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A NOTE ON THE COHOMOLOGY RING OF THE ORIENTED GRASSMANN MANIFOLDS $\widetilde{G}_{n,4}$

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Abstract. We use known results on the characteristic rank of the canonical 4–plane bundle over the oriented Grassmann manifold $\widetilde{G}_{n,4}$ to compute the generators of the $\mathbb{Z}_2$–cohomology groups $H^j(\widetilde{G}_{n,4})$ for $n = 8, 9, 10, 11$. Drawing from the similarities of these examples with the general description of the cohomology rings of $\widetilde{G}_{n,3}$ we conjecture some predictions.

1. Introduction

Let us denote $G_{n,k}$ the Grassmann manifold of k–dimensional vector subspaces in $\mathbb{R}^n$, i.e. the space $O(n)/(O(k) \times O(n-k))$. Next, denote $\widetilde{G}_{n,k}$ the *oriented* Grassmann manifold of *oriented* k-dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n-k))$. We may suppose that $k \leq n-k$ for both of them.

The manifolds $G_{n,k}$ and $\widetilde{G}_{n,k}$ come equipped with their canonical k-plane bundles, which we denote $\gamma_{n,k}$ and $\widetilde{\gamma}_{n,k}$ respectively.

For the Grassmann manifold $G_{n,k}$ there is a concise description of its $\mathbb{Z}_2$-cohomology ring as a quotient ring of a polynomial ring (see [2])

$$H^*(G_{n,k};\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \dots, w_k]/I_{n,k}\,, \tag{1.1}$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \dots, \bar{w}_n$, where each $\bar{w}_i$ denotes the i-dimensional component of the formal power series

$$1 + (w_1 + w_2 + \cdots + w_k) + (w_1 + w_2 + \cdots + w_k)^2 + (w_1 + w_2 + \cdots + w_k)^3 + \cdots .$$

Each indeterminate w_i is a representative of the ith Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical k-plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

However, the cohomology ring of the oriented Grassmann manifold $\widetilde{G}_{n,k}$ is not fully generated by the characteristic classes $w_i(\widetilde{\gamma}_{n,k})$ and is not known in general. There are descriptions of $H^*(\widetilde{G}_{n,k};\mathbb{Z}_2)$ for spheres $\widetilde{G}_{n,1} \cong S^{n-1}$, complex quadrics $\widetilde{G}_{n,2}$, and in [1] for $\widetilde{G}_{n,3}$ as well.

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In this paper we begin the study of the $\mathbb{Z}_2$-cohomology ring of $\widetilde{G}_{n,4}$ by considering the cases $n = 8, 9, 10, 11$. We will abbreviate $H^j(X; \mathbb{Z}_2)$ to $H^j(X)$, denote $w_i = w_i(\gamma_{n,k})$ and $\widetilde{w}_i = w_i(\widetilde{\gamma}_{n,k})$ as usual.

The paper is organized as follows. In the second section we review the general strategy on how to approach the study of $H^*(\widetilde{G}_{n,k})$. It contains the tools which will be used later to perform the computations. The third section contains the main result of the paper, which is the complete description of all cohomology groups of $\widetilde{G}_{n,4}$ for $n = 8, 9, 10, 11$, along with partial information about the ring structure of $H^*(\widetilde{G}_{n,4})$. Some conjectures are also discussed based on these results.

2. Preliminaries

To obtain information about $H^j(\widetilde{G}_{n,4})$, we first need to recall some general facts about $H^j(\widetilde{G}_{n,k})$. We proceed similarly as in [4].

There is a covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n, k) \neq (2, 1)$. To this 2-fold covering, there is an associated line bundle ξ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence ([6, Corollary 12.3])

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} \tag{2.1}$$

where $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel-Whitney class $w_1 = w_1(\gamma_{n,k})$.

Since the pullback $p^*\gamma_{n,k}$ is isomorphic to $\widetilde{\gamma}_{n,k}$, the covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$ induces the ring homomorphism $p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$, which maps each Stiefel-Whitney class w_i to $\widetilde{w}_i$.

Consequently, the image $\operatorname{Im}(p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$-vector space $H^j(\widetilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel-Whitney characteristic classes of $\widetilde{\gamma}_{n,k}$. We will call it *the characteristic subspace* and denote it $C(j; n, k)$. Moreover (see [9]), the image $\operatorname{Im}(p^*)$ of the ring homomorphism $p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$ is a *self-annihilating* subspace of $H^*(\widetilde{G}_{n,k})$. That is, we have the following.

Lemma 2.1. *For any $\widetilde{x} \in C(j; n, k)$ and $\widetilde{y} \in C(j'; n, k)$ we have $\widetilde{x}\widetilde{y} = 0$ if $j + j' = k(n-k) = \dim(\widetilde{G}_{n,k})$.*

From the exactness of the sequence (2.1), we have $\widetilde{w}_1 = p^*(w_1) = 0$ and it is clear that a monomial $\widetilde{w}_2^{a_2}\widetilde{w}_3^{a_3}\dots\widetilde{w}_k^{a_k} = p^*(w_2^{a_2}w_3^{a_3}\dots w_k^{a_k})$ is zero in $H^j(\widetilde{G}_{n,k})$ if and only if $w_2^{a_2}w_3^{a_3}\dots w_k^{a_k}$ is a w_1-multiple of some polynomial in $H^*(G_{n,k})$. Let us therefore denote $g_i \in \mathbb{Z}_2[w_2, \dots, w_k]$ the reduction of the polynomial $\bar{w}_i$ (see (1.1)) modulo w_1 and by $J_{n,k}$ the ideal in $\mathbb{Z}_2[w_2, \dots, w_k]$ generated by $g_{n-k+1}, \dots, g_n$. The following lemma is a formal restatement of the previous observation.

Lemma 2.2. *Monomial $\widetilde{w}_2^{a_2}\widetilde{w}_3^{a_3}\dots\widetilde{w}_k^{a_k} \in C(j; n, k)$ is equal to zero iff $w_2^{a_2}w_3^{a_3}\dots w_k^{a_k} \in J_{n,k}$.*

The question whether $C(j; n, k)$ is equal to $H^j(\widetilde{G}_{n,k})$ is related to the notion of the characteristic rank of a vector bundle, which was defined in [3], [7].

Definition 2.3. Let X be a connected, finite CW-complex and ξ a real vector bundle over X. The *characteristic rank* of the vector bundle ξ, $\operatorname{charrank}(\xi)$, is the greatest integer q, $0 \le q \le \dim(X)$, such that every cohomology class in $H^j(X)$ for $0 \le j \le q$ can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(\xi)$ of ξ.

This implies that the characteristic rank of $\widetilde{\gamma}_{n,k}$ is equal to the greatest integer q, such that the homomorphism $p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k})$ is surjective (that is $C(j;n,k) = H^j(\widetilde{G}_{n,k})$) for all j, $0 \le j \le q$, or equivalently, by (2.1), that the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective for all j, $0 \le j \le q$.

Hence, in order to compute the characteristic rank of $\widetilde{\gamma}_{n,k}$, it is necessary to study the kernel of $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$. Let us denote $b_j(X)$ the jth $\mathbb{Z}_2$–Betti number of a manifold X and then define $\alpha_j(\widetilde{G}_{n,k}) = b_j(\widetilde{G}_{n,k}) - \dim(C(j;n,k))$, the codimension of the subspace $C(j;n,k) \subseteq H^j(\widetilde{G}_{n,k})$.

There is a useful upper bound for this number described in the next proposition.

Proposition 2.4 ([5, Proposition 2.4. (3)])**.** *For a non-negative integer x, we associate with $H^{n-k+x+1}(G_{n,k})$ $(2 \le k \le n-k)$ the set*

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}\,.$$

If $x \le n-k-1$ and there are t linearly independent elements in the set $N_x(G_{n,k})$, then

$$\alpha_{n-k+x}(\widetilde{G}_{n,k}) \le |N_x(G_{n,k})| - t\,,$$

where $|N_x(G_{n,k})|$ is the cardinality of the set $N_x(G_{n,k})$.

When $j \le \operatorname{charrank}(\widetilde{\gamma}_{n,k})$ we have (see [4, (3)])

$$b_j(\widetilde{G}_{n,k}) = b_j(G_{n,k}) - b_{j-1}(G_{n,k}) \tag{2.2}$$

and the Betti numbers for $G_{n,k}$ are readily calculable from the Poincaré polynomial [2]

$$P_t(G_{n,k}) = \frac{(1-t^{n-k+1})\cdots(1-t^n)}{(1-t)\cdots(1-t^k)}\,. \tag{2.3}$$

3. Computations

Recently, the number $\operatorname{charrank}(\widetilde{\gamma}_{n,4})$ was completely determined [8] and adjusted to our notation we have the following.

Theorem 3.1 ([8, Theorem 6.6])**.** *Let $n \ge 8$ be an integer. If $t \ge 3$ is the unique integer such that $2^{t-1} < n \le 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{n,4}) = \min\left\{4n - 3\cdot 2^{t-1} - 5, 2^t - 5\right\}\,.$$

For better clarity of the forthcoming proofs we first list generators of the ideals $J_{n,4}$ and derive some additional relations in cohomology implied by Lemma 2.2.

Lemma 3.2. *We have*

$$\begin{aligned} J_{8,4} = J_{9,4} &= (g_6, g_7, g_8) = (w_2^3 + w_3^2, w_2^2 w_3, w_2^4 + w_2 w_3^2 + w_2^2 w_4 + w_4^2),\\ J_{10,4} &= (w_2^2 w_3, w_2^4 + w_2 w_3^2 + w_2^2 w_4 + w_4^2, w_3^3, w_2^5 + w_3^2 w_4 + w_2 w_4^2),\\ J_{11,4} &= (w_2^4 + w_2 w_3^2 + w_2^2 w_4 + w_4^2, w_3^3, w_2^5 + w_3^2 w_4 + w_2 w_4^2, w_2^4 w_3 + w_3 w_4^2). \end{aligned}$$

Additionally, in $H^*(\widetilde{G}_{8,4})$ and $H^*(\widetilde{G}_{9,4})$ we have

$$\widetilde{w}_4^2 = \widetilde{w}_2^2 \widetilde{w}_4\,, \qquad \widetilde{w}_3^3 = 0\,, \qquad \widetilde{w}_2^5 = 0\,,$$
$$\widetilde{w}_3 \widetilde{w}_4^2 = 0\,, \qquad \widetilde{w}_2^2 \widetilde{w}_4^2 = \widetilde{w}_2^4 \widetilde{w}_4\,.$$

In $H^*(\widetilde{G}_{10,4})$ we have

$$\widetilde{w}_2^3 \widetilde{w}_4 = \widetilde{w}_3^2 \widetilde{w}_4\,, \qquad \widetilde{w}_3 \widetilde{w}_4^2 = 0\,, \qquad \widetilde{w}_2^6 + \widetilde{w}_2^4 \widetilde{w}_4 = \widetilde{w}_2^2 \widetilde{w}_4^2\,,$$
$$\widetilde{w}_4^3 = \widetilde{w}_2^2 \widetilde{w}_4^2\,.$$

In $H^*(\widetilde{G}_{11,4})$ we have

$$\widetilde{w}_2^2 \widetilde{w}_3^2 + \widetilde{w}_2^3 \widetilde{w}_4 = \widetilde{w}_3^2 \widetilde{w}_4\,, \qquad \widetilde{w}_2^2 \widetilde{w}_3 \widetilde{w}_4 = 0\,, \qquad \widetilde{w}_2^3 \widetilde{w}_4^2 = \widetilde{w}_3^2 \widetilde{w}_4^2\,,$$
$$\widetilde{w}_2^7 = \widetilde{w}_2^4 \widetilde{w}_3^2, \qquad \widetilde{w}_2^7 = \widetilde{w}_2^5 \widetilde{w}_4\,, \qquad \widetilde{w}_2 \widetilde{w}_4^3 = 0\,,$$
$$\widetilde{w}_2^5 \widetilde{w}_3^2 = \widetilde{w}_2^4 \widetilde{w}_4^2\,, \qquad \widetilde{w}_2^6 \widetilde{w}_4 = \widetilde{w}_2^4 \widetilde{w}_4^2\,,$$

Proof. Direct computation of the polynomials $g_i \in \mathbb{Z}_2[w_2, w_3, w_4]$ shows that $g_5 = 0$ and $g_6, \ldots, g_{11}$ are as claimed. Since $g_5 = 0$ we have $(g_5, g_6, g_7, g_8) = (g_6, g_7, g_8)$. However $g_9 = w_3^3 = w_2 g_7 + w_3 g_6$, thus also $(g_6, g_7, g_8, g_9) = (g_6, g_7, g_8)$. By definition, both $J_{8,4}$ and $J_{9,4}$ are equal to (g_6, g_7, g_8).

Now, since $J_{8,4} = J_{9,4}$, by Lemma 2.2 the relations in $H^*(\widetilde{G}_{8,4})$ and $H^*(\widetilde{G}_{9,4})$ for the elements of the characteristic subspace will be the same. We will check them in $H^*(\widetilde{G}_{8,4})$. The proof for $H^*(\widetilde{G}_{9,4})$ is identical. Since $w_2 g_6 + g_8 \in J_{8,4}$, we have $\widetilde{w}_2^2 \widetilde{w}_4 + \widetilde{w}_4^2 = 0$, which is equivalent to $\widetilde{w}_4^2 = \widetilde{w}_2^2 \widetilde{w}_4$. We have already shown that $w_3^3 \in J_{8,4}$, thus $\widetilde{w}_3^3 = 0$. We have $w_2^5 = w_2^2 g_6 + w_3 g_7 \in J_{8,4}$, therefore we obtain $\widetilde{w}_2^5 = 0$. Next $\widetilde{w}_3 \widetilde{w}_4^2 = \widetilde{w}_2^2 \widetilde{w}_3 \widetilde{w}_4$ by the first relation and the latter is zero because $w_4 g_7 \in J_{8,4}$. Finally, $\widetilde{w}_2^2 \widetilde{w}_4^2 = \widetilde{w}_2^4 \widetilde{w}_4$ by the same reason.

In $H^*(\widetilde{G}_{10,4})$ we have $\widetilde{w}_2^3 \widetilde{w}_4 + \widetilde{w}_3^2 \widetilde{w}_4 = 0$ since $w_2^3 w_4 + w_3^2 w_4 = w_3 g_7 + w_2 g_8 + g_{10} \in J_{10,4}$. It is easy to check that $w_3 w_4^2 = (w_2^2 + w_4) g_7 + w_3 g_8 + w_2 g_9 \in J_{10,4}$. Next $\widetilde{w}_2^6 + \widetilde{w}_2^4 \widetilde{w}_4 = \widetilde{w}_2^6 + \widetilde{w}_2 \widetilde{w}_3^2 \widetilde{w}_4 = \widetilde{w}_2^2 \widetilde{w}_4^2$ by the first relation and $w_2 g_{10} \in J_{10,4}$. Finally $\widetilde{w}_4^3 = \widetilde{w}_2^4 \widetilde{w}_4 + \widetilde{w}_2 \widetilde{w}_3^2 \widetilde{w}_4 + \widetilde{w}_2^2 \widetilde{w}_4^2$ since $w_4 g_8 \in J_{10,4}$ and the first two summands are equal as they are $\widetilde{w}_2$-multiples of equal classes.

In $H^*(\widetilde{G}_{11,4})$ we have $\widetilde{w}_2^2 \widetilde{w}_3^2 + \widetilde{w}_2^3 \widetilde{w}_4 = \widetilde{w}_3^2 \widetilde{w}_4$, since $w_2^2 w_3^2 + w_2^3 w_4 + w_3^2 w_4 = w_2 g_8 + g_{10} \in J_{11,4}$. Then we have $w_2^2 w_3 w_4 = w_3 g_8 + w_2 g_9 + g_{11} \in J_{11,4}$. Next, we have $w_2^3 w_4^2 + w_3^2 w_4^2 = (w_3^2 + w_2 w_4) g_8 + w_2 w_3 g_9 + w_4 g_{10} + w_3 g_{11} \in J_{11,4}$. Since $w_2^2 g_{10} \in J_{11,4}$, we have $\widetilde{w}_2^7 = \widetilde{w}_2^2 \widetilde{w}_3^2 \widetilde{w}_4 + \widetilde{w}_2^3 \widetilde{w}_4^2$, but the first summand is zero and the last is equal to $\widetilde{w}_3^2 \widetilde{w}_4^2$ by the third relation, which is equal to $\widetilde{w}_2^4 \widetilde{w}_3^2$, because $w_3 g_{11} \in J_{11,4}$. Since $w_2^3 g_8 + w_3 g_{11} \in J_{11,4}$, we have $\widetilde{w}_2^7 + \widetilde{w}_2^5 \widetilde{w}_4 + \widetilde{w}_2^3 \widetilde{w}_4^2 + \widetilde{w}_3^2 \widetilde{w}_4^2 = 0$, but the last two summands are equal by the third relation. Since $w_2 w_4 g_8 + w_2^2 g_{10} \in J_{11,4}$, we have $\widetilde{w}_2^7 + \widetilde{w}_2^5 \widetilde{w}_4 + \widetilde{w}_2 \widetilde{w}_4^3 = 0$, but the first two summands are equal by the previous relation.

Since $w_2w_3g_{11} \in J_{11,4}$, we have $\widetilde{w}_2^5\widetilde{w}_3^2 = \widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4^2$ and the RHS is equal to $\widetilde{w}_2^4\widetilde{w}_4^2$ by the third relaton. Since $w_2w_4g_{10} \in J_{11,4}$, we have $\widetilde{w}_2^6\widetilde{w}_4 + \widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4^2 + \widetilde{w}_2^2\widetilde{w}_4^3 = 0$, but the last summand is $\widetilde{w}_2$–multiple of zero and the middle one is equal to $\widetilde{w}_2^4\widetilde{w}_4^2$ and we obtain the desired result. □

Theorem 3.3. *We have the following generators of $H^j(\widetilde{G}_{8,4})$.*

j	*gen.*	j	*gen.*
0	$\widetilde{w}_0$	9	$a_4\widetilde{w}_2\widetilde{w}_3, \widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
1	–	10	$a_4\widetilde{w}_2^3, a_4\widetilde{w}_2\widetilde{w}_4, \widetilde{w}_2^3\widetilde{w}_4$
2	$\widetilde{w}_2$	11	$a_4\widetilde{w}_3\widetilde{w}_4$
3	$\widetilde{w}_3$	12	$a_4\widetilde{w}_2^4, a_4\widetilde{w}_2^2\widetilde{w}_4, \widetilde{w}_2^4\widetilde{w}_4$
4	$a_4, \widetilde{w}_2^2, \widetilde{w}_4$	13	$a_4\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
5	$\widetilde{w}_2\widetilde{w}_3$	14	$a_4\widetilde{w}_2^3\widetilde{w}_4$
6	$a_4\widetilde{w}_2, \widetilde{w}_2^3, \widetilde{w}_2\widetilde{w}_4$	15	–
7	$a_4\widetilde{w}_3, \widetilde{w}_3\widetilde{w}_4$	16	$a_4\widetilde{w}_2^4\widetilde{w}_4$
8	$a_4\widetilde{w}_2^2, a_4\widetilde{w}_4, \widetilde{w}_2^4, \widetilde{w}_2^2\widetilde{w}_4$		

where a_4 is an element in $H^4(\widetilde{G}_{8,4}) \setminus C(4;8,4)$.

Proof. We have charrank$(\widetilde{\gamma}_{8,4}) = 3$, so for $j \leq 3$ we have $C(j;8,4) = H^j(\widetilde{G}_{8,4})$, but $C(4;8,4) \subset H^4(\widetilde{G}_{8,4})$ is a proper subspace and thus for the codimension we have $\alpha_4(\widetilde{G}_{8,4}) = b_j(\widetilde{G}_{n,k}) - \dim(C(j;n,k)) \geq 1$. On the other hand, from Proposition 2.4 we have $\alpha_4(\widetilde{G}_{8,4}) \leq 1$ since $x = 0$ and $N_0(G_{8,4}) = \{g_5\}$ is a one element set. Let us denote $a_4 \in H^4(\widetilde{G}_{8,4})$ an element outside $C(4;8,4)$.

Now, let us first list all generators of $C(j;8,4)$ with the help of Lemma 3.2 before continuing further with $H^j(\widetilde{G}_{8,4})$. Note that $\widetilde{w}_2^3 = \widetilde{w}_3^2$ and $\widetilde{w}_2^2\widetilde{w}_3 = 0$, since $g_6, g_7 \in J_{8,4}$. Also note that if $\widetilde{x} \in C(j;8,4)$ is a nonzero element, it must be a $\widetilde{w}_i$–multiple of some nonzero element in $C(j-i;8,4)$ for some $i \in \{2,3,4\}$.

In $C(5;8,4)$ there is only one nonzero element, $\widetilde{w}_2\widetilde{w}_3$.

In $C(6;8,4)$ there are two, $\widetilde{w}_2^3$ and $\widetilde{w}_2\widetilde{w}_4$, because $\widetilde{w}_3^2 = \widetilde{w}_2^3$.

In $C(7;8,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3 = 0$ and thus $\widetilde{w}_3\widetilde{w}_4$ is the only generator.

In $C(8;8,4)$ we have $\widetilde{w}_2^4 = \widetilde{w}_2\widetilde{w}_3^2$ and $\widetilde{w}_2^2\widetilde{w}_4 = \widetilde{w}_4^2$ as the two generators.

In $C(9;8,4)$ we have $\widetilde{w}_2^3\widetilde{w}_3 = \widetilde{w}_3^3 = 0$, so $\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$ is the only nonzero element.

In $C(10;8,4)$ we have $\widetilde{w}_2^5 = \widetilde{w}_2^2\widetilde{w}_3^2 = 0$ and $\widetilde{w}_2^3\widetilde{w}_4 = \widetilde{w}_3^2\widetilde{w}_4 = \widetilde{w}_2\widetilde{w}_4^2$ as the generator.

In $C(11;8,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3\widetilde{w}_4 = 0$, $\widetilde{w}_2^4\widetilde{w}_3 = 0$, $\widetilde{w}_3\widetilde{w}_4^2 = 0$.

In $C(12;8,4)$ there is one generator $\widetilde{w}_2^4\widetilde{w}_4$ equal to both $\widetilde{w}_2^2\widetilde{w}_4^2$ and $\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4$.

By Poincaré duality, to each nonzero element $\widetilde{x} \in H^j(\widetilde{G}_{8,4})$ there exists a nonzero element $\widetilde{y} \in H^{16-j}(\widetilde{G}_{8,4})$ such that $\widetilde{x}\widetilde{y} \neq 0$. Thus Lemma 2.1 implies $C(j;8,4) = 0$ for all $j > 12$. Additionally, the dual to element $\widetilde{w}_2^4\widetilde{w}_4$ must be a_4. Hence $a_4\widetilde{w}_2^4\widetilde{w}_4 = a_4\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4 \neq 0$.

It remains to determine $\alpha_j(\widetilde{G}_{8,4})$ for $j = 5, 6, 7, 8$. For $j \le 7$ Proposition 2.4 with $x = j-4 \le 3$ implies that $\alpha_j(\widetilde{G}_{8,4}) \le |N_{j-4}(G_{8,4})| - t_{j-4}$, where t_{j-4} is the maximal number of linearly independent elements of $N_{j-4}(G_{8,4})$. For $N_1(G_{8,4}) = \{g_6\}$ we have $t_1 = 1$. For $N_2(G_{8,4}) = \{w_2g_5, g_7\}$ we have $t_2 = 1$, since $g_5 = 0$. For $N_3(G_{8,4}) = \{w_3g_5, w_2g_6, g_8\}$ we have $t_3 = 2$ as all $w_2g_6, g_8, w_2g_6 + g_8$ are nonzero. Thus $\alpha_5(\widetilde{G}_{8,4}) \le 0$, $\alpha_6(\widetilde{G}_{8,4}) \le 1$ and $\alpha_7(\widetilde{G}_{8,4}) \le 1$. On the other hand, we have shown $a_4\widetilde{w}_2 \in H^6(\widetilde{G}_{8,4})$ and $a_4\widetilde{w}_3 \in H^7(\widetilde{G}_{8,4})$ to be nonzero and dual to $\widetilde{w}_2^3\widetilde{w}_4$ and $\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$ respectively. Therefore $\alpha_6(\widetilde{G}_{8,4}), \alpha_7(\widetilde{G}_{8,4}) \ne 0$ and both must be equal to 1.

We determine $\alpha_8(\widetilde{G}_{8,4})$ with the help of the Euler characteristic. For the Grassmann manifold $G_{8,4}$ we can compute its Euler characteristic from the Poincaré polynomial (2.3) to obtain $\chi(G_{8,4}) = 6$. As $\widetilde{G}_{8,4}$ is a 2–fold cover, we have $\chi(\widetilde{G}_{8,4}) = 2 \cdot \chi(G_{8,4}) = 12$. By this point we know the Betti numbers $b_0(\widetilde{G}_{8,4}), \ldots, b_7(\widetilde{G}_{8,4})$. Poincaré duality and a simple calculation yields $b_8(\widetilde{G}_{8,4}) = 4$. Consequently, $\alpha_8(\widetilde{G}_{8,4}) = 2$ and since we already know $a_4\widetilde{w}_2^2$ and $a_4\widetilde{w}_4$ are nonzero, we only need to show that they are distinct. That is done by considering their products with $\widetilde{w}_2^4$ and realizing one is zero while the other is not.

By obvious adjustment of the last argument we also prove that $a_4\widetilde{w}_2^3 \ne a_4\widetilde{w}_2\widetilde{w}_4$ and $a_4\widetilde{w}_2^4 \ne a_4\widetilde{w}_2^2\widetilde{w}_4$. All the remaining numbers $\alpha_j(\widetilde{G}_{8,4})$ are now determined by Poincaré duality combined with the knowledge of all $C(j; 8, 4)$ and the obvious generators suffice to produce the required values. □

Theorem 3.4. *We have the following generators of $H^j(\widetilde{G}_{9,4})$.*

j	*gen.*	j	*gen.*
0	$\widetilde{w}_0$	11	$a_8\widetilde{w}_3$
1	–	12	$a_8\widetilde{w}_2^2, a_8\widetilde{w}_4, \widetilde{w}_2^4\widetilde{w}_4$
2	$\widetilde{w}_2$	13	$a_8\widetilde{w}_2\widetilde{w}_3$
3	$\widetilde{w}_3$	14	$a_8\widetilde{w}_2^3, a_8\widetilde{w}_2\widetilde{w}_4$
4	$\widetilde{w}_2^2, \widetilde{w}_4$	15	$a_8\widetilde{w}_3\widetilde{w}_4$
5	$\widetilde{w}_2\widetilde{w}_3$	16	$a_8\widetilde{w}_2^4, a_8\widetilde{w}_2^2\widetilde{w}_4$
6	$\widetilde{w}_2^3, \widetilde{w}_2\widetilde{w}_4$	17	$a_8\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
7	$\widetilde{w}_3\widetilde{w}_4$	18	$a_8\widetilde{w}_2^3\widetilde{w}_4$
8	$\widetilde{a}_8, w_2^4, \widetilde{w}_2^2\widetilde{w}_4$	19	–
9	$\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$	20	$a_8\widetilde{w}_2^4\widetilde{w}_4$
10	$a_8\widetilde{w}_2, \widetilde{w}_2^3\widetilde{w}_4$		

where a_8 is an element in $H^8(\widetilde{G}_{9,4}) \setminus C(8; 9, 4)$.

Proof. First, as $J_{9,4} = J_{8,4}$, we have $C(j; 9, 4) = C(j; 8, 4)$ for all j.

We have $\operatorname{charrank}(\widetilde{\gamma}_{9,4}) = 7$ so $H^j(\widetilde{G}_{9,4}) = C(j; 9, 4) = C(j; 8, 4)$ for $j \le 7$ and $\alpha_8(\widetilde{G}_{9,4}) \ge 1$. To obtain an upper bound for $\alpha_8(\widetilde{G}_{9,4})$ we consider $N_3(G_{9,4}) =$

$\{w_3g_6, w_2g_7, g_9\}$. Any two elements from this set are linearly independent, which means $\alpha_8(\widetilde{G}_{9,4}) = 1$. Denote a_8 an element in $H^8(\widetilde{G}_{9,4}) \setminus C(8; 9, 4)$.

Clearly, the Poincaré dual to $\widetilde{w}_2^4\widetilde{w}_4$ is a_8 and similarly as before we have $a_8\widetilde{w}_2^4\widetilde{w}_4 = a_8\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4 \neq 0$.

Next, $N_4(G_{9,4}) = \left\{w_2^2g_6, w_4g_6, w_3g_7, w_2g_8\right\}$. We will show that these four elements are linearly independent. Suppose that for some $c_i \in \mathbb{Z}_2$, $1 \leq i \leq 4$ we have

$$c_1w_2^2g_6 + c_2w_4g_6 + c_3w_3g_7 + c_4w_2g_8 = 0.$$

Since every element in $\mathbb{Z}_2[w_2, w_3, w_4]$ is of order 2, the equation implies

$$(c_1 + c_3 + c_4)w_2^2g_6 + c_2w_4g_6 + c_3(w_3g_7 + w_2^2g_6) + c_4(w_2g_8 + w_2^2g_6) = 0\,,$$
$$(c_1 + c_3 + c_4)(w_2^5 + w_2^2w_3^2) + c_2w_4g_6 + c_3w_2^5 + c_4(w_2^3w_4 + w_2w_4^2) = 0\,.$$

If $c_1 + c_3 + c_4$ or c_3 or both are nonzero, the LHS is not divisible by w_4, which is a contradiction. Thus $c_1 + c_3 + c_4 = c_3 = 0$ and the equation simplifies so much,we immediately deduce $c_2 = c_4 = 0$. Which in turn implies $c_1 = 0$. We have proved independence and so $\alpha_9(\widetilde{G}_{9,4}) = 0$.

Now that Betti numbers $b_0(\widetilde{G}_{9,4}), \ldots, b_9(\widetilde{G}_{9,4})$ are known, from calculating $\chi(G_{9,4}) = 6$ and $\chi(\widetilde{G}_{9,4}) = 12$ we obtain $b_{10}(\widetilde{G}_{9,4}) = 2$.

This gives enough information to quickly determine all $\alpha_j(\widetilde{G}_{9,4})$ and all are once again covered by the obvious generators derived from $a_8\widetilde{w}_2^4\widetilde{w}_4 = a_8\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4 \neq 0$. $\square$

Theorem 3.5. *We have the following generators of $H^j(\widetilde{G}_{10,4})$.*

j	*gen.*	j	*gen.*
0	$\widetilde{w}_0$	13	$-$
1	$-$	14	$a_{12}\widetilde{w}_2, b_{12}\widetilde{w}_2$
2	$\widetilde{w}_2$	15	$a_{12}\widetilde{w}_3$
3	$\widetilde{w}_3$	16	$a_{12}\widetilde{w}_2^2, a_{12}\widetilde{w}_4, b_{12}\widetilde{w}_2^2$
4	$\widetilde{w}_2^2, \widetilde{w}_4$	17	$a_{12}\widetilde{w}_2\widetilde{w}_3$
5	$\widetilde{w}_2\widetilde{w}_3$	18	$a_{12}\widetilde{w}_2^3, a_{12}\widetilde{w}_2\widetilde{w}_4, b_{12}\widetilde{w}_2^3$
6	$\widetilde{w}_2^3, \widetilde{w}_3^2, \widetilde{w}_2\widetilde{w}_4$	19	$a_{12}\widetilde{w}_3\widetilde{w}_4$
7	$\widetilde{w}_3\widetilde{w}_4$	20	$a_{12}\widetilde{w}_2^4, b_{12}\widetilde{w}_2^4$
8	$\widetilde{w}_2^4, \widetilde{w}_2\widetilde{w}_3^2, \widetilde{w}_2^2\widetilde{w}_4$	21	$a_{12}\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
9	$\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$	22	$a_{12}\widetilde{w}_3^2\widetilde{w}_4 = b_{12}\widetilde{w}_2^5$
10	$\widetilde{w}_2^5, \widetilde{w}_2^3\widetilde{w}_4$	23	$-$
11	$-$	24	$a_{12}\widetilde{w}_2^4\widetilde{w}_4$
12	$a_{12}, b_{12}, \widetilde{w}_2^6, \widetilde{w}_2^4\widetilde{w}_4$		

where a_{12}, b_{12} are linearly independent elements in $H^{12}(\widetilde{G}_{9,4}) \setminus C(12; 9, 4)$ such that $a_{12}\widetilde{w}_2^4\widetilde{w}_4 \neq 0$, $a_{12}\widetilde{w}_2^5 = 0$, $b_{12}\widetilde{w}_2^6 \neq 0$ and $b_{12}\widetilde{w}_2^4\widetilde{w}_4 = 0$.

Proof. As before, we begin with determining the generators of $C(j;10,4)$. For $j\leq 6$ we have $C(j;10,4)$ as stated since there are no relations.

In $C(7;10,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3=0$ and thus $\widetilde{w}_3\widetilde{w}_4$ is the only generator.

In $C(8;10,4)$ we have $\widetilde{w}_2^4+\widetilde{w}_2\widetilde{w}_3^2+\widetilde{w}_2^2\widetilde{w}_4=\widetilde{w}_4^2$ as the only relation, hence there are the three generators.

In $C(9;10,4)$ we have $\widetilde{w}_2^3\widetilde{w}_3=0$ and $\widetilde{w}_3^3=0$, so $\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$ is the only generator.

In $C(10;10,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3^2=0$, $\widetilde{w}_2^3\widetilde{w}_4=\widetilde{w}_3^2\widetilde{w}_4$, and among $\widetilde{w}_2^5$, $\widetilde{w}_2^3\widetilde{w}_4$, $\widetilde{w}_2\widetilde{w}_4^2$ either is equal to the sum of the other two since $g_{10}\in J_{10,4}$. Thus there are two generators.

In $C(11;10,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3\widetilde{w}_4$, $\widetilde{w}_2^4\widetilde{w}_3$ both multiples of $\widetilde{w}_2^2\widetilde{w}_3=0$. Then $\widetilde{w}_2\widetilde{w}_3^3$ is a multiple of $\widetilde{w}_3^3=0$ and $\widetilde{w}_3\widetilde{w}_4^2=0$ by Lemma 3.2.

In $C(12;10,4)$ we have $\widetilde{w}_2^4\widetilde{w}_4=\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4$ and $\widetilde{w}_2^6+\widetilde{w}_2^4\widetilde{w}_4=\widetilde{w}_2^2\widetilde{w}_4^2=\widetilde{w}_4^3$ by Lemma 3.2. Thus there are two generators, e.g. $\widetilde{w}_2^6$ and $\widetilde{w}_2^4\widetilde{w}_4$.

We know that $\operatorname{charrank}(\widetilde{\gamma}_{10,4})=11$, so by Poincaré duality and Lemma 2.1 we have $C(j;10,4)=0$ for $j\geq 13$.

Also $\alpha_j(\widetilde{G}_{10,4})=0$ for $j\leq 11$, so by now we have determined all Betti numbers except for $b_{12}(\widetilde{G}_{10,4})$. We calculate it from the Euler characteristic $\chi(\widetilde{G}_{10,4})=20$. The result is $b_{12}(\widetilde{G}_{10,4})=4$.

So $\alpha_{12}(\widetilde{G}_{10,4})=2$ and there are two linearly independent elements in $H^{12}(\widetilde{G}_{9,4})\setminus C(12;9,4)$. By Poincaré duality we can start by arbitrarily picking a pair (a'_{12},b'_{12}) such that $a'_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$ and $b'_{12}\widetilde{w}_2^6\neq 0$. Then we adjust b'_{12} based on the fact that $H^{24}(\widetilde{G}_{10,4})\cong\mathbb{Z}_2$. If $b'_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$, we define $b_{12}=a'_{12}+b'_{12}$, so that $b_{12}\widetilde{w}_2^4\widetilde{w}_4=0$. Otherwise let $b_{12}=b'_{12}$

Similarly, since $H^{22}(\widetilde{G}_{10,4})\cong H^2(\widetilde{G}_{10,4})\cong\mathbb{Z}_2$, if $a'_{12}\widetilde{w}_2^5\neq 0$, we define $a_{12}=a'_{12}+b_{12}$, else $a_{12}=a'_{12}$, so that $a_{12}\widetilde{w}_2^5=0$. We have $a_{12},b_{12},\widetilde{w}_2^6,\widetilde{w}_2^4\widetilde{w}_4$ as generators of $H^{12}(\widetilde{G}_{10,4})$.

Next, we have $H^{13}(\widetilde{G}_{10,4})=H^{11}(\widetilde{G}_{10,4})=0$.

Since $a_{12}\widetilde{w}_2$ and $b_{12}\widetilde{w}_2$ have different products with $\widetilde{w}_2^3\widetilde{w}_4$ they are distinct and therefore linearly independent.

We have $a_{12}\widetilde{w}_3\neq 0$, since $a_{12}\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4=a_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$.

By considering the products of nonzero elements $a_{12}\widetilde{w}_2^2,a_{12}\widetilde{w}_4,b_{12}\widetilde{w}_2^2$ with $\widetilde{w}_2^4,\widetilde{w}_2\widetilde{w}_3^2,\widetilde{w}_2^2\widetilde{w}_4$ we see that they are independent. Indeed, suppose that for some $c_1,c_2,c_3\in\mathbb{Z}_2$ we have $c_1a_{12}\widetilde{w}_2^2+c_2a_{12}\widetilde{w}_4+c_3b_{12}\widetilde{w}_2^2=0$. Multiplying both sides by $\widetilde{w}_2\widetilde{w}_3^2$ and recalling that $\widetilde{w}_2^2\widetilde{w}_3=0$, we obtain $c_2=0$. Then multiplying by $\widetilde{w}_2^4$ yields $c_3=0$ and multiplying by $w_2^2\widetilde{w}_4$ gives $c_1=0$.

We have $a_{12}\widetilde{w}_2\widetilde{w}_3\neq 0$, since $a_{12}\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4=a_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$.

Elements $a_{12}\widetilde{w}_2^3,a_{12}\widetilde{w}_2\widetilde{w}_4,b_{12}\widetilde{w}_2^3$ are nonzero. They prove to be independent upon considering their products with $\widetilde{w}_2^3,\widetilde{w}_3^2,\widetilde{w}_2\widetilde{w}_4$.

We have $a_{12}\widetilde{w}_3\widetilde{w}_4\neq 0$, since $a_{12}\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4=a_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$.

Elements $a_{12}\widetilde{w}_2^4,b_{12}\widetilde{w}_2^4$ are nonzero. They prove to be independent upon considering their products with $\widetilde{w}_2^2,\widetilde{w}_4$.

We have $a_{12}\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4\neq 0$, since $a_{12}\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4=a_{12}\widetilde{w}_2^4\widetilde{w}_4\neq 0$.

We have $a_{12}\widetilde{w}_3^2\widetilde{w}_4\neq 0$ and $b_{12}\widetilde{w}_2^5\neq 0$, therefore they are equal.

□

Before we examine the last case, we separately prove one important piece of information.

Lemma 3.6. *The set $N_6(G_{11,4})$ is linearly independent.*

Proof. We partition the set $N_6(G_{11,4})$ into two disjoint sets $N_6^+(G_{11,4})$ and $N_6^-(G_{11,4})$, where $N_6^+(G_{11,4}) = \{w_2^3 g_8, w_3^2 g_8, w_2 w_4 g_8, w_3 g_{11}\}$ and $N_6^-(G_{11,4}) = \{w_2 w_3 g_9, w_2^2 g_{10}, w_4 g_{10}\}$. To prove linear independence of $N_6(G_{11,4})$ we have to prove that no nontrivial linear combination of elements of $N_6^+(G_{11,4})$ is equal to any linear combination of elements of $N_6^-(G_{11,4})$ and vice versa. All elements of $N_6^+(G_{11,4})$ are polynomials with an even number of terms, thus any linear combination of them will have an even number of terms, since each term is of order 2. All elements of $N_6^-(G_{11,4})$ are polynomials with an odd number of terms, so the only nontrivial linear combinations worth considering are $w_2 w_3 g_9 + w_2^2 g_{10}$, $w_2 w_3 g_9 + w_4 g_{10}$ and $w_2^2 g_{10} + w_4 g_{10}$, that is the polynomials

$$\begin{aligned} R_1 &= w_2^7 + w_2 w_3^4 + w_2^2 w_3^2 w_4 + w_2^3 w_4^2, \\ R_2 &= w_2 w_3^4 + w_2^5 w_4 + w_3^2 w_4^2 + w_2 w_4^3, \\ R_3 &= w_2^7 + w_2^5 w_4 + w_2^2 w_3^2 w_4 + w_2^3 w_4^2 + w_3^2 w_4^2 + w_2 w_4^3. \end{aligned}$$

Since none of these are zero polynomials, the set $N_6^-(G_{11,4})$ is linearly independent. It remains to show that there is no nontrivial linear comination of elements of $N_6^+(G_{11,4})$ equal to R_1, R_2, R_3 or zero.

First, let us consider combinations without $w_2^3 g_8$, that is for any $c_1, c_2, c_3 \in \mathbb{Z}_2$ the expression

$$L_{0,c_1,c_2,c_3} = c_1 w_3^2 g_8 + c_2 w_2 w_4 g_8 + c_3 w_3 g_{11},$$

where

$$\begin{array}{rlllll} w_3^2 g_8 = & w_2^4 w_3^2 & + w_2 w_3^4 & & + w_2^2 w_3^2 w_4 & + w_3^2 w_4^2\,, \\ w_2 w_4 g_8 = & & & w_2^5 w_4 & + w_2^2 w_3^2 w_4 + w_2^3 w_4^2 & + w_2 w_4^3\,, \\ w_3 g_{11} = & w_2^4 w_3^2 + & & & & + w_3^2 w_4^2\,. \end{array}$$

Since none of them contain w_2^7 we have that $L_{0,c_1,c_2,c_3} \neq R_1, R_3$. For L_{0,c_1,c_2,c_3} to be equal to zero, first c_2 must be zero and then c_1 and c_3 also. To have $L_{0,c_1,c_2,c_3} = R_2$, we need $c_1 = 1$ to obtain $w_2 w_3^4$ and $c_2 = 1$ to obtain $w_2^5 w_4$. But then no choice of c_3 will make $L_{0,1,1,c_3} = R_2$ true. In the end, $L_{0,c_1,c_2,c_3} \neq R_1, R_2, R_3$ and $L_{0,c_1,c_2,c_3} = 0$ only for $L_{0,0,0,0}$.

Now let us consider combinations containing $w_2^3 g_8$, that is for any $c_1, c_2, c_3 \in \mathbb{Z}_2$ the expression.

$$L_{1,c_1,c_2,c_3} = w_2^3 g_8 + c_1 w_3^2 g_8 + c_2 w_2 w_4 g_8 + c_3 w_3 g_{11}\,.$$

Since w_2^7 is always a term in L_{1,c_1,c_2,c_3}, we only need to consider if it is possible for L_{1,c_1,c_2,c_3} to be equal to R_1 or R_3. Subtracting $w_2^3 g_8$ from both sides of the considered equations, it is the same as considering whether L_{0,c_1,c_2,c_3} is equal to either of the following

$$R_1 + w_2^3 g_8 = w_2^4 w_3^2 + w_2 w_3^4 + w_2^5 w_4 + w_2^2 w_3^2 w_4\,,$$
$$R_3 + w_2^3 g_8 = w_2^4 w_3^2 + w_2^2 w_3^2 w_4 + w_3^2 w_4^2 + w_2 w_4^3\,.$$

Each of $w_3^2 g_8, w_2 w_4 g_8, w_3 g_{11}$ contains an even number of terms from the set $\{w_2^5 w_4, w_2 w_4^3\}$, hence every L_{0,c_1,c_2,c_3} will too. However both $R_1 + w_2^3 g_8, R_3 + w_2^3 g_8$ contain exactly one such term. Thus for any choice of indices $c_1, c_2, c_3 \in \mathbb{Z}_2, i \in \{1,3\}$ the equality $L_{0,c_1,c_2,c_3} = R_i + w_2^3 g_8$ and equivalently $L_{1,c_1,c_2,c_3} = R_i$ is impossible.

In conclusion, the only case when a linear combination of elements of $N_6^+(G_{11,4})$ is equal to some linear combination of elements of $N_6^-(G_{11,4})$ is when they are both trivial. With that we have proved $N_6(G_{11,4})$ is a linearly independent set. $\square$

Theorem 3.7. *We have $H^j(\widetilde{G}_{11,4}) \cong H^j(\widetilde{G}_{10,4})$ for $j \le 6$ and there are following generators of the remaining $H^j(\widetilde{G}_{11,4})$.*

j	*gen.*	j	*gen.*
7	$\widetilde{w}_2^2\widetilde{w}_3, \widetilde{w}_3\widetilde{w}_4$	18	$a_{12}\widetilde{w}_2^3, a_{12}\widetilde{w}_2\widetilde{w}_4, a_{12}\widetilde{w}_3^2$
8	$\widetilde{w}_2^4, \widetilde{w}_2\widetilde{w}_3^2, \widetilde{w}_2^2\widetilde{w}_4$	19	$a_{12}\widetilde{w}_2^2\widetilde{w}_3, a_{12}\widetilde{w}_3\widetilde{w}_4$
9	$\widetilde{w}_2^3\widetilde{w}_3, \widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$	20	$a_{12}\widetilde{w}_2^4, a_{12}\widetilde{w}_2^2\widetilde{w}_4, a_{12}\widetilde{w}_2\widetilde{w}_3^2$
10	$\widetilde{w}_2^5, \widetilde{w}_2^2\widetilde{w}_3^2, \widetilde{w}_2^3\widetilde{w}_4$	21	$a_{12}\widetilde{w}_2^3\widetilde{w}_3, a_{12}\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$
11	$\widetilde{w}_2^4\widetilde{w}_3$	22	$a_{12}\widetilde{w}_2^5, a_{12}\widetilde{w}_2^3\widetilde{w}_4, a_{12}\widetilde{w}_2^2\widetilde{w}_3^2$
12	$a_{12}, \widetilde{w}_2^6, \widetilde{w}_2^3\widetilde{w}_3^2, \widetilde{w}_2^4\widetilde{w}_4$	23	$a_{12}\widetilde{w}_2^4\widetilde{w}_3$
13	$\widetilde{w}_2^5\widetilde{w}_3$	24	$a_{12}\widetilde{w}_2^6, a_{12}\widetilde{w}_2^2\widetilde{w}_4^2$
14	$a_{12}\widetilde{w}_2, \widetilde{w}_2^7$	25	$a_{12}\widetilde{w}_2^5\widetilde{w}_3$
15	$a_{12}\widetilde{w}_3$	26	$a_{12}\widetilde{w}_2^7$
16	$a_{16}, a_{12}\widetilde{w}_2^2, a_{12}\widetilde{w}_4, \widetilde{w}_2^8$	27	–
17	$a_{12}\widetilde{w}_2\widetilde{w}_3$	28	$a_{12}\widetilde{w}_2^8 = a_{16}\widetilde{w}_2^4\widetilde{w}_4$

where a_{12} is an element in $H^{12}(\widetilde{G}_{11,4}) \setminus C(12;11,4)$ such that $a_{12}\widetilde{w}_2^8 \neq 0$ and a_{16} is an element in $H^{16}(\widetilde{G}_{11,4}) \setminus C(16;11,4)$ such that $a_{16}\widetilde{w}_2^4\widetilde{w}_4 \neq 0$, $a_{16}\widetilde{w}_2^6 = 0$ and $a_{16}\widetilde{w}_2^3\widetilde{w}_3^2 = 0$.

Proof. We have $\operatorname{charrank}(\widetilde{\gamma}_{11,4}) = 11$, so $H^j(\widetilde{G}_{11,4}) = C(j;11,4)$ for $j \le 11$ and $C(j;11,4) \cong C(j;10,4) = H^j(\widetilde{G}_{10,4})$ for $j \le 6$, since neither $J_{11,4}$ or $J_{10,4}$ produce relations in cohomology in dimensions lower than seven.

In $C(7;11,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3$ and $\widetilde{w}_3\widetilde{w}_4$ as generators.

In $C(8;11,4)$ we have $\widetilde{w}_2^4 + \widetilde{w}_2\widetilde{w}_3^2 + \widetilde{w}_2^2\widetilde{w}_4 = \widetilde{w}_4^2$ as the only relation, hence there are the three generators.

In $C(9;11,4)$ we have $\widetilde{w}_3^3 = 0$ and $\widetilde{w}_2^3\widetilde{w}_3$, $\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$ are generators.

In $C(10;11,4)$ we have only $\widetilde{w}_2^5, \widetilde{w}_2^2\widetilde{w}_3^2, \widetilde{w}_2^3\widetilde{w}_4$ as generators, since $\widetilde{w}_3^2\widetilde{w}_4 = \widetilde{w}_2^2\widetilde{w}_3^2 + \widetilde{w}_2^3\widetilde{w}_4$ and $\widetilde{w}_2\widetilde{w}_4^2$ is the sum of all three generators, because $w_2g_8 \in J_{11,4}$.

In $C(11;11,4)$ we have $\widetilde{w}_2\widetilde{w}_3^3 = 0$ and $\widetilde{w}_2^2\widetilde{w}_3\widetilde{w}_4 = 0$, so $\widetilde{w}_2^4\widetilde{w}_3 = \widetilde{w}_3\widetilde{w}_4^2$ is the generator.

In $C(12;11,4)$ we have $\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4 = \widetilde{w}_2^6 + \widetilde{w}_2^2\widetilde{w}_4^2$, since $w_2g_{10} \in J_{11,4}$, with $\widetilde{w}_2^2\widetilde{w}_4^2 = \widetilde{w}_2^6 + \widetilde{w}_2^3\widetilde{w}_3^2 + \widetilde{w}_2^4\widetilde{w}_4$, since $w_2^2g_8 \in J_{11,4}$, so only $\widetilde{w}_2^6$, $\widetilde{w}_2^3\widetilde{w}_3^2$, $\widetilde{w}_2^4\widetilde{w}_4$ are generators.

In $C(13;11,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3^3 = 0$, $\widetilde{w}_2^3\widetilde{w}_3\widetilde{w}_4 = 0$ and $\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4^2 = \widetilde{w}_2^5\widetilde{w}_3$, since $w_2g_{11} \in J_{11,4}$, so $\widetilde{w}_2^5\widetilde{w}_3$ is the only generator.

In $C(14;11,4)$ we have $\widetilde{w}_2^2\widetilde{w}_3^2\widetilde{w}_4 = 0$. Thus also $\widetilde{w}_2^7 = \widetilde{w}_2^3\widetilde{w}_4^2$, since $w_2^2g_{10} \in J_{11,4}$. And we already know $\widetilde{w}_2^7 = \widetilde{w}_2^4\widetilde{w}_3^2$ and $\widetilde{w}_2^7 = \widetilde{w}_2^5\widetilde{w}_4$.

In $C(15;11,4)$ we have $\widetilde{w}_2^6\widetilde{w}_3 = \widetilde{w}_2^2\widetilde{w}_3\widetilde{w}_4^2$, since $w_2^2g_{11} \in J_{11,4}$, but the latter is zero. Also $\widetilde{w}_2^3\widetilde{w}_3^3 = 0$ and $\widetilde{w}_2^4\widetilde{w}_3\widetilde{w}_4 = 0$.

In $C(16;11,4)$ we have $\widetilde{w}_2^3\widetilde{w}_3^2\widetilde{w}_4 = 0$, $\widetilde{w}_2^5\widetilde{w}_3^2 = \widetilde{w}_2^4\widetilde{w}_4^2 = \widetilde{w}_2^6\widetilde{w}_4$ and $\widetilde{w}_2^8 = \widetilde{w}_2^5\widetilde{w}_3^2$ as it is a $\widetilde{w}_2$–multiple of a known equality.

From $\operatorname{charrank}(\widetilde{\gamma}_{11,4}) = 11$, Poincaré duality and Lemma 2.1 we have $C(j;11,4) = 0$ for $j \geq 17$.

Also, we have $\alpha_j(\widetilde{G}_{11,4}) = 0$ for $j \leq 11$ and $\alpha_{12}(\widetilde{G}_{11,4}) \geq 1$. Let us consider $N_5(G_{11,4}) = \{w_2w_3g_8, w_2^2g_9, w_4g_9, w_3g_{10}, w_2g_{11}\}$. We will show that $\{w_2w_3g_8, w_2^2g_9, w_4g_9, w_2g_{11}\}$ is a linearly independent subset and therefore $\alpha_{12}(\widetilde{G}_{11,4}) = 1$. Suppose that for some $c_i \in \mathbb{Z}_2$,$1 \leq i \leq 4$ we have

$$c_1w_2w_3g_8 + c_2w_2^2g_9 + c_3w_4g_9 + c_4w_2g_{11} = 0\,.$$

Considering that $w_4g_9 = w_3^3w_4$ is not divisible by w_2, we immediately see that $c_3 = 0$ and

$$c_1w_3g_8 + c_2w_2g_9 + c_4g_{11} = 0\,.$$

Since both g_8 and g_{11} have an even number of terms, the same is true for $c_1w_3g_8$ and c_4g_{11}, thus the parity of number of terms in LHS is the same as parity of number of terms in $c_2w_2g_9$. But $g_9 = w_3^3$, therefore $c_2 = 0$. Finally, we deduce $c_1 = c_4 = 0$ as well. In conclusion, there is one generator in $H^{12}(\widetilde{G}_{11,4})\setminus C(12;11,4)$, some a_{12}. By Poincaré duality we have $a_{12}\widetilde{w}_2^8 \neq 0$.

By Lemma 3.6 and Proposition 2.4 we have $\alpha_{13}(\widetilde{G}_{11,4}) = 0$.

We have $\chi(G_{11,4}) = 10$, so from a simple calculation we obtain $b_{14}(\widetilde{G}_{11,4}) = 2$. Hence $\alpha_{14}(\widetilde{G}_{11,4}) = 1$ and $a_{12}\widetilde{w}_2$ is the obvious generator.

To finish the proof, recall that $\widetilde{w}_2^8 = \widetilde{w}_2^5\widetilde{w}_3^2 = \widetilde{w}_2^4\widetilde{w}_4^2 = \widetilde{w}_2^6\widetilde{w}_4$ and the second term is equal to $\widetilde{w}_2\widetilde{w}_3^2\widetilde{w}_4^2$ by $w_2w_3g_{11} \in J_{11,4}$. Also $\widetilde{w}_3^3 = 0$, $\widetilde{w}_2^2\widetilde{w}_3\widetilde{w}_4 = 0$ and $\widetilde{w}_2\widetilde{w}_4^3 = 0$. So $a_{12}\widetilde{w}_3$ is nonzero. Next, $a_{12}\widetilde{w}_2^2$ and $a_{12}\widetilde{w}_4$ are nonzero and distinct, since their $\widetilde{w}_2^3\widetilde{w}_3^2$–multiples are not equal. But there is one more generator in $H^{16}(\widetilde{G}_{11,4}) \setminus C(16;11,4)$, some a_{16}.

Next, $a_{12}\widetilde{w}_2\widetilde{w}_3 \neq 0$. Nonzero elements $a_{12}\widetilde{w}_2^3$, $a_{12}\widetilde{w}_2\widetilde{w}_4$ and $a_{12}\widetilde{w}_3^2$ are found out to be independent after considering their multiples with $\widetilde{w}_2^5$, $\widetilde{w}_2^2\widetilde{w}_3^2$ and $\widetilde{w}_2^3\widetilde{w}_4$. By obvious adjustement of this argument triples $a_{12}\widetilde{w}_2^4$, $a_{12}\widetilde{w}_2^2\widetilde{w}_4$, $a_{12}\widetilde{w}_2\widetilde{w}_3^2$ and $a_{12}\widetilde{w}_2^5$, $a_{12}\widetilde{w}_2^3\widetilde{w}_4$, $a_{12}\widetilde{w}_2^2\widetilde{w}_3^2$ prove to be independent as well.

Nonzero elements $a_{12}\widetilde{w}_2^2\widetilde{w}_3$ and $a_{12}\widetilde{w}_3\widetilde{w}_4$ prove to be independent after considering their $\widetilde{w}_2^3\widetilde{w}_3$-multiples. Similarly, elements $a_{12}\widetilde{w}_2^3\widetilde{w}_3$ and $a_{12}\widetilde{w}_2\widetilde{w}_3\widetilde{w}_4$ are independent. Also $a_{12}\widetilde{w}_2^6$ and $a_{12}\widetilde{w}_2^2\widetilde{w}_4^2$, after considering $\widetilde{w}_4$–multiples. The rest is obvious.

Lastly, we will show that it is possible to choose a_{16} in such a way, that $a_{16}\widetilde{w}_2^4\widetilde{w}_4 \neq 0$, $a_{16}\widetilde{w}_2^6 = 0$ and $a_{16}\widetilde{w}_2^3\widetilde{w}_3^2 = 0$ simultaneously. Start with picking a'_{16} as any element, such that $(a'_{16}, a_{12}\widetilde{w}_2^2, a_{12}\widetilde{w}_4, \widetilde{w}_2^8)$ is a basis for $H^{16}(\widetilde{G}_{11,4})$. The matrix of the cup product bilinear pairing $H^{16}(\widetilde{G}_{11,4}) \times H^{12}(\widetilde{G}_{11,4}) \to \mathbb{Z}_2$ with respect to bases $(a'_{16}, a_{12}\widetilde{w}_2^2, a_{12}\widetilde{w}_4, \widetilde{w}_2^8)$ and $(a_{12}, \widetilde{w}_2^6, \widetilde{w}_2^3\widetilde{w}_3^2, \widetilde{w}_2^4\widetilde{w}_4)$ is

$$\begin{pmatrix} * & * & * & * \\ * & 1 & 1 & 1 \\ * & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where the stars represent unknown values. By Poincaré duality, the rows of this matrix are linearly independent, so there are following options for the last three values in the first row.

If we have $(*\ 1\ 0\ 0)$, then we define $a_{16} = a'_{16} + a_{12}\widetilde{w}_4$.

If we have $(*\ 0\ 0\ 1)$, then we define $a_{16} = a'_{16}$.

If we have $(*\ 1\ 1\ 0)$, then we define $a_{16} = a'_{16} + a_{12}\widetilde{w}_2^2$.

If we have $(*\ 0\ 1\ 1)$, then we define $a_{16} = a'_{16} + a_{12}\widetilde{w}_2^2 + a_{12}\widetilde{w}_4$. □

Now that we are done with the examples, we are ready to discuss some patterns. Similar to the case $k = 3$ studied in [1] we predict there will be indecomposable element a_{2^t} in $H^{2^t}(\widetilde{G}_{2^t+1,4})$ reflecting the case for $H^*(\widetilde{G}_{2^t,3})$.

It appears that for $2^t + 1 < n \leq 2^{t+1} - 4$ there are apart from Stiefel-Whitney classes $\widetilde{w}_2, \widetilde{w}_3, \widetilde{w}_4$ at least two aditional indecomposable elements $a_{4n-3\cdot 2^t-4} \in H^{4n-3\cdot 2^t-4}(\widetilde{G}_{n,4})$ and $a_{2^{t+1}-4} \in H^{2^{t+1}-4}(\widetilde{G}_{n,4})$. Note that previously mentioned a_{2^t} can be thought of as also being of the form $a_{4n-3\cdot 2^t-4}$ for $n = 2^t + 1$.

From observing that the Poincaré dual to those $a_{4n-3\cdot 2^{t-1}-4}$ in our examples for $n = 9, 10, 11$ was always of the form $\widetilde{w}_2^4\widetilde{w}_4$, we may reasonably anticipate these duals will exhibit some kind of stability in general.

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