=1}^{n} x_i^2 - n = \min\{a_i\} - n > 0\,.$$

We see that the polynomial $p(x_i)$ is always positive on the unit sphere of vectors $X$ such that $\|X\| = \sum_{i=1}^{n} x_i^2 = 1$ and the equation $p(x_i) = 0$ has no nontrivial solution $x_1, \dots, x_n$. Consequently, the system (3) has no other solution than those in the statement. □

Obviously, the estimates in the above proof are rather rough and the statement is valid for more general choice of parameters $a_1, \dots, a_n$. To show that the statement is not valid for arbitrary Randers metric based on a Riemannian metric with just one homogeneous geodesic, we analyze in more detail the example in dimension 3, with the assumption $a_2 = -a_1$.

**Proposition 7.** *Let $(N, F)$ be the 3-dimensional homogeneous Randers space constructed above, with parameters $a_1$, $a_2$ such that $a_2 = -a_1$ and with a Randers metric $F$ determined by a vector $V = kE_1$. Then $(N, F)$ admits at least 6 homogeneous geodesics through the origin $e \in N$.*

**Proof.** The system of equations (3), for the Randers metric $F$ becomes

$$\begin{aligned} x_3 a_1(x_1 + k\|X\|) + x_3 x_2 &= 0\,, \\ x_3 x_2 a_2 &= 0\,, \\ x_1 a_1(x_1 + k\|X\|) + (x_1 + x_2 a_2) x_2 &= 0\,. \end{aligned} \tag{4}$$

If $x_3 \neq 0$, we obtain the two solutions from Theorem 6. If $x_3 = 0$, we assume $\|X\| = 1$ and we put $a_2 = -a_1$ into the third equation of the system (4). We denote this parameter simply by $a$ and the equation becomes

$$a x_1^2 + a k x_1 + x_1 x_2 - a x_2^2 = 0\,. \tag{5}$$

We now introduce circular coordinates $x_1 = \cos(t), x_2 = \sin(t)$ and the equation (5) becomes $f(t) = 0$ for the smooth function

$$f(t) = a\big(\cos(2t) + k\cos(t)\big) + \frac{1}{2}\sin(2t)\,.$$

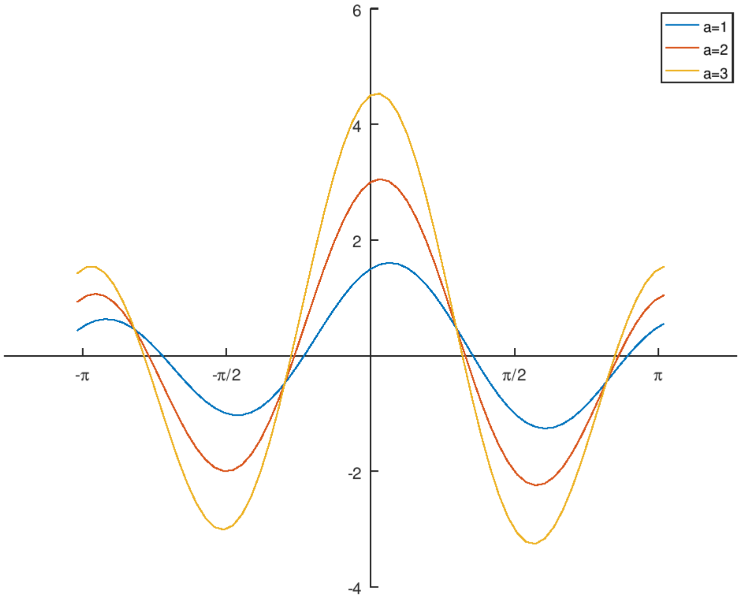
We denote by $t_1, \dots, t_4$ the values of the parameter $t$ from the interval $(-\pi, \pi)$, for which $t_i < t_{i+1}$ and

$$\cos(2t_i) + k\cos(t_i) = 0\,, \qquad i = 1, \dots, 4\,.$$

Because $k > 0$, it holds $t_1 \in (-\pi, -\frac{3\pi}{4})$, $t_2 \in (-\frac{\pi}{2}, -\frac{\pi}{4})$, $t_3 \in (\frac{\pi}{4}, \frac{\pi}{2})$, $t_4 \in (\frac{3\pi}{4}, \pi)$ and $t_1 = -t_4$, $t_2 = -t_3$. Obviously, it holds

$$f(t_i) = \frac{1}{2}\sin(2t_i)\,, \qquad i = 1, \dots, 4\,,$$

for any $a$ and any $k$. Further, it holds $f(t_1) > 0$, $f(t_2) < 0$, $f(t_3) > 0$, $f(t_4) < 0$. From the continuity and the periodicity of the function $f(t)$, it follows that there are at least 4 distinct solutions of the equation $f(t) = 0$ on the interval $\langle -\pi, \pi \rangle$. For the illustration, we include the picture of the function $f(t)$ for three different values of the parameter $a$, whose value is chosen as 1, 2, or 3, respectively and for $k = \frac{1}{2}$. The intersections of all the graphs in the picture are at the points with coordinates $[t_i, \frac{1}{2}\sin(2t_i)]$.




Consequently, each solution $\bar{t}$ of the equation $f(t) = 0$ corresponds to a solution $(x_1, x_2, x_3) = (\cos(\bar{t}), \sin(\bar{t}), 0)$ of the system of equations (4) and it determines an initial vector of a homogeneous geodesic. □

We have seen that the explicit description of all homogeneous geodesics of a given Randers metric is not easy, even in a simple situation in dimension 3.

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