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REMARKS ON NATURAL DIFFERENTIAL OPERATORS WITH TENSOR FIELDS

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Abstract. We study natural differential operators transforming two tensor fields into a tensor field. First, it is proved that all bilinear operators are of order one, and then we give the full classification of such operators in several concrete situations.

Introduction

In differential geometry, many natural differential operators transforming two tensor fields into a tensor field are used. For instance, the Frölicher-Nijenhuis bracket of two tangent-valued forms (see [2]), the Shouten bracket of two multi-vector fields (see [6]), the Lie derivative of a form with respect to a tangent-valued form (see [3]) and so on. The common property of all such operators is that they are $\mathbb{R}$-bilinear and of order one.

In the present paper, we shall discuss such operators in the case that one of the input tensor fields φ is of type $(1,p)$ and the second input tensor field ψ is of type (r,s). We shall prove that for $p>1$, $s>r$, any natural differential operator Φ transforming φ and ψ into a $(r,s+p)$-tensor field is $\mathbb{R}$-bilinear and of order one. If we assume that the operator is bilinear, then it is of order one for any p, r, s. Choice of the tensor field φ of type $(1,p)$ is motivated by the paper [8] where operators of the above type were studied under some special properties of the input fields. In addition to the result of [8], we give the full classification of operators without the assumption of special properties of the input fields.

We shall give as examples full classification of natural bilinear operators transforming a vector field X or a (1,1)-tensor field φ or a (1,2)-tensor field S and a tensor field ψ into tensor fields.

We assume that all operators are natural in the sense of [3]. We use the general properties of such operators. To classify natural differential operators on tensor fields we use the method of an auxiliary linear symmetric connection K, [4, p. 144], and the second-order reduction theorem, [7, p. 165]. We assume that a k-order natural operator also depends on a symmetric linear connection K. Then, according

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to the second reduction theorem, such operator is factorized through the covariant derivatives up to the order k and covariant derivatives of the curvature tensor of K up to the order $(k-2)$. Finally, we assume that the operator is independent of K.

All manifolds and mappings are assumed to be smooth.

1. Preliminaries

Let M be an m-dimensional manifold and (x^i) local coordinates on M. We shall denote as ∂_i and d^i local bases of vector fields and 1-forms.

First of all, we shall discuss the order of natural operators transforming two tensor fields φ and ψ into tensor fields. We shall assume that φ is a tensor field of type $(1,p)$.

Theorem 1.1. *All finite order natural differential operators transforming a $(1,p)$, $p>1$, tensor field φ and an (r,s), $s>r$, tensor field ψ into $(r,s+p)$ tensor fields $\Phi(\varphi,\psi)$ are $\mathbb{R}$-bilinear and of order 1.*

If we assume that the operator Φ is $\mathbb{R}$-bilinear we can consider weaker conditions on types of tensor fields φ and ψ.

Theorem 1.2. *All finite order $\mathbb{R}$-bilinear natural differential operators transforming a $(1,p)$-tensor field φ and an (r,s)-tensor field ψ into $(r,s+p)$-tensor fields $\Phi(\varphi,\psi)$ are of order 1.*

Proof of Theorem 1.1. Let us assume a k-order, $k\geq 1$, natural differential operator

$$\Phi\colon C^\infty(T^{(1,p)}M)\times C^\infty(T^{(r,s)}M)\to C^\infty(T^{(r,s+p)}M)\,,$$

where $p>1$ and $s>r$. Then the associated fibred morphism (denoted by the same symbol)

$$\Phi\colon J^k(T^{(1,p)}M)\times_M J^k(T^{(r,s)}M)\to T^{(r,s+p)}M$$

is an equivariant mapping with respect to the actions of the $(k+1)$-order differential group $G^{k+1}_m=\mathrm{reg}J^k_0(\mathbb{R}^m,\mathbb{R}^m)_0$ on the standard fibres of $J^k(T^{(1,p)}M)$, $J^k(T^{(r,s)}M)$ and $T^{(r,s+p)}M$. The restriction of the action of G^{k+1}_m to constant multiples of the unite element of G^{k+1}_m implies that Φ has to satisfy the following condition

$$\begin{aligned}k^{s-r+p}\Phi(j^k\varphi,j^k\psi)=\Phi(&k^{p-1}\varphi,k^p\partial\varphi,\dots,k^{p+k-1}\partial^k\varphi,\\ &k^{s-r}\psi,k^{s-r+1}\partial\psi,\dots,k^{s-r+k}\partial^k\psi)\end{aligned}\tag{1.1}$$

for all $k\in\mathbb{R}^+$.

All exponents in the equation (1.1) are positive integers which implies, from the homogeneous function theorem (see [3, p. 213]), that the operator Φ is a polynomial of orders a_l in $\partial^l\varphi$ and b_l in $\partial^l\psi$ such that

$$\sum_{l=0}^{k}\big((p+l-1)\,a_l+(s-r+l)\,b_l\big)=s-r+p\,.\tag{1.2}$$

Since all coefficients in (1.2) are positive there are only two solutions in non-negative integers: a) $a_0=1$ $b_1=1$ and the others $a_i,\ b_i$ are vanishing , b) $a_1=1\,,\ b_0=1$

and the others $a_i,\ b_i$ are vanishing. These solutions correspond to $\mathbb{R}$-bilinear 1st order operators. $\square$

Proof of Theorem 1.2. Let p, r, s are arbitrary. If we assume that the operator is $\mathbb{R}$-bilinear then it is a polynomial of orders a_l in $\partial^l\varphi$ and b_l in $\partial^l\psi$ such that the equation (1.2) is satisfied. But now some coefficients in (1.2) can be negative or vanishing. There are only two solutions in natural numbers which corresponds to $\mathbb{R}$-bilinear operators: a) $a_0 = 1\ \ b_1 = 1$ and the others $a_i,\ b_i$ are vanishing, b) $a_1 = 1\,,\ b_0 = 1$ and the others $a_i,\ b_i$ are vanishing. Hence all finite order natural $\mathbb{R}$-bilinear differential operators are of order 1. $\square$

According to Theorems 1.1 and 1.2 all $\mathbb{R}$-bilinear natural differential operators Φ are of the form

$$\begin{aligned}\Phi(\varphi,\psi) = \Big(&A^{i_1\dots i_r m_1\dots m_p t_1\dots t_{s+1}}_{j_1\dots j_{s+p} k q_1\dots q_r}\,\varphi^k_{m_1\dots m_p}\,\partial_{t_{s+1}}\psi^{q_1\dots q_r}_{t_1\dots t_s}\\ &+ B^{i_1\dots i_r m_1\dots m_{p+1} t_1\dots t_s}_{j_1\dots j_{s+p} k q_1\dots q_r}\,\psi^{q_1\dots q_r}_{t_1\dots t_s}\,\partial_{m_{p+1}}\varphi^k_{m_1\dots m_p}\Big)\\ &\partial_{i_1}\otimes\dots\otimes\partial_{i_r}\otimes d^{j_1}\otimes\dots\otimes d^{j_{s+p}}\,,\end{aligned}\tag{1.3}$$

where $A^{i_1\dots i_r m_1\dots m_p t_1\dots t_{s+1}}_{j_1\dots j_{s+p} k q_1\dots q_r}$ and $B^{i_1\dots i_r m_1\dots m_{p+1} t_1\dots t_s}_{j_1\dots j_{s+p} k q_1\dots q_r}$ are absolute invariant tensors (see [3, p. 214]). Such absolute invariant tensors are all possible linear combinations, with real coefficients, of tensor products of the identity $\mathbb{I}$ of TM, i.e.

$$A^{i_1\dots i_r m_1\dots m_p t_1\dots t_{s+1}}_{j_1\dots j_{s+p} k q_1\dots q_r} = \sum_\sigma a_\sigma\,\delta^{i_1}_{\sigma(j_1)}\dots\delta^{t_{s+1}}_{\sigma(q_r)}\tag{1.4}$$

and

$$B^{i_1\dots i_r m_1\dots m_{p+1} t_1\dots t_s}_{j_1\dots j_{s+p} k q_1\dots q_r} = \sum_\sigma b_\sigma\,\delta^{i_1}_{\sigma(j_1)}\dots\delta^{t_s}_{\sigma(q_r)}\,,\tag{1.5}$$

$a_\sigma\,,\ b_\sigma\in\mathbb{R}$, where σ runs all permutations of $(r+s+p+1)$ indices.

Moreover, to obtain natural operators, coefficients a_σ, b_σ have to satisfy some identities. To calculate these identities, we use the method of an auxiliary linear symmetric connection K, [4, p. 144], and the second reduction theorem, [7, p. 165]. We assume that the operator Φ also depends on K. Then, by the second reduction theorem, the operator is factorized via the covariant derivatives of φ and ψ with respect to K. So, we replace derivatives of tensor fields with covariant derivatives and assume that the operator is independent of K which gives a system of homogeneous linear equations for a_σ and b_σ.

Remark 1.1. Let us note that natural differential operators satisfy the naturality condition, and, moreover, they are local, see [3, p. 143].

Then the assumptions of Theorem 1.1 immediately ensure that all local operators (with values in any vector bundle) must be of finite order (see the Peetre-like theorem, [3, p. 176]).

On the other hand bilinear operators satisfying the infinitesimal version of naturality, i.e., commuting with Lie derivatives, can be completely described without the locality assumption (see [1]). In particular, the bilinear operators on vector fields producing vector fields and satisfying the Jacobi identity, i.e., commuting with Lie derivatives, are just the Lie bracket, up to constant multiples, without any locality assumption.

2. Natural $\mathbb{R}$-bilinear operators transforming vector fields X and tensor fields ψ into tensor fields of the same type as ψ

According to Theorem 1.2 all such natural $\mathbb{R}$-bilinear operators are of order 1.

2.1. **Operator $\Phi(X,-)$ applied to vector fields.** It is very well known that the Lie bracket is unique, up to a constant multiple, natural $\mathbb{R}$-bilinear operator transforming two vector fields into a vector field. We shall reprove this fact to demonstrate the method of an auxiliary linear symmetric connection.

Theorem 2.1. *All natural $\mathbb{R}$-bilinear differential operators transforming two vector fields into vector fields are constant multiples of the Lie bracket.*

Proof. Let X and $\psi = Y$ be vector fields. Then from (1.3)–(1.5)

$$\Phi(X,Y) = \Phi^i\,\partial_i\,,$$

where

$$\Phi^i = a_1\,X^m\,\partial_m Y^i + a_2\,X^i\,\partial_m Y^m + b_1\,Y^m\,\partial_m X^i + b_2\,Y^i\,\partial_m X^m\,.$$

Let us assume a natural differential operator Ψ transforming vector fields X, Y and a linear symmetric connection K into vector fields. Then, according to the second reduction theorem, [7, p. 165], this operator factorizes through covariant derivatives ∇X and ∇Y and it is an $\mathbb{R}$-bilinear operator. In coordinates we obtain

$$\begin{aligned}\Psi^i &= a_1\,X^m\,\nabla_m Y^i + a_2\,X^i\,\nabla_m Y^m + b_1\,Y^m\,\nabla_m X^i + b_2\,Y^i\,\nabla_m X^m\\ &= a_1\,X^m\,(\partial_m Y^i - K_m{}^i{}_p Y^p) + a_2\,X^i\,(\partial_m Y^m - K_m{}^m{}_p Y^p)\\ &\quad + b_1\,Y^m\,(\partial_m X^i - K_m{}^i{}_p X^p) + b_2\,Y^i\,(\partial_m X^m - K_m{}^m{}_p X^p)\,,\end{aligned}$$

where $K_j{}^i{}_k$ are the symbols of K. The part of Ψ^i independent of K coincides with Φ^i, so we obtain for Φ^i the following identity

$$0 = (a_1\,X^m\,K_m{}^i{}_p + a_2\,X^i\,K_m{}^m{}_p)\,Y^p + (b_1\,Y^m\,K_m{}^i{}_p + b_2\,Y^i\,K_m{}^m{}_p)\,X^p\,.$$

It is easy to see that this identity is satisfied if and only if

$$a_1 + b_1 = 0\,,\quad a_2 = 0 = b_2\,.$$

So

$$\Phi^i = a_1\,(X^m\,\partial_m Y^i - Y^m\,\partial_m X^i) = a_1\,[X,Y]^i$$

and $\Phi(X,Y)$ is a constant multiple of the Lie bracket $[X,Y]$. □

2.2. **Operator $\Phi(X,-)$ applied to 1-forms.**

Theorem 2.2. *All natural $\mathbb{R}$-bilinear operators transforming a vector field X and a 1-form ψ into 1-forms are linear combinations, with real coefficients, of two operators*

$$d(\psi(X))\,,\quad i_X d\psi\,.$$

Proof. Let X be a vector field and ψ be a 1-form. Then by (1.3)–(1.5)

$$\Phi(X,\psi)=\Phi_i\,d^i\,,$$

where

$$\Phi_i=a_1\,X^m\,\partial_m\psi_i+a_2\,X^m\,\partial_i\psi_m+b_1\,\psi_i\,\partial_m X^m+b_2\,\psi_m\,\partial_i X^m\,.$$

Now, we replace partial derivatives with covariant derivatives with respect to an auxiliary linear symmetric connection K and assume that the operator is independent of K. We obtain the following identity

$$0=(a_1+a_2-b_2)\,X^m\,K_m{}^p{}_i\psi_p-b_1\,\psi_i\,K_m{}^m{}_p X^p\,.$$

So, we have

$$a_1+a_2-b_2=0\,,\quad b_1=0$$

and

$$\Phi_i=a_1\,X^m\,(\partial_m\psi_i-\partial_i\psi_m)+b_2\,(X^m\,\partial_i\psi_m+\psi_m\,\partial_i X^m)$$

which is the coordinate expression of $a_1\,i_X d\psi+b_2\,d(\psi(X))$. □

Remark 2.1. In differential geometry the Lie derivative $L_X\psi=i_X d\psi+d i_X\psi$ is very often used, but according to Theorem 2.2 any linear combination of $d(\psi(X))$, $i_X d\psi$ is a natural 1-form.

2.3. **Operator $\Phi(X,-)$ applied to (0,2)-tensor fields.** We assume a $(0,2)$-tensor field ψ.

Theorem 2.3. *All natural $\mathbb{R}$-bilinear differential operators transforming a vector field X and a $(0,2)$-tensor field ψ into $(0,2)$-tensor fields are linear combinations, with real coefficients, of four operators*

$$L_X\psi\,,\quad L_X\widetilde{\psi}\,,\quad d(X\lrcorner\psi)\,,\quad d(X\lrcorner\widetilde{\psi})\,,$$

where $\widetilde{\psi}$ is the $(0,2)$-tensor field given as $\widetilde{\psi}(Y,Z)=\psi(Z,Y)$ and $(X\lrcorner\psi)(Y)=\psi(X,Y)$ for any vector fields X, Y, Z.

Proof. Let X be a vector field and ψ be a (0,2)-tensor field. Then by (1.3)–(1.5)

$$\Phi(X,\psi)=\Phi_{ij}\,d^i\otimes d^j\,,$$

where

$$\begin{aligned}\Phi_{ij}=&\,a_1\,X^m\,\partial_m\psi_{ij}+a_2\,X^m\,\partial_m\psi_{ji}+a_3\,X^m\,\partial_i\psi_{mj}+a_4\,X^m\,\partial_i\psi_{jm}\\&+a_5\,X^m\,\partial_j\psi_{im}+a_6\,X^m\,\partial_j\psi_{mi}+b_1\,\psi_{ij}\,\partial_m X^m+b_2\,\psi_{ji}\,\partial_m X^m\\&+b_3\,\psi_{mj}\,\partial_i X^m+b_4\,\psi_{jm}\,\partial_i X^m+b_5\,\psi_{im}\,\partial_j X^m+b_6\,\psi_{mi}\,\partial_j X^m\,.\end{aligned}$$

Now, we replace partial derivatives with covariant derivatives with respect to an auxiliary linear symmetric connection K and assume that the operator is independent of K. We obtain the following identity

$$\begin{aligned} 0 &= (a_1 + a_3 - b_3)\, X^m\, K_m{}^p{}_i\, \psi_{pj} + (a_2 + a_4 - b_4)\, X^m\, K_m{}^p{}_j\, \psi_{pi} \\ &\quad + (a_1 + a_5 - b_5)\, X^m\, K_m{}^p{}_j\, \psi_{ip} + (a_2 + a_6 - b_6)\, X^m\, K_m{}^p{}_i\, \psi_{jp} \\ &\quad + (a_3 + a_6)\, X^m\, K_i{}^p{}_j\, \psi_{mp} + (a_4 + a_5)\, X^m\, K_i{}^p{}_j\, \psi_{pm} \\ &\quad - b_1\, \psi_{ij}\, K_m{}^m{}_p X^p - b_2\, \psi_{ji}\, K_m{}^m{}_p X^p\,. \end{aligned}$$

The above identity is satisfied if and only if $b_1 = 0 = b_2$ and the following system of homogeneous linear equations is satisfied

$$\begin{aligned} a_1 + a_3 - b_3 &= 0\,, & a_1 + a_5 - b_5 &= 0\,, \\ a_2 + a_4 - b_4 &= 0\,, & a_2 + a_6 - b_6 &= 0\,, \\ a_4 + a_5 &= 0\,, & a_3 + a_6 &= 0\,. \end{aligned}$$

Then we get

$$\begin{aligned} \Phi_{ij} &= a_1 \big(X^m\, \partial_m \psi_{ij} + \psi_{mj}\, \partial_i X^m + \psi_{im}\, \partial_j X^m\big) \\ &\quad + a_2 \big(X^m\, \partial_m \psi_{ji} + \psi_{mi}\, \partial_j X^m + \psi_{jm}\, \partial_i X^m\big) \\ &\quad + a_3 \big(X^m\, \partial_i \psi_{mj} + \psi_{mj}\, \partial_i X^m - X^m\, \partial_j \psi_{mi} - \psi_{mi}\, \partial_j X^m\big) \\ &\quad + a_4 \big(X^m\, \partial_i \psi_{jm} + \psi_{jm}\, \partial_i X^m - X^m\, \partial_j \psi_{im} - \psi_{im}\, \partial_j X^m\big) \end{aligned}$$

which is the coordinate expression of a linear combination of $L_X\psi$, $L_X\widetilde{\psi}$, $d(X \lrcorner \psi)$, $d(X \lrcorner \widetilde{\psi})$. □

Remark 2.2. Let us note that in above Theorem 2.3 we have used the Lie derivation of any $(0,2)$-tensor field defined as

$$(L_X\psi)(Y,Z) = X.\psi(Y,Z) - \psi([X,Y],Z) - \psi(Y,[X,Z]),$$

for any vector fields X, Y, Z. In the case that ψ is a 2-form this Lie derivative coincides with $L_X\psi = i_X d\psi + d i_X \psi$.

3. Natural $\mathbb{R}$-bilinear operators transforming (1,1)-tensor fields φ and $(*,*)$-tensor fields ψ into $(*,*+1)$-tensor fields

A (1,1) tensor field φ can be considered as a linear mapping $\varphi\colon TM \to TM$. As $\operatorname{Tr}\varphi$ we assume the contraction and $\mathbb{I}\colon TM \to TM$ is the identity. We do not assume special properties of φ.

3.1. **Operator $\Phi(\varphi, -)$ applied to (1,1)-tensor fields.** Full classification of natural $\mathbb{R}$-bilinear operators transforming two (1,1)-tensor fields into (1,2)-tensor fields was done in [4, p. 152] by using the other method. We recall this classification.

Theorem 3.1. *All natural $\mathbb{R}$-bilinear differential operators transforming $(1,1)$-tensor fields φ and ψ into $(1,2)$-tensor fields form a 15 parameter family of operators* *given as a linear combination of the following operators*

$$\begin{gathered} d(\operatorname{Tr}\varphi)\otimes\psi\,,\quad \psi\otimes d(\operatorname{Tr}\varphi)\,,\quad d(\operatorname{Tr}\psi)\otimes\varphi\,,\quad \varphi\otimes d(\operatorname{Tr}\psi)\,,\\ (\operatorname{Tr}\psi)\,d(\operatorname{Tr}\varphi)\otimes\mathbb{I}\,,\quad (\operatorname{Tr}\psi)\,\mathbb{I}\otimes d(\operatorname{Tr}\varphi)\,,\quad (\operatorname{Tr}\varphi)\,d(\operatorname{Tr}\psi)\otimes\mathbb{I}\,,\\ (\operatorname{Tr}\varphi)\,\mathbb{I}\otimes d(\operatorname{Tr}\psi)\,,\quad (d(\operatorname{Tr}\varphi)\circ\psi)\otimes\mathbb{I}\,,\quad \mathbb{I}\otimes(d(\operatorname{Tr}\varphi)\circ\psi)\,,\\ (d(\operatorname{Tr}\psi)\circ\varphi)\otimes\mathbb{I}\,,\quad \mathbb{I}\otimes(d(\operatorname{Tr}\psi)\circ\varphi)\,,\quad d\big(\operatorname{Tr}(\varphi\circ\psi)\big)\otimes\mathbb{I}\,,\\ \mathbb{I}\otimes d\big(\operatorname{Tr}(\varphi\circ\psi)\big)\,,\quad N(\varphi,\psi)\,, \end{gathered}$$

where $\mathbb{I}$ is the identity of TM and $N(\varphi,\psi)$ is the Frölicher-Nijenhuis bracket. □

Remark 3.1. It is very well known that the Frölicher-Nijenhuis bracket, [2], has values in tangent-valued forms. If we assume operators transforming φ and ψ into tangent-valued 2-forms we obtain 8 parameter family generated by

$$\begin{gathered} d(\operatorname{Tr}\varphi)\wedge\psi\,,\quad d(\operatorname{Tr}\psi)\wedge\varphi\,.\\ (\operatorname{Tr}\psi)\,d(\operatorname{Tr}\varphi)\wedge\mathbb{I}\,,\quad (\operatorname{Tr}\varphi)\,d(\operatorname{Tr}\psi)\wedge\mathbb{I}\,,\\ (d(\operatorname{Tr}\varphi)\circ\psi)\wedge\mathbb{I}\,,\quad (d(\operatorname{Tr}\psi)\circ\varphi)\wedge\mathbb{I}\,,\\ d\big(\operatorname{Tr}(\varphi\circ\psi)\big)\wedge\mathbb{I}\,,\quad N(\varphi,\psi)\,. \end{gathered}$$

3.2. **Operator $\Phi(\varphi,-)$ applied to 1-forms.**

Lemma 3.1. *We have the following 6 canonical 1st order natural $\mathbb{R}$-bilinear differential operators*

$$\begin{gathered} (\operatorname{Tr}\varphi)\,d\psi\,,\quad \psi\otimes d(\operatorname{Tr}\varphi)\,,\quad d(\operatorname{Tr}\varphi)\otimes\psi\,,\\ d\psi\circ_1\varphi\,,\quad d\psi\circ_2\varphi\,,\quad d(\psi\circ\varphi)\,, \end{gathered}$$

where $(d\psi\circ_1\varphi)(X,Y)=d\psi(\varphi(X),Y)$ and $(d\psi\circ_2\varphi)(X,Y)=d\psi(X,\varphi(Y))$ for any vector fields X, Y. □

Remark 3.2. We have the following independent operators with values in 2-forms

$$(\operatorname{Tr}\varphi)\,d\psi\,,\quad \psi\wedge d(\operatorname{Tr}\varphi)\,,\quad \operatorname{Alt}(d\psi\circ_1\varphi)\,,\quad d(\psi\circ\varphi)$$

which follows from $\operatorname{Alt}(d\psi\circ_1\varphi)=\operatorname{Alt}(d\psi\circ_2\varphi)$, where Alt is the antisymmetrisation.

Theorem 3.2. *All natural $\mathbb{R}$-bilinear differential operators transforming φ and ψ into a $(0,2)$ tensor fields form a six parameter family of operators which is a linear combination of operators from Lemma 3.1.*

Proof. According to (1.3)–(1.5)

$$\Phi(\varphi,\psi)=\Phi_{ij}\,d^i\otimes d^j\,,$$

where

$$\begin{aligned} \Phi_{ij}=a_1\,\varphi^m_m\,\partial_i\psi_j+a_2\,\varphi^m_m\,\partial_j\psi_i+a_3\,\varphi^m_i\,\partial_m\psi_j+a_4\,\varphi^m_i\,\partial_j\psi_m\\ +a_5\,\varphi^m_j\,\partial_m\psi_i+a_6\,\varphi^m_j\,\partial_i\psi_m\\ +b_1\,\psi_i\,\partial_m\varphi^m_j+b_2\,\psi_i\,\partial_j\varphi^m_m+b_3\,\psi_j\,\partial_m\varphi^m_i+b_4\,\psi_j\,\partial_i\varphi^m_m\\ +b_5\,\psi_m\,\partial_i\varphi^m_j+b_6\,\psi_m\,\partial_j\varphi^m_i\,. \end{aligned}$$

In order to calculate relations for coefficients a_i, b_i, $i = 1, \dots, 6$, we use the method of an auxiliary linear symmetric connection K, [4, p. 144]. We replace derivatives of tensor fields with covariant derivatives and assume that the operator is independent of K. Then we get

$$\begin{aligned} 0 = {} & \varphi^m_m \big(a_1 + a_2\big)\, K_i{}^p{}_j\, \psi_p \\ & + \varphi^m_i \big[\big(a_3 + a_4 - b_6\big)\, K_m{}^p{}_j\, \psi_p - b_3\, K_p{}^p{}_m\, \psi_j\big] \\ & + \varphi^m_j \big[\big(a_5 + a_6 - b_5\big)\, K_m{}^p{}_i\, \psi_p - b_1\, K_p{}^p{}_m\, \psi_i\big] \\ & + \varphi^m_p \big[b_1\, K_m{}^p{}_j\, \psi_i + b_3\, K_m{}^p{}_i\, \psi_j + (b_5 + b_6)\, K_i{}^p{}_j\, \psi_m\big]\,. \end{aligned}$$

Then b_2 and b_4 are arbitrary, $b_1 = b_3 = 0$ and

$$b_6 = -b_5\,, \qquad a_2 = -a_1\,, \qquad a_4 = -a_3 - b_5\,, \qquad a_6 = -a_5 + b_5\,.$$

Hence

$$\begin{aligned} \Phi_{ij} = {} & a_1\, \varphi^m_m \big(\partial_i \psi_j - \partial_j \psi_i\big) \\ & + a_3\, \varphi^m_i \big(\partial_m \psi_j - \partial_j \psi_m\big) + a_5\, \varphi^m_j \big(\partial_m \psi_i - \partial_i \psi_m\big) \\ & + b_2\, \psi_i\, \partial_j \varphi^m_m + b_4\, \psi_j\, \partial_i \varphi^m_m \\ & + b_5 \big(\varphi^m_j\, \partial_i \psi_m - \varphi^m_i\, \partial_j \psi_m + \psi_m\, (\partial_i \varphi^m_j - \partial_j \varphi^m_i)\big)\,. \end{aligned}$$

which is the coordinate expression of a linear combination of operators from Lemma 3.1. □

Corollary 3.1. *If the 1-form ψ is closed, then all $\mathbb{R}$-bilinear 1st order natural differential operators form the 3-parameter family of operators generated by*

$$\psi \otimes d(\operatorname{Tr} \varphi)\,, \quad d(\operatorname{Tr} \varphi) \otimes \psi\,, \quad d(\psi \circ \varphi)\,.$$

Moreover, we have 2 independent operators $\psi \wedge d(\operatorname{Tr} \varphi)$ and $d(\psi \circ \varphi)$ with values in 2-forms.

Remark 3.3. We can define others natural $\mathbb{R}$-bilinear operators on φ and ψ. But, according to Theorem 3.2, they have to be obtained as linear combinations of operators from Lemma 3.1.

Let X, Y be vector fields, in [8] the operator Φ was defined as follows

$$\Phi(\varphi, \psi)(X, Y) = (L_{\varphi(X)} \psi - L_X(\psi \circ \varphi))(Y)$$

which can be expressed as the linear combination of operators from Lemma 3.1

$$\Phi(\varphi, \psi) = d\psi \circ_1 \varphi - d(\psi \circ \varphi)\,.$$

Further, according to [3, p. 69], we can define the Lie derivative of ψ with respect to φ as

$$L_\varphi \psi = [i_\varphi, d]\psi = i_\varphi d\psi - d i_\varphi \psi$$

which is a 2-form. It is easy to see that

$$L_\varphi \psi = d\psi \circ_1 \varphi + d\psi \circ_2 \varphi - d(\psi \circ \varphi)\,.$$

For the identity of TM we have

$$L_{\mathbb{I}}(\psi \circ \varphi) = i_{\mathbb{I}} d(\psi \circ \varphi) - d i_{\mathbb{I}}(\psi \circ \varphi) = d(\psi \circ \varphi)$$

and we obtain, [8],

$$2\ \mathrm{Alt}\,\Phi(\varphi,\psi) = L_\varphi\psi - L_{\mathbb{I}}(\psi\circ\varphi)\,.$$

3.3. **Operator $\Phi(\varphi,-)$ applied to (0,2) tensor fields.** Let us denote as $\mathrm{Alt}\,\psi$ the antisymmetric part of ψ, i.e. in coordinates

$$\mathrm{Alt}\,\psi = \tfrac{1}{2}\,(\psi_{ij}-\psi_{ji})\,d^i\otimes d^j = \psi_{ij}\,d^i\wedge d^j\,.$$

First of all, we describe several types of 1st order natural $\mathbb{R}$-bilinear operators which are given by the tensorial operations (permutation of indices, tensor product, contraction, exterior differential).

Lemma 3.2. *$\psi\otimes d(\mathrm{Tr}\,\varphi)$ defines six independent natural $\mathbb{R}$-bilinear differential operators given by permutations of subindices, so for vector fields X,Y,Z we have operators*

$$\begin{gathered}\psi(X,Y)\,d(\mathrm{Tr}\,\varphi)(Z)\,,\quad \psi(Y,X)\,d(\mathrm{Tr}\,\varphi)(Z)\,,\quad \psi(X,Z)\,d(\mathrm{Tr}\,\varphi)(Y)\,,\\ \psi(Z,X)\,d(\mathrm{Tr}\,\varphi)(Y)\,,\quad \psi(Y,Z)\,d(\mathrm{Tr}\,\varphi)(X)\,,\quad \psi(Z,Y)\,d(\mathrm{Tr}\,\varphi)(X)\,.\end{gathered}$$

Moreover, $\mathrm{Alt}\,\psi\wedge d(\mathrm{Tr}\,\varphi)$ is the unique operator with values in 3-foms.

Corollary 3.2. *If the tensor field ψ is symmetric or antisymmetric then we get three independent operators from Lemma 3.2*

$$\psi(X,Y)\,d(\mathrm{Tr}\,\varphi)(Z)\,,\ \psi(X,Z)\,d(\mathrm{Tr}\,\varphi)(Y)\,,\ \psi(Y,Z)\,d(\mathrm{Tr}\,\varphi)(X))\,.$$

Lemma 3.3. *We have the following six independent natural $\mathbb{R}$-bilinear differential operators*

$$\begin{gathered}(\mathrm{Tr}\,\varphi)\,d(\mathrm{Alt}\,\psi)\,,\quad d(\mathrm{Alt}\,\psi)\circ_1\varphi\,,\quad d(\mathrm{Alt}\,\psi)\circ_2\varphi\,,\quad d(\mathrm{Alt}\,\psi)\circ_3\varphi\,,\\ d(\mathrm{Alt}(\psi\circ_1\varphi))\,,\quad d(\mathrm{Alt}(\psi\circ_2\varphi))\,.\end{gathered}$$

Corollary 3.3. *1. If ψ is symmetric then $\mathrm{Alt}\,\psi=0$ and $\mathrm{Alt}(\psi\circ_1\varphi) = -\,\mathrm{Alt}(\psi\circ_2\varphi)$ and we have the unique operator from Lemma 3.3*

$$d(\mathrm{Alt}(\psi\circ_1\varphi))\,.$$

2. If ψ is antisymmetric then $\mathrm{Alt}\,\psi=\psi$ and $\mathrm{Alt}(\psi\circ_1\varphi)=\mathrm{Alt}(\psi\circ_2\varphi)$ and we have five independent operators from Lemma 3.3

$$\begin{gathered}(\mathrm{Tr}\,\varphi)\,d\psi\,,\quad d\psi\circ_1\varphi\,,\quad d\psi\circ_2\varphi\,,\quad d\psi\circ_3\varphi\,,\\ d(\mathrm{Alt}(\psi\circ_1\varphi))\,.\end{gathered}$$

From $\mathrm{Alt}(d\psi\circ_1\varphi)=\mathrm{Alt}(d\psi\circ_2\varphi)=\mathrm{Alt}(d\psi\circ_3\varphi)$ we have 3 operators with values in 3-forms.

Moreover, if ψ is a closed 2-form, then there is the unique operator from Lemma 3.3

$$d(\mathrm{Alt}(\psi\circ_1\varphi))$$

which has values in 3-forms.

If the tensor fields φ and ψ satisfy

$$(\psi \circ_1 \varphi)(X,Y) = (\psi \circ_2 \varphi)(X,Y) \quad \Leftrightarrow \quad \psi(\varphi(X),Y) = \psi(X,\varphi(Y))$$

then ψ is said to be *pure* with respect to φ. Natural $\mathbb{R}$-bilinear differential operators $\Phi(\varphi,\psi)$ on pure tensor fields were studied in [5, 8]. We recall the main result.

Theorem 3.3. *Let ψ is pure with respect to φ. Then*

$$\Phi(\varphi,\psi)(X,Y,Z) = \big(L_{\varphi(X)}\psi - L_X(\psi \circ_1 \varphi)\big)(Y,Z)$$

is a $(0,3)$-tensor field. □

The above operator for pure tensor fields can be generalized for any tensor field ψ.

Theorem 3.4. *For any vector fields X, Y, Z the operators*

$$\begin{aligned}\Phi_1(\varphi,\psi)(X,Y,Z) = {} & \big(L_{\varphi(X)}\psi - L_X(\psi \circ_1 \varphi)\big)(Y,Z) \\ & - \big(L_{\varphi(Z)}\psi - L_Z(\psi \circ_1 \varphi)\big)(Y,X) \\ & + (\psi \circ_2 \varphi)(Y,[X,Z]) - (\psi \circ_1 \varphi)(Y,[X,Z])\end{aligned}$$

and

$$\begin{aligned}\Phi_2(\varphi,\psi)(X,Y,Z) = {} & \big(L_{\varphi(X)}\psi - L_X(\psi \circ_2 \varphi)\big)(Y,Z) \\ & - \big(L_{\varphi(Y)}\psi - L_Y(\psi \circ_2 \varphi)\big)(X,Z) \\ & - (\psi \circ_2 \varphi)([X,Y],Z) + (\psi \circ_1 \varphi)([X,Y],Z)\end{aligned}$$

are $(0,3)$-tensor fields with the coordinate expressions

$$\begin{aligned}\Phi_1(\varphi,\psi)(X,Y,Z) = {} & \big(\varphi^m_i\,\partial_m\psi_{jk} + \varphi^m_j\,(\partial_k\psi_{mi} - \partial_i\psi_{mk}) - \varphi^m_k\,\partial_m\psi_{ji} \\ & + \psi_{mi}\,(\partial_k\varphi^m_j - \partial_j\varphi^m_k) + \psi_{jm}\,(\partial_k\varphi^m_i - \partial_i\varphi^m_k) \\ & + \psi_{mk}\,(\partial_j\varphi^m_i - \partial_i\varphi^m_j)\big)\,X^i\,Y^j\,Z^k\end{aligned}$$

and

$$\begin{aligned}\Phi_2(\varphi,\psi)(X,Y,Z) = {} & \big(\varphi^m_i\,\partial_m\psi_{jk} - \varphi^m_j\,\partial_m\psi_{ik} + \varphi^m_k\,(\partial_j\psi_{im} - \partial_i\psi_{jm}) \\ & - \psi_{im}\,(\partial_k\varphi^m_j - \partial_j\varphi^m_k) + \psi_{jm}\,(\partial_k\varphi^m_i - \partial_i\varphi^m_k) \\ & + \psi_{mk}\,(\partial_j\varphi^m_i - \partial_i\varphi^m_j)\big)\,X^i\,Y^j\,Z^k\,,\end{aligned}$$

respectively.

Proof. It is easy to prove it in coordinates. □

Remark 3.4. Any linear combination of the above operators is an $\mathbb{R}$-bilinear operator, for instance

$$\begin{aligned}&\big(6\,d(\mathrm{Alt}\,\psi) \circ_3 \varphi - 6\,d(\mathrm{Alt}(\psi \circ_1 \varphi)) - \Phi_1(\varphi,\psi)\big)(X,Y,Z) = \\ &\quad = \big(\varphi^m_i\,(\partial_j\psi_{mk} - \partial_k\psi_{mj} - \partial_m\psi_{jk}) + \varphi^m_k\,(\partial_i\psi_{jm} - \partial_j\psi_{im} + \partial_m\psi_{ij}) \\ &\qquad + (\psi_{jm} + \psi_{mj})\,(\partial_i\varphi^m_k - \partial_k\varphi^m_i)\big)\,X^i\,Y^j\,Z^k\,,\end{aligned}$$

$$\begin{aligned}&\big(6\,d(\operatorname{Alt}\psi)\circ_1\varphi - 6\,d(\operatorname{Alt}(\psi\circ_2\varphi)) - \Phi_2(\varphi,\psi)\big)(X,Y,Z) =\\ &\quad= \big(\varphi_i^m\,(\partial_k\psi_{mj} - \partial_j\psi_{mk} - \partial_m\psi_{kj}) + \varphi_j^m\,(\partial_i\psi_{km} - \partial_k\psi_{im} + \partial_m\psi_{ik})\\ &\qquad + (\psi_{km}+\psi_{mk})\,(\partial_i\varphi_j^m - \partial_j\varphi_i^m)\big)\,X^i\,Y^j\,Z^k\,,\end{aligned}$$

$$\begin{aligned}&\big(-6\,d(\operatorname{Alt}\psi)\circ_2\varphi + 6\,d(\operatorname{Alt}\psi)\circ_3\varphi\\ &\qquad - \Phi_1(\varphi,\psi) + \Phi_2(\varphi,\psi)\big)(X,Y,Z) =\\ &\quad= \big(\varphi_j^m\,(\partial_i\psi_{km} - \partial_k\psi_{im} - \partial_m\psi_{ki}) + \varphi_k^m\,(\partial_j\psi_{mi} - \partial_i\psi_{mj} + \partial_m\psi_{ij})\\ &\qquad + (\psi_{im}+\psi_{mi})\,(\partial_j\varphi_k^m - \partial_k\varphi_j^m)\big)\,X^i\,Y^j\,Z^k\,,\end{aligned}$$

and

$$\begin{aligned}&\big(\Phi_1(\varphi,\psi) + \Phi_2(\varphi,\psi)\big)(X,Y,Z) =\\ &\quad= \big(2\,\varphi_i^m\,\partial_m\psi_{jk} + \varphi_j^m\,(\partial_k\psi_{mi} - \partial_i\psi_{mk} - \partial_m\psi_{ik})\\ &\qquad + \varphi_k^m\,(\partial_j\psi_{im} - \partial_i\psi_{jm} - \partial_m\psi_{ji})\\ &\qquad + (\psi_{im}-\psi_{mi})\,(\partial_j\varphi_k^m - \partial_k\varphi_j^m) - 2\,\psi_{jm}\,(\partial_i\varphi_k^m - \partial_k\varphi_i^m)\\ &\qquad - 2\,\psi_{mk}\,(\partial_i\varphi_j^m - \partial_j\varphi_i^m)\big)\,X^i\,Y^j\,Z^k\end{aligned}$$

are such operators which we shall need later.

Theorem 3.5. *All natural $\mathbb{R}$-bilinear differential operators Φ transforming a $(1,1)$-tensor field φ and a $(0,2)$-tensor field ψ into $(0,3)$-tensor fields form a 14-parameter family which is a linear combination of operators described in Lemma 3.3, Lemma 3.2 and Theorem 3.4.*

Proof. By Theorem 1.2 and (1.3)–(1.5) we get that all natural $\mathbb{R}$-bilinear differential operators are of the form

$$\Phi(\varphi,\psi) = \Phi_{ijk}\,d^i\otimes d^j\otimes d^k\,,$$

where

$$\begin{aligned}\Phi_{ijk} &= a_1\,\varphi_m^m\,\partial_i\psi_{jk} + a_2\,\varphi_m^m\,\partial_i\psi_{kj} + a_3\,\varphi_m^m\,\partial_j\psi_{ik} + a_4\,\varphi_m^m\,\partial_j\psi_{ki}\\ &\qquad + a_5\,\varphi_m^m\,\partial_k\psi_{ij} + a_6\,\varphi_m^m\,\partial_k\psi_{ji}\\ &\quad + a_7\,\varphi_i^m\,\partial_m\psi_{jk} + a_8\,\varphi_i^m\,\partial_m\psi_{kj} + a_9\,\varphi_i^m\,\partial_j\psi_{mk} + a_{10}\,\varphi_i^m\,\partial_j\psi_{km}\\ &\qquad + a_{11}\,\varphi_i^m\,\partial_k\psi_{mj} + a_{12}\,\varphi_i^m\,\partial_k\psi_{jm}\\ &\quad + a_{13}\,\varphi_j^m\,\partial_i\psi_{mk} + a_{14}\,\varphi_j^m\,\partial_i\psi_{km} + a_{15}\,\varphi_j^m\,\partial_m\psi_{ik} + a_{16}\,\varphi_j^m\,\partial_m\psi_{ki}\\ &\qquad + a_{17}\,\varphi_j^m\,\partial_k\psi_{im} + a_{18}\,\varphi_j^m\,\partial_k\psi_{mi}\\ &\quad + a_{19}\,\varphi_k^m\,\partial_i\psi_{jm} + a_{20}\,\varphi_k^m\,\partial_i\psi_{mj} + a_{21}\,\varphi_k^m\,\partial_j\psi_{im} + a_{22}\,\varphi_k^m\,\partial_j\psi_{mi}\\ &\qquad + a_{23}\,\varphi_k^m\,\partial_m\psi_{ij} + a_{24}\,\varphi_k^m\,\partial_m\psi_{ji}\\ &\quad + b_1\,\psi_{ij}\,\partial_k\varphi_m^m + b_2\,\psi_{ji}\,\partial_k\varphi_m^m + b_3\,\psi_{ik}\,\partial_j\varphi_m^m + b_4\,\psi_{ki}\,\partial_j\varphi_m^m\\ &\qquad + b_5\,\psi_{jk}\,\partial_i\varphi_m^m + b_6\,\psi_{kj}\,\partial_i\varphi_m^m\end{aligned}$$

$$
\begin{aligned}
&+ b_7\,\psi_{mj}\,\partial_k\varphi_i^m + b_8\,\psi_{jm}\,\partial_k\varphi_i^m + b_9\,\psi_{mk}\,\partial_j\varphi_i^m + b_{10}\,\psi_{km}\,\partial_j\varphi_i^m \\
&\quad + b_{11}\,\psi_{jk}\,\partial_m\varphi_i^m + b_{12}\,\psi_{kj}\,\partial_m\varphi_i^m \\
&+ b_{13}\,\psi_{im}\,\partial_k\varphi_j^m + b_{14}\,\psi_{mi}\,\partial_k\varphi_j^m + b_{15}\,\psi_{ik}\,\partial_m\varphi_j^m + b_{16}\,\psi_{ki}\,\partial_m\varphi_j^m \\
&\quad + b_{17}\,\psi_{mk}\,\partial_i\varphi_j^m + b_{18}\,\psi_{km}\,\partial_i\varphi_j^m \\
&+ b_{19}\,\psi_{ij}\,\partial_m\varphi_k^m + b_{20}\,\psi_{ji}\,\partial_m\varphi_k^m + b_{21}\,\psi_{im}\,\partial_j\varphi_k^m + b_{22}\,\psi_{mi}\,\partial_j\varphi_k^m \\
&\quad + b_{23}\,\psi_{jm}\,\partial_i\varphi_k^m + b_{24}\,\psi_{mj}\,\partial_i\varphi_k^m\,.
\end{aligned}
$$

In order to calculate relations for coefficients a_i, b_i, $i = 1, \dots, 24$, we use the method of an auxiliary linear symmetric connection K, [4, p. 144]. We replace derivatives of tensor fields with covariant derivatives and assume that the operator is independent of K. Then we get

$$
\begin{aligned}
0 = \varphi_m^m \big[&(a_1 + a_3)\,K_i{}^p{}_j\,\psi_{pk} + (a_2 + a_5)\,K_i{}^p{}_k\,\psi_{pj} + (a_2 + a_4)\,K_i{}^p{}_j\,\psi_{kp} \\
&+ (a_1 + a_6)\,K_i{}^p{}_k\,\psi_{jp} + (a_3 + a_5)\,K_j{}^p{}_k\,\psi_{ip} + (a_4 + a_6)\,K_j{}^p{}_k\,\psi_{pi}\big] \\
+ \varphi_i^m \big[&(a_7 + a_9 - b_9)\,K_j{}^p{}_m\,\psi_{pk} + (a_8 + a_{10} - b_{10})\,K_j{}^p{}_m\,\psi_{kp} \\
&+ (a_7 + a_{12} - b_8)\,K_m{}^p{}_k\,\psi_{jp} + (a_8 + a_{11} - b_7)\,K_m{}^p{}_k\,\psi_{pj} \\
&+ (a_9 + a_{11})\,K_j{}^p{}_k\,\psi_{mp} + (a_{10} + a_{12})\,K_j{}^p{}_k\,\psi_{pm} \\
&- b_{12}\,K_p{}^p{}_m\,\psi_{kj} - b_{11}\,K_p{}^p{}_m\,\psi_{jk}\big] \\
+ \varphi_j^m \big[&(a_{13} + a_{15} - b_{17})\,K_i{}^p{}_m\,\psi_{pk} + (a_{14} + a_{16} - b_{18})\,K_i{}^p{}_m\,\psi_{kp} \\
&+ (a_{15} + a_{17} - b_{13})\,K_m{}^p{}_k\,\psi_{ip} + (a_{16} + a_{18} - b_{14})\,K_m{}^p{}_k\,\psi_{pi} \\
&+ (a_{13} + a_{18})\,K_i{}^p{}_k\,\psi_{mp} + (a_{14} + a_{17})\,K_i{}^p{}_k\,\psi_{pm} \\
&- b_{16}\,K_p{}^p{}_m\,\psi_{ki} - b_{15}\,K_p{}^p{}_m\,\psi_{ik}\big] \\
+ \varphi_k^m \big[&(a_{20} + a_{23} - b_{24})\,K_i{}^p{}_m\,\psi_{pj} + (a_{19} + a_{24} - b_{23})\,K_i{}^p{}_m\,\psi_{jp} \\
&+ (a_{21} + a_{23} - b_{21})\,K_m{}^p{}_j\,\psi_{ip} + (a_{22} + a_{24} - b_{22})\,K_m{}^p{}_j\,\psi_{pi} \\
&+ (a_{20} + a_{22})\,K_i{}^p{}_j\,\psi_{mp} + (a_{19} + a_{21})\,K_i{}^p{}_j\,\psi_{pm} \\
&- b_{20}\,K_p{}^p{}_m\,\psi_{ji} - b_{19}\,K_p{}^p{}_m\,\psi_{ij}\big] \\
+ \varphi_p^m \big[&b_{19}\,K_m{}^p{}_k\,\psi_{ij} + b_{20}\,K_m{}^p{}_k\,\psi_{ji} + b_{15}\,K_m{}^p{}_j\,\psi_{ik} + b_{16}\,K_m{}^p{}_j\,\psi_{ki} \\
&+ b_{11}\,K_m{}^p{}_i\,\psi_{jk} + b_{12}\,K_m{}^p{}_i\,\psi_{kj} \\
&+ (b_{13} + b_{21})\,K_j{}^p{}_k\,\psi_{im} + (b_{14} + b_{22})\,K_j{}^p{}_k\,\psi_{mi} + (b_8 + b_{23})\,K_i{}^p{}_k\,\psi_{jm} \\
&+ (b_7 + b_{24})\,K_i{}^p{}_k\,\psi_{mi} + (b_{10} + b_{18})\,K_i{}^p{}_j\,\psi_{km} + (b_9 + b_{17})\,K_i{}^p{}_j\,\psi_{mk}\big]\,.
\end{aligned}
$$

So, the operator is independent of K if and only if the following conditions for coefficients are satisfied:

I: Coefficients $b_1, \dots, b_6$ are arbitrary and we obtain that the corresponding part of the operator $\Phi(\varphi, \psi)$ is a linear combination of operators from Lemma 3.2. We shall put $B_i = b_i$, $i = 1, \dots, 6$.

II: For part staying with φ_m^m we get that the coefficients $a_1, \dots, a_6$ satisfy the conditions

$$\begin{array}{cccccccc}
a_1 & & +a_3 & & & & = & 0\,, \\
a_1 & & & & & +a_6 & = & 0\,, \\
& a_2 & & & +a_5 & & = & 0\,, \\
& a_2 & & +a_4 & & & = & 0\,, \\
& & a_3 & & +a_5 & & = & 0\,, \\
& & & a_4 & & +a_6 & = & 0\,.
\end{array}$$

This system of equations has one free variable and putting $a_6 = B_7$ and the others free variables are vanishing we get a multiple of the operator

$$\Phi_{ijk} = \varphi_m^m \big(\partial_i\psi_{jk} - \partial_i\psi_{kj} + \partial_j\psi_{ki} - \partial_j\psi_{ik} + \partial_k\psi_{ij} - \partial_k\psi_{ji}\big)$$

which is a multiple of the operator

$$(\operatorname{Tr}\varphi)\, d(\operatorname{Alt}\psi)$$

from Lemma 3.3.

III: For part staying with φ_m^p we get the following conditions. The coefficients $b_{11} = b_{12} = b_{15} = b_{16} = b_{19} = b_{20} = 0$.

Further

$$\begin{array}{ccc}
b_{24} = -b_7\,, & b_{23} = -b_8\,, & b_{22} = -b_{14}\,, \\
b_{21} = -b_{13}\,, & b_{18} = -b_{10}\,, & b_{17} = -b_9\,.
\end{array}$$

IV: For part staying with φ_i^p the coefficients $a_7, \dots, a_{12}$ satisfy

$$\begin{array}{cccccccc}
a_7 & & +a_9 & & & & = & b_9\,, \\
a_7 & & & & & +a_{12} & = & b_8\,, \\
& a_8 & & & +a_{11} & & = & b_7\,, \\
& a_8 & & +a_{10} & & & = & b_{10}\,, \\
& & a_9 & & +a_{11} & & = & 0\,, \\
& & & a_{10} & & +a_{12} & = & 0\,.
\end{array}$$

V: For part staying with φ_j^p the coefficients $a_{13}, \dots, a_{18}$ satisfy

$$\begin{array}{cccccccc}
a_{13} & & +a_{15} & & & & = & b_{17}\,, \\
a_{13} & & & & & +a_{18} & = & 0\,, \\
& a_{14} & & & +a_{17} & & = & 0\,, \\
& a_{14} & & +a_{16} & & & = & b_{18}\,, \\
& & a_{15} & & +a_{17} & & = & b_{13}\,, \\
& & & a_{16} & & +a_{18} & = & b_{14}\,.
\end{array}$$

VI: For part staying with φ^p_k the coefficients $a_{19},\dots,a_{24}$ satisfy

$$\begin{array}{cccccccl}
a_{19} & & +a_{21} & & & & = & 0\,,\\
a_{19} & & & & & +a_{24} & = & b_{23}\,,\\
 & a_{20} & & & +a_{23} & & = & b_{24}\,,\\
 & a_{20} & & +a_{22} & & & = & 0\,,\\
 & & a_{21} & & +a_{23} & & = & b_{21}\,,\\
 & & & a_{22} & & +a_{24} & = & b_{22}\,.
\end{array}$$

The above systems IV–VI of linear equations we modify to

$$\begin{array}{cccccccc}
a_7 & & +a_9 & & & & = & b_9\,,\\
 & a_8 & & +a_{10} & & & = & b_{10}\,,\\
 & & a_9 & & +a_{11} & & = & 0\,,\\
 & & & a_{10} & & +a_{12} & = & 0\,,\\
 & & & & a_{11} & +a_{12} & = & b_7-b_{10}\,,\\
 & & & & & 0 & = & b_7-b_{10}-b_8+b_9\,.
\end{array}$$

$$\begin{array}{cccccccc}
a_{13} & & +a_{15} & & & & = & b_{17}\,,\\
 & a_{14} & & +a_{16} & & & = & b_{18}\,,\\
 & & a_{15} & & +a_{17} & & = & b_{13}\,,\\
 & & & a_{16} & & +a_{18} & = & b_{14}\,,\\
 & & & & a_{17} & +a_{18} & = & b_{13}-b_{17}\,,\\
 & & & & & 0 & = & b_{14}-b_{18}-b_{13}+b_{17}\,.
\end{array}$$

$$\begin{array}{cccccccc}
a_{19} & & +a_{21} & & & & = & 0\,,\\
 & a_{20} & & +a_{22} & & & = & 0\,,\\
 & & a_{21} & & +a_{23} & & = & b_{21}\,,\\
 & & & a_{22} & & +a_{24} & = & b_{22}\,,\\
 & & & & a_{23} & +a_{24} & = & b_{21}+b_{23}\,,\\
 & & & & & 0 & = & b_{22}+b_{24}-b_{21}-b_{23}\,,
\end{array}$$

So, for coefficients b_7, b_8, b_9, b_{10}, b_{13}, b_{14}, b_{17}, b_{18}, b_{21}, b_{22}, b_{23}, b_{24} we have a system of homogeneous linear equations with 4 independent variables. We choose as these free variables $b_{18}=B_8$, $b_{22}=B_9$, $b_{23}=B_{10}$ and $b_{24}=B_{11}$. We obtain

$$\begin{aligned}
b_7 &= -B_{11}\,, & b_{18} &= B_8\,,\\
b_8 &= -B_{10}\,, & b_{17} &= B_8-B_{10}+B_{11}\,,\\
b_9 &= -B_8-B_{10}+B_{11}\,, & b_{21} &= B_9-B_{10}+B_{11}\,,\\
b_{10} &= -B_8\,, & b_{22} &= B_9\,,\\
b_{13} &= -B_9+B_{10}-B_{11}\,, & b_{23} &= B_{10}\,,\\
b_{14} &= -B_9\,, & b_{24} &= B_{11}\,.
\end{aligned}$$

Now, putting a_{12} as a free variable B_{12}, we get from the system of equations IV

$$\begin{aligned}
a_7 &= -B_{12}-B_{10}\,, & a_{10} &= -B_{12}\,,\\
a_8 &= B_{12}-B_8\,, & a_{11} &= -B_{12}+B_8-B_{11}\,,\\
a_9 &= B_{12}-B_8+B_{11}\,, & a_{12} &= B_{12}\,.
\end{aligned}$$

Further, putting a_{18} as a free variable B_{13}, we get from the system of equations V

$$\begin{aligned} a_{13} &= -B_{13}\,, & a_{16} &= -B_{13} - B_9\,,\\ a_{14} &= B_{13} + B_8 + B_9\,, & a_{17} &= -B_{13} - B_8 - B_9\,,\\ a_{15} &= B_{13} + B_8 + B_{10} - B_{11}\,, & a_{18} &= B_{13}\,. \end{aligned}$$

Finally, putting a_{24} as a free variable B_{14}, we get from the system of equations VI

$$\begin{aligned} a_{19} &= -B_{14} + B_{10}\,, & a_{22} &= -B_{14} + B_9\,,\\ a_{20} &= B_{14} - B_9\,, & a_{23} &= -B_{14} + B_9 + B_{11}\,,\\ a_{21} &= B_{14} - B_{10}\,, & a_{24} &= B_{14}\,. \end{aligned}$$

Let us put $B_{12} = 1$ and the others free variables are vanishing. We get

$$\Phi_{ijk} = -\varphi_i^m \left(\partial_m\psi_{jk} - \partial_m\psi_{kj} + \partial_j\psi_{km} - \partial_j\psi_{mk} + \partial_k\psi_{mj} - \partial_k\psi_{jm}\right)$$

which is the coordinate expression for a multiple of

$$d(\operatorname{Alt}\psi) \circ_1 \varphi\,.$$

Similarly for $B_{13} = 1$ (respective $B_{14} = 1$) and the others free variables vanishing we get multiples of $d(\operatorname{Alt}\psi) \circ_2 \varphi$ (respective $d(\operatorname{Alt}\psi) \circ_3 \varphi$).

If we put $B_8 = 1$ and the others free variables vanishing we get

$$\begin{aligned} \Phi_{ijk} = {}& \varphi_i^m \left(\partial_k\psi_{mj} - \partial_m\psi_{kj} - \partial_j\psi_{mk}\right) + \varphi_j^m \left(\partial_i\psi_{km} + \partial_m\psi_{ik} - \partial_k\psi_{im}\right)\\ &+ \left(\psi_{mk} + \psi_{km}\right)\left(\partial_i\varphi_j^m - \partial_j\varphi_i^m\right). \end{aligned}$$

According to Remark 3.4 this operator corresponds in coordinates to

$$6\,d(\operatorname{Alt}\psi) \circ_1 \varphi - 6\,d(\operatorname{Alt}(\psi \circ_2 \varphi)) - \Phi_2(\varphi,\psi)\,.$$

If we put $B_9 = 1$ and the others free variables vanishing we get

$$\begin{aligned} \Phi_{ijk} = {}& \varphi_j^m \left(\partial_i\psi_{km} - \partial_m\psi_{ki} - \partial_k\psi_{im}\right) + \varphi_k^m \left(-\partial_i\psi_{mj} + \partial_j\psi_{mi} + \partial_m\psi_{ij}\right)\\ &+ \left(\psi_{im} + \psi_{mi}\right)\left(\partial_j\varphi_k^m - \partial_k\varphi_j^m\right). \end{aligned}$$

According to Remark 3.4 this operator corresponds in coordinates to

$$-6\,d(\operatorname{Alt}\psi) \circ_2 \varphi + 6\,d(\operatorname{Alt}\psi) \circ_3 \varphi - \Phi_1(\varphi,\psi) + \Phi_2(\varphi,\psi)\,.$$

If we put $B_{10} = 1$ and the others free variables vanishing we get

$$\begin{aligned} \Phi_{ijk} = {}& -\varphi_i^m\,\partial_m\psi_{jk} + \varphi_j^m\,\partial_m\psi_{ik} + \varphi_k^m \left(\partial_i\psi_{jm} - \partial_j\psi_{im}\right)\\ &+ \psi_{jm}\left(\partial_i\varphi_k^m - \partial_k\varphi_i^m\right) - \psi_{mk}\left(\partial_j\varphi_i^m - \partial_i\varphi_j^m\right)\\ &+ \psi_{im}\left(\partial_k\varphi_j^m - \partial_j\varphi_k^m\right). \end{aligned}$$

If we put $B_{11} = 1$ and the others free variables vanishing we get

$$\begin{aligned} \Phi_{ijk} = {}& \varphi_i^m \left(\partial_j\psi_{mk} - \partial_k\psi_{mj}\right) - \varphi_j^m\,\partial_m\psi_{ik} + \varphi_k^m\,\partial_m\psi_{ij}\\ &+ \psi_{mj}\left(\partial_i\varphi_k^m - \partial_k\varphi_i^m\right) + \psi_{mk}\left(\partial_j\varphi_i^m - \partial_i\varphi_j^m\right)\\ &+ \psi_{im}\left(\partial_j\varphi_k^m - \partial_k\varphi_j^m\right). \end{aligned}$$

Then the sum of the last 2 operators, i.e. $B_{10} = 1 = B_{11}$, gives

$$\Phi_{ijk} = \varphi_i^m\,(\partial_j\psi_{mk} - \partial_k\psi_{mj} - \partial_m\psi_{jk}) + \varphi_k^m\,(\partial_m\psi_{ij} + \partial_i\psi_{jm} - \partial_j\psi_{im}) \\ + (\psi_{jm} + \psi_{mj})\,(\partial_i\varphi_k^m - \partial_k\varphi_i^m)\,.$$

According to Remark 3.4 this operator corresponds in coordinates to

$$6\,d(\operatorname{Alt}\psi)\circ_3\varphi - 6\,d(\operatorname{Alt}(\psi\circ_1\varphi)) - \Phi_1(\varphi,\psi)\,.$$

On the other side for $B_{10} = 1\,,\ B_{11} = -1$ we have

$$\Phi_{ijk} = \varphi_i^m\,(-\partial_j\psi_{mk} + \partial_k\psi_{mj} - \partial_m\psi_{jk}) + 2\,\varphi_j^m\,\partial_m\psi_{ik} \\ + \varphi_k^m\,(-\partial_m\psi_{ij} + \partial_i\psi_{jm} - \partial_j\psi_{im}) + 2\,\psi_{im}\,(\partial_k\varphi_j^m - \partial_j\varphi_k^m) \\ + (\psi_{jm} - \psi_{mj})\,(\partial_i\varphi_k^m - \partial_k\varphi_i^m) - 2\,\psi_{mk}\,(\partial_j\varphi_i^m - \partial_i\varphi_j^m)$$

which, according to Remark 3.4, corresponds in coordinates to

$$\Phi_1(\varphi,\psi) + \Phi_2(\varphi,\psi)\,.$$

So all 14 independent operators are generated by 14 independent operators described in Lemma 3.3, Lemma 3.2 and Theorem 3.4. □

Remark 3.5. Let ψ be a 2-form. According to [3, p. 69] we can define the Lie derivative of ψ with respect to φ as

$$L_\varphi\psi = [i_\varphi, d]\psi = i_\varphi d\psi - d i_\varphi\psi$$

which is a 3-form. It is easy to see that

$$L_\varphi\psi = d\psi\circ_1\varphi + d\psi\circ_2\varphi + d\psi\circ_3\varphi - 2\,d(\operatorname{Alt}(\psi\circ_1\varphi))\,.$$

4. Natural operators transforming (1,2)-tensor fields S and 1-forms ψ into (0,3)-tensor fields

Let us recall that, according to Theorem 1.1, all natural operators transforming (1,2)-tensor fields S and 1-forms ψ into (0,3)-tensor fields are $\mathbb{R}$-bilinear and of order 1.

4.1. **General case.** We shall denote by $C^1_i S$, $i = 1, 2$, the contraction with respect to the corresponding indices.

Lemma 4.1. *We have 6 canonical natural differential operators given by $C^1_i S\otimes d\psi$, $i = 1, 2$, namely*

$$(C^1_1 S)(X)\,d\psi(Y,Z)\,,\quad (C^1_1 S)(Y)\,d\psi(X,Z)\,,\quad (C^1_1 S)(Z)\,d\psi(X,Y)\,,$$

$$(C^1_2 S)(X)\,d\psi(Y,Z)\,,\quad (C^1_2 S)(Y)\,d\psi(X,Z)\,,\quad (C^1_2 S)(Z)\,d\psi(X,Y)\,.$$

Lemma 4.2. *We have 6 canonical natural differential operators given by the composition of S with $d\psi$, namely*

$$d\psi(S(X,Y),Z)\,,\quad d\psi(S(Y,X),Z)\,,\quad d\psi(S(X,Z),Y)\,,$$

$$d\psi(S(Z,X),Y)\,,\quad d\psi(S(Y,Z),X)\,,\quad d\psi(S(Z,Y),X)\,.$$

Lemma 4.3. *We have 6 canonical natural differential operators given by $\psi \otimes d(C_i^1 S)$, $i = 1, 2$, namely*

$$\psi(X)d(C_1^1 S)(Y,Z)\,, \quad \psi(Y)d(C_1^1 S)(X,Z)\,, \quad \psi(Z)d(C_1^1 S)(X,Y)\,,$$

$$\psi(X)d(C_2^1 S)(Y,Z)\,, \quad \psi(Y)d(C_2^1 S)(X,Z)\,, \quad \psi(Z)d(C_2^1 S)(X,Y\,.$$

Lemma 4.4. *Let us assume the antisymmetric part* $\operatorname{Alt} S$ *of* S *with the coordinate expression* $\operatorname{Alt} S = \frac{1}{2}(S^i_{jk} - S^i_{kj})\,\partial_i \otimes d^j \otimes d^k$. *Then*

$$d(\psi \circ \operatorname{Alt} S) \tag{4.1}$$

is a first order natural $\mathbb{R}$*-bilinear differential operator with values in 3-forms.*

Corollary 4.1. *If the 1-form ψ is closed then we have only 7 operators from Lemma 4.3 and Lemma 4.4.*

Theorem 4.1. *All natural differential operators transforming a $(1,2)$-tensor field S and a 1-form ψ into $(0,3)$-tensor fields form a 19-parameter family of operators described in Lemmas 4.1–4.4.*

Proof. According to Theorem 1.1 and (1.3)–(1.5)

$$\Phi(S,\psi) = \Phi_{ijk}\, d^i \otimes d^j \otimes d^k\,,$$

where

$$\begin{aligned}
\Phi_{ijk} = {} & a_1\, S^m_{mi}\,\partial_j\psi_k + a_2\, S^m_{mi}\,\partial_k\psi_j + a_3\, S^m_{mj}\,\partial_i\psi_k + a_4\, S^m_{mj}\,\partial_k\psi_i \\
& \quad + a_5\, S^m_{mk}\,\partial_i\psi_j + a_6\, S^m_{mk}\,\partial_j\psi_i \\
& + a_7\, S^m_{im}\,\partial_j\psi_k + a_8\, S^m_{im}\,\partial_k\psi_j + a_9\, S^m_{jm}\,\partial_i\psi_k + a_{10}\, S^m_{jm}\,\partial_k\psi_i \\
& \quad + a_{11}\, S^m_{km}\,\partial_i\psi_j + a_{12}\, S^m_{km}\,\partial_j\psi_i \\
& + a_{13}\, S^m_{ij}\,\partial_m\psi_k + a_{14}\, S^m_{ji}\,\partial_m\psi_k + a_{15}\, S^m_{ik}\,\partial_m\psi_j + a_{16}\, S^m_{ki}\,\partial_m\psi_j \\
& \quad + a_{17}\, S^m_{jk}\,\partial_m\psi_i + a_{18}\, S^m_{kj}\,\partial_m\psi_i \\
& + a_{19}\, S^m_{ij}\,\partial_k\psi_m + a_{20}\, S^m_{ji}\,\partial_k\psi_m + a_{21}\, S^m_{ik}\,\partial_j\psi_m + a_{22}\, S^m_{ki}\,\partial_j\psi_m \\
& \quad + a_{23}\, S^m_{jk}\,\partial_i\psi_m + a_{24}\, S^m_{kj}\,\partial_i\psi_m \\
& + b_1\,\psi_i\,\partial_j S^m_{km} + b_2\,\psi_i\,\partial_j S^m_{mk} + b_3\,\psi_i\,\partial_k S^m_{jm} + b_4\,\psi_i\,\partial_k S^m_{mj} \\
& \quad + b_5\,\psi_i\,\partial_m S^m_{jk} + b_6\,\psi_i\,\partial_m S^m_{kj} \\
& + b_7\,\psi_j\,\partial_i S^m_{km} + b_8\,\psi_j\,\partial_i S^m_{mk} + b_9\,\psi_j\,\partial_k S^m_{im} + b_{10}\,\psi_j\,\partial_k S^m_{mi} \\
& \quad + b_{11}\,\psi_j\,\partial_m S^m_{ik} + b_{12}\,\psi_j\,\partial_m S^m_{ki} \\
& + b_{13}\,\psi_k\,\partial_i S^m_{jm} + b_{14}\,\psi_k\,\partial_i S^m_{mj} + b_{15}\,\psi_k\,\partial_j S^m_{im} + b_{16}\,\psi_k\,\partial_j S^m_{mi} \\
& \quad + b_{17}\,\psi_k\,\partial_m S^m_{ij} + b_{18}\,\psi_k\,\partial_m S^m_{ji} \\
& + b_{19}\,\psi_m\,\partial_i S^m_{jk} + b_{20}\,\psi_m\,\partial_i S^m_{kj} + b_{21}\,\psi_m\,\partial_j S^m_{ik} + b_{22}\,\psi_m\,\partial_j S^m_{ki} \\
& \quad + b_{23}\,\psi_m\,\partial_k S^m_{ij} + b_{24}\,\psi_m\,\partial_k S^m_{ji}\,.
\end{aligned}$$

In order to calculate relations for coefficients a_i, b_i, $i = 1,\dots,24$, we use the method of an auxiliary linear symmetric connection K, [4]. We replace derivatives of tensor fields with covariant derivatives and assume that the operator is independent of K. Then we get

$$
\begin{aligned}
0 = \psi_p \big[& S^m_{mi}\,(a_1+a_2)\,K_k{}^p{}_j + S^m_{im}\,(a_7+a_8)\,K_k{}^p{}_j \\
& + S^m_{mj}\,(a_3+a_4)\,K_k{}^p{}_i + S^m_{jm}\,(a_9+a_{10})\,K_k{}^p{}_i \\
& + S^m_{mk}\,(a_5+a_6)\,K_j{}^p{}_i + S^m_{km}\,(a_{11}+a_{12})\,K_j{}^p{}_i \\
& + S^m_{ij}\,(a_{13}+a_{19}-b_{23})\,K_k{}^p{}_m + S^m_{ji}\,(a_{14}+a_{20}-b_{24})\,K_k{}^p{}_m \\
& + S^m_{ik}\,(a_{15}+a_{21}-b_{21})\,K_j{}^p{}_m + S^m_{ki}\,(a_{16}+a_{22}-b_{22})\,K_j{}^p{}_m \\
& + S^m_{jk}\,(a_{17}+a_{23}-b_{19})\,K_i{}^p{}_m + S^m_{kj}\,(a_{18}+a_{24}-b_{20})\,K_i{}^p{}_m \big] \\
+ \psi_m \big[& (b_{19}+b_{21})\,K_i{}^p{}_j\,S^m_{pk} + (b_{19}+b_{24})\,K_i{}^p{}_k\,S^m_{jp} \\
& + (b_{20}+b_{23})\,K_i{}^p{}_k\,S^m_{pj} + (b_{20}+b_{22})\,K_i{}^p{}_j\,S^m_{kp} \\
& + (b_{21}+b_{23})\,K_j{}^p{}_k\,S^m_{ip} + (b_{22}+b_{24})\,K_j{}^p{}_k\,S^m_{pi} \big] \\
+ \psi_i \big[& (b_1+b_3)\,K_j{}^p{}_k\,S^m_{pm} + (b_2+b_4)\,K_j{}^p{}_k\,S^m_{mp} \\
& + b_5\,(K_m{}^p{}_j\,S^m_{pk} + K_m{}^p{}_k\,S^m_{jp} - K_m{}^m{}_p\,S^p_{jk}) \\
& + b_6\,(K_m{}^p{}_k\,S^m_{pj} + K_m{}^p{}_j\,S^m_{kp} - K_m{}^m{}_p\,S^p_{kj}) \big] \\
+ \psi_j \big[& (b_7+b_9)\,K_i{}^p{}_k\,S^m_{pm} + (b_8+b_{10})\,K_i{}^p{}_k\,S^m_{mp} \\
& + b_{11}\,(K_m{}^p{}_i\,S^m_{pk} + K_m{}^p{}_k\,S^m_{ip} - K_m{}^m{}_p\,S^p_{ik}) \\
& + b_{12}\,(K_m{}^p{}_k\,S^m_{pi} + K_m{}^p{}_i\,S^m_{kp} - K_m{}^m{}_p\,S^p_{ki}) \big] \\
+ \psi_k \big[& (b_{13}+b_{15})\,K_i{}^p{}_j\,S^m_{pm} + (b_{14}+b_{16})\,K_i{}^p{}_j\,S^m_{mp} \\
& + b_{17}\,(K_m{}^p{}_i\,S^m_{pj} + K_m{}^p{}_j\,S^m_{ip} - K_m{}^m{}_p\,S^p_{ij}) \\
& + b_{18}\,(K_m{}^p{}_j\,S^m_{pi} + K_m{}^p{}_i\,S^m_{jp} - K_m{}^m{}_p\,S^p_{ji}) \big]\,.
\end{aligned}
$$

So, we get $b_3 = -b_1$, $b_4 = -b_2$, $b_9 = -b_7$, $b_{10} = -b_8$, $b_{16} = -b_{14}$, $b_{15} = -b_{13}$ and $b_5 = b_6 = b_{11} = b_{12} = b_{17} = b_{18} = 0$. This corresponds to linear combination of operators from Lemma 4.3.

Further $a_2 = -a_1$, $a_4 = -a_3$, $a_8 = -a_7$, $a_{10} = -a_9$, $a_6 = -a_5$ and $a_{12} = -a_{11}$ which gives a linear combinations of operators from Lemma 4.1.

For coefficients $b_{19}, \dots, b_{24}$ we obtain the following system of homogeneous linear equations.

$$
\begin{array}{cccccccc}
b_{19} & & +b_{21} & & & & = & 0\,, \\
b_{19} & & & & & +b_{24} & = & 0\,, \\
 & b_{20} & & & +b_{23} & & = & 0\,, \\
 & b_{20} & & +b_{22} & & & = & 0\,, \\
 & & b_{21} & & +b_{23} & & = & 0\,, \\
 & & & b_{22} & & +b_{24} & = & 0\,.
\end{array}
$$

This system of equations has one free variable and if we put $b_{24} = -B$ we obtain $b_{19} = b_{22} = b_{23} = B$ and $b_{20} = b_{21} = b_{24} = -B$.

Finally, we have the system of linear equations

$$
\begin{aligned}
a_{13} + a_{19} - b_{23} &= 0\,, & a_{14} + a_{20} - b_{24} &= 0\,,\\
a_{15} + a_{21} - b_{21} &= 0\,, & a_{16} + a_{22} - b_{22} &= 0\,,\\
a_{17} + a_{23} - b_{19} &= 0\,, & a_{18} + a_{24} - b_{20} &= 0\,.
\end{aligned}
$$

It gives the following operators

$$
\begin{aligned}
\Phi_{ijk} = {} & a_{13}\, S^m_{ij}\,(\partial_m\psi_k - \partial_k\psi_m) + B\,(S^m_{ij}\,\partial_k\psi_m + \psi_m\,\partial_k S^m_{ij})\\
& + a_{14}\, S^m_{ji}\,(\partial_m\psi_k - \partial_k\psi_m) - B\,(S^m_{ji}\,\partial_k\psi_m + \psi_m\,\partial_k S^m_{ji})\\
& + a_{15}\, S^m_{ik}\,(\partial_m\psi_j - \partial_j\psi_m) - B\,(S^m_{ik}\,\partial_j\psi_m + \psi_m\,\partial_j S^m_{ik})\\
& + a_{16}\, S^m_{ki}\,(\partial_m\psi_j - \partial_j\psi_m) + B\,(S^m_{ki}\,\partial_j\psi_m + \psi_m\,\partial_j S^m_{ki})\\
& + a_{17}\, S^m_{jk}\,(\partial_m\psi_i - \partial_i\psi_m) + B\,(S^m_{jk}\,\partial_i\psi_m + \psi_m\,\partial_i S^m_{jk})\\
& + a_{18}\, S^m_{kj}\,(\partial_m\psi_i - \partial_i\psi_m) - B\,(S^m_{kj}\,\partial_i\psi_m + \psi_m\,\partial_i S^m_{kj})\,.
\end{aligned}
$$

Now, if we put $B = 0$, we get a linear combination of operators from Lemma 4.2.

Finally, putting $B = 1$ and the others free variables are vanishing, we obtain

$$
\begin{aligned}
\Phi_{ijk} = {} & S^m_{ij}\,\partial_k\psi_m + \psi_m\,\partial_k S^m_{ij} - S^m_{ji}\,\partial_k\psi_m - \psi_m\,\partial_k S^m_{ji}\\
& - S^m_{ik}\,\partial_j\psi_m - \psi_m\,\partial_j S^m_{ik} + S^m_{ki}\,\partial_j\psi_m + \psi_m\,\partial_j S^m_{ki}\\
& + S^m_{jk}\,\partial_i\psi_m + \psi_m\,\partial_i S^m_{jk} - S^m_{kj}\,\partial_i\psi_m - \psi_m\,\partial_i S^m_{kj}\\
& + \psi_m\,\big(\partial_i S^m_{jk} - \partial_i S^m_{kj} - \partial_j S^m_{ik} + \partial_j S^m_{ki} + \partial_k S^m_{ij} - \partial_k S^m_{ji}\big)\\
= {} & (S^m_{ij} - S^m_{ji})\,\partial_k\psi_m + (S^m_{ki} - S^m_{ik})\,\partial_j\psi_m + (S^m_{jk} - S^m_{kj})\,\partial_i\psi_m\\
& + \psi_m\,\big(\partial_i S^m_{jk} - \partial_i S^m_{kj} + \partial_j S^m_{ki} - \partial_j S^m_{ik} + \partial_k S^m_{ij} - \partial_k S^m_{ji}\big)\,.
\end{aligned}
\tag{4.2}
$$

The operator $\Phi(S,\psi)$ defined by (4.2) is a multiple of

$$d(\psi \circ \operatorname{Alt} S)$$

described in Lemma 4.4. So all natural operators are linear combinations of 19 operators from Lemmas 4.1–4.4. □

4.2. **The case of tangent-valued 2-forms.** Now, we assume that S is a tangent valued 2-form, i.e. $\operatorname{Alt} S = S$. Then there are 3 independent operators given by Lemma 4.1, 3 independent operators given by Lemma 4.2 and 3 independent operators given by Lemma 4.3.

Remark 4.1. If S is a tangent-valued 2-form, then we have the Yano-Ako operator, [8], defined as

$$
\begin{aligned}
\Phi(S,\psi)(X,Y,Z) = {} & (L_{S(X,Y)}\psi)(Z) - (L_X(\psi\circ S))(Z,Y)\\
& - (L_Y(\psi\circ S))(X,Z) + (\psi\circ S)([X,Y],Z)\,.
\end{aligned}
$$

We can express this operator as the linear combination of the basic operators in the form

$$\Phi(S,\psi)(X,Y,Z) = d(\psi\circ S)(X,Z,Y) + d\psi(S(X,Y),Z)\,.$$

Remark 4.2. Let as assume that S is a tangent-valued 2-form. According to [2] and [3, p. 69] we can define the Lie derivative of ψ with respect to S as

$$L_S\psi = [i_S, d]\psi = i_S d\psi + d i_S\psi$$

which is a 3-form. It is easy to see that

$$\begin{aligned}(L_S\psi)(X,Y,Z) = d\psi(S(X,Y),Z) + d\psi(S(Y,Z),X) + d\psi(S(Z,X),Y)\\ + d(\psi\circ S)(X,Y,Z)\,.\end{aligned}$$

Remark 4.3. If S is a tangent-valued 2-form, then we can consider the Lie derivation of $\psi\circ S$ with respect to the identity tensor $\mathbb{I}$ and obtain the tangent valued 3-form

$$L_{\mathbb{I}}(\psi\circ S) = i_{\mathbb{I}} d(\psi\circ S) - d i_{\mathbb{I}}(\psi\circ S) = d(\psi\circ S)\,.$$

The Yano-Ako operator is antisymmetric in the first two arguments. On the other hand $L_S\psi$ and $L_{\mathbb{I}}(\psi\circ S)$ have values in 3-forms. If we assume the antisymmetrization of the Yano-Ako operator we get the following identity, [8],

$$3\,\mathrm{Alt}\,\Phi(\psi,S) = L_S\psi + 2\,L_{\mathbb{I}}(\psi\circ S)\,.$$

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