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SEWN SPHERE COHOMOLOGIES FOR VERTEX ALGEBRAS

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Abstract. We define sewn elliptic cohomologies for vertex algebras by sewing procedure for coboundary operators.

1. Introduction

In [5] the author had introduced the notion of a cohomology of grading-restricted vertex algebras [3]. The main construction of coboundary operator was given using considerations of rational functions obtained as matrix elements for such vertex algebras [6, 1, 2]. As we know [8] from the theory of correlation functions for vertex algebras, matrix elements corresponds to the choice of formal parameters for vertex operators to be local coordinates on the complex sphere. On can consider more complicated situation when local coordinates for vertex operators are taken on a complex sphere sewn to itself [7]. This procedure would change coboundary operators and enrich the cohomological structure of corresponding vertex algebras.

In this paper we introduce the cohomology on sewn complex sphere and define coboundary operators by means of matrix elements and action of combinations of vertex and intertwining operators on a special class of maps associated to a grading-restricted vertex algebra.

Let us first recall the set up for self-sewing the complex sphere [7]. Consider the construction of a torus $\Sigma^{(1)}$ formed by self-sewing a handle to a Riemann sphere $\Sigma^{(0)}$. This is given by Yamada formalism [7], or so-called we refer to as the ρ-formalism. Let z_1, z_2 be local coordinates in the neighbourhood of two separated points p_1 and p_2 on the sphere. Consider two disks $|z_a| \le r_a$, for $r_a > 0$ and $a = 1, 2$. Note that r_1, r_2 must be sufficiently small to ensure that the disks do not intersect. Introduce a complex parameter ρ where $|\rho| \le r_1 r_2$ and excise the disks $\{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\} \subset \Sigma^{(0)}$, to form a twice-punctured sphere $\widehat{\Sigma}^{(0)} = \Sigma^{(0)} \backslash \bigcup_{a=1,2} \{z_a : |z_a| < |\rho| r_{\bar{a}}^{-1}\}$. We use the convention $\bar{1} = 2$, $\bar{2} = 1$. We define annular regions $\mathcal{A}_a \subset \widehat{\Sigma}^{(g)}$ with $\mathcal{A}_a = \{z_a : |\rho| r_{\bar{a}}^{-1} \le |z_a| \le r_a\}$ and identify them as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \rho\,, \tag{1.1}$$

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to form a torus $\Sigma^{(1)} = \widehat{\Sigma}^{(0)} \backslash \{\mathcal{A}_1 \cup \mathcal{A}_2\} \cup \mathcal{A}$. The sewing relation (1.1) can be considered to be a parameterization of a cylinder connecting the punctured Riemann surface to itself.

When one treats correlation functions for a vertex algebra V [1,2,8] on the torus obtained as a result of sewing a sphere to itself, one starts from matrix elements $\langle \mathbf{1}_V, Y(v_1, z_1) \ldots Y(v_1, z_n) \mathbf{1}_V \rangle$ where $v_1, \ldots, v_n \in V$, $z_1, \ldots, z_n$ on $\Sigma^{(0)}$ and pass to matrix elements $\sum\limits_{w \in W; k \geq 0} \rho^k \langle \bar{w}, Y_W(\bar{u}, \eta_1) Y_W(v_1, z_1) \ldots Y_W(v_1, z_n) Y_W(u, \eta_2) w \rangle$, reproducing the trace of product of vertex operators on the torus, where $\bar{w}$ is dual to w with respect to a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on W, $Y_W(v_1, z_1) \ldots Y_W(v_1, z_n)$ are vertex operators in a V-module W, and η_1, $\eta_2 \in \mathbb{C}$ are coordinates of points on the sphere where a handle is attached, and ρ as introduced above.

2. Functional formulation for matrix elements

Let us recall the functional formulation for matrix elements for a grading-restricted vertex algebra [5] (see Appendix 5). Let V be a grading-restricted vertex algebra and W a grading-restricted generalized V-module. Let $\overline{W}$ be the algebraic completion of W, that is,

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{(n)} = (W')^* \,.$$

A $\overline{W}$-valued rational function in $z_1, \ldots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$ is a map

$$\begin{aligned} f : F_n\mathbb{C} \quad &\rightarrow \quad \overline{W}, \\ (z_1, \ldots, z_n) \quad &\mapsto \quad f(z_1, \ldots, z_n), \end{aligned}$$

such that for any $w' \in W'$, matrix element $\langle w', f(z_1, \ldots, z_n) \rangle$ is a rational function in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$. Denote the space of all $\overline{W}$-valued rational functions in $z_1, \ldots, z_n$ by $\widetilde{W}_{z_1, \ldots, z_n}$. If a meromorphic function $f(z_1, \ldots, z_n)$ on a region in C^n can be analytically extended to a rational function in $z_1, \ldots, z_n$, we will use [5] $R(f(z_1, \ldots, z_n))$ to denote this rational function. For each $(z_1, \ldots, z_n, \zeta) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$, $w \in W$ and $w' \in W'$, we have an element

$$E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta) \mathbf{1}_V) \in \overline{W} \,, \tag{2.1}$$

given by

$$\begin{aligned} &\langle w', E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta) \mathbf{1}) \rangle \\ &\qquad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n) Y^W_{WV}(w, \zeta) \mathbf{1}_V \rangle) \,, \end{aligned}$$

where $Y^W_{WV}(w, \zeta)$ is the intertwining operator. It is a linear map

$$\begin{aligned} Y^W_{WV} \colon W \otimes V \quad &\rightarrow \quad W[[z, z^{-1}]] \,, \\ w \otimes v \quad &\mapsto \quad Y^W_{WV}(w, z) v \,, \end{aligned}$$

defined by

$$Y^W_{WV}(w, z) v = e^{zL(-1)} Y_W(v, -z) w \,,$$

for $v \in V$ and $w \in W$.

Let $\Phi\colon V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$, be a map composable [5] (see Appendix 7) with m vertex operators. We then define

$$\Phi(E^{(l_1)}_{V;\,\mathbf{1}_V} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}_V})\colon V^{\otimes m+n} \to \widetilde{W}_{z_1,\dots,z_{m+n}}\,,$$

by

$$\begin{aligned}\Phi(E^{(l_1)}_{V;\,\mathbf{1}_V} \otimes \cdots \otimes E^{(l_n)}_{V;\,\mathbf{1}_V})(v_1 \otimes \cdots \otimes v_{m+n-1})\\ = \Phi\big(E^{(l_1)}_{V;\mathbf{1}_V}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots\\ \otimes E^{(l_n)}_{V;\mathbf{1}_V}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n})\big)\,.\end{aligned}$$

Finally, for $\zeta \in \mathbb{C}$ we introduce the special action of an E-element of the form (2.1) on Φ by adding of intertwiner operators with formal parameters associated to coordinates of insertion of a handle to the sphere:

$$\begin{aligned}&E((v_1,z_1) \otimes \cdots \otimes (v_m,z_m); w,\zeta) \circ \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))(z_{m+1},\dots,z_{m+n})\\ &\quad = R(\langle \mathbf{1}_W, Y_W(v_1,z_1)\dots, Y_W(v_m,z_m) Y^W_{WV}(\Phi(v_{m+1}\\ &\qquad \otimes \cdots \otimes v_{m+n}))(z_{m+1},\dots,z_{m+n}) Y^W_{WV}(w,\zeta)\mathbf{1}_V, -\zeta)\mathbf{1}_V)\rangle)\,.\end{aligned} \tag{2.2}$$

This action provides passing from a matrix element to the trace on sewn sphere. Note that this action can be combined with (2.2) (see (3.1)). The action (2.2) allows to define the coboundary operators for bicomplexes constructed for grading-restricted vertex algebras. The idea is to use E-operators involved in [5] in order to define coboundary operators on the self-sewn sphere in terms of original matrix elements.

3. Sewn elliptic cohomology of a grading-restricted vertex algebra

In this section we define (in lines of [5]) the sewn elliptic cohomology associated to a grading-restricted vertex algebra. Let V be a grading-restricted vertex algebra and W a V-module. For fixed m, and $n \in \mathbb{Z}_+$, let $C^n_m(V,W)$ be the vector spaces of all linear maps from $V^{\otimes m} \to \widetilde{W}_{z_1,\dots,z_m}$ composable with m vertex operators, satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. Let $C^0_m(V,W) = W$. Then we have

$$C^n_m(V,W) \subset C^n_{m-1}(V,W)\,.$$

We then formulate

Proposition 1. *For η_1, $\eta_2 \in \mathbb{C}$ and arbitrary $\zeta_i \in \mathbb{C}$, $i = 1,\dots,n$ and let $u \in V$ be such that $\lim_{\eta_1 \to 0} Y^{\dagger}(u,\eta_1)\mathbf{1}_W = \bar{w}$. Then the operator*

$$\begin{aligned}\delta^n_m(\Phi) = \sum_{w \in W_{(k)};k\geq 0} &\rho^k \lim_{\eta_1 \to 0} [E((\bar{w},\eta_1) \otimes (v_1,z_1); w,\eta_2) \circ \Phi\\ &+ \sum_{i=1}^{n} (-1)^i E((\bar{w},\eta_1); w,\eta_2) \circ \Phi(E^{(2)}_{V;\mathbf{1}_V}(v_i, z_i - \zeta_i; v_{i+1}, z_{i+1} - \zeta_i))\\ &+ (-1)^{n+1} E((\bar{w},\eta_1) \otimes (v_{n+1},z_{n+1}); w,\eta_2) \circ \Phi\big]\,,\end{aligned} \tag{3.1}$$

for $\Phi \in C^n_m(V,W)$*, defines a coboundary operator*

$$\delta^n_m : C^l_m(V,W) \to C^{n+1}_{m-1}(V,W)\,.$$

(here we omit dependence on $(\eta_1,\,\eta_2)$*).*

In [**?**] dagger means the dual vertex operator with respect to the bilinear form on W. With coboundary operator (3.1) one defines the sewn sphere l-th cohomology $H^l_m(V,W)$ of V with coefficient in W and composable with m vertex operators to be

$$H^l_m(V,W) = \ker \delta^l_m / \mathrm{im}\ \delta^{l-1}_{m+1}\,.$$

Proof. In (3.1) $w \in W_{(n)}$, $\bar{w}$ is dual to w with respect to a non-degenerate non-vanishing bilinear form on W, and η_1, $\eta_2 \in \mathbb{C}$ are complex coordinates of two points on the sphere where cylinder is attached to form a torus. Let $v_1, \ldots, v_{n+1} \in V$, and $(z_1, \ldots, z_{n+1}) \in F_{n+1}\mathbb{C}$, and $\Phi \in C^n_m(V,W)$, and let us denote

$$\begin{aligned}
\Phi_{2,n+1} &= \Phi(v_2 \otimes \cdots \otimes v_{n+1})(z_2, \ldots, z_{n+1})\,,\\
\Phi_i &= \Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes (Y_V(v_i, z_i - \zeta_i) Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1}_V)\\
&\qquad \otimes v_{i+2} \otimes \cdots \otimes v_{n+1})(z_1, \ldots, z_{i-1}, \zeta_i, z_{i+2}, \ldots, z_{n+1})\,,\\
\Phi_{1,n} &= \Phi(v_1 \otimes \cdots \otimes v_n)(z_1, \ldots, z_n)\,.
\end{aligned} \tag{3.2}$$

We consider

$$\begin{aligned}
&\delta^n_m \Phi(v_1 \otimes \cdots \otimes v_{n+1})(z_1, \ldots, z_{n+1})\\
&\quad = \sum_{w \in W_{(k)};k\geq 0} \rho^k \lim_{\eta_1 \to 0} [R(\langle \mathbf{1}_W\,, Y_W(u, \eta_1) Y_W(v_1, z_1 - \eta_2)\\
&\qquad \times Y^W_{WV}(\Phi_{2,n+1} Y^W_{WV}(w, \eta_2)\mathbf{1}_V, -\eta_2)\mathbf{1}_V\rangle)\\
&\qquad + \sum_{i=1}^{n} (-1)^i R(\langle \mathbf{1}_W, Y^W_{WV}(\Phi_i Y^W_{WV}(w, \eta_2)\mathbf{1}_V, -\eta_2)\mathbf{1}_V)\rangle)\\
&\qquad + (-1)^{n+1} R(\langle \mathbf{1}_W, Y_W(\bar{u}_1, \eta_1) Y_W(v_{n+1}, z_{n+1} - \eta_2)\\
&\qquad \times Y^W_{WV}(\Phi_{1,n} Y^W_{Wv}(w, \eta_2)\mathbf{1}_V, -\eta_2)\mathbf{1}_V)\rangle)]\ .
\end{aligned}$$

Note that due to

$$Y_V(v_i, z_i - \zeta_i) Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1}_V = Y_V(v_i, z_i - z_{i+1}) v_{i+1},$$

in (3.2), the last expression is independent of ζ_i. When we take $\zeta_i = z_{i+1}$ for $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
\delta^n_m \Phi = &\sum_{w \in W_{(k)};k\geq 0} \rho^k \lim_{\eta_1 \to 0} R(\langle Y^\dagger_W(u, \eta_1)\mathbf{1}_W, [Y_W(v_1, z_1 - \eta_2) e^{-\eta_2 L(-1)} \Phi_{2,n+1} e^{\eta_2 L(-1)}\\
&+ \sum_{i=1}^{n} (-1)^i e^{-\eta_2 L(-1)} \Phi_i e^{\eta_2 L(-1)}\\
&+ (-1)^{n+1} Y_W(v_{n+1}, z_{n+1} - \eta_2) e^{-\eta_2 L(-1)} \Phi_{1,n} e^{\eta_2 L(-1)}] Y_W(\mathbf{1}_V, -\eta_2) w\rangle)\,.
\end{aligned}$$

By performing the summation for all $w \in W_{(k)}$ to obtain trace function over W. Using the $L(-1)$ property (6.1) of maps Φ we finally find

$$\delta_m^n \Phi(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{n+1}) = \sum_{k \geq 0} \rho^k \ \mathrm{Tr}_W [Y_W(v_1, z_1 - \eta_2) \Phi_{2,n+1}$$
$$+ \sum_{i=1}^{n} (-1)^i \Phi_i + (-1)^{n+1} Y_W(v_{n+1}, z_{n+1} - \eta_2) \Phi_{1,n}] .$$

By Proposition 3.10 of [5], $\delta_m^n(\Phi)$ is composable with $m-1$ vertex operators and has the $L(-1)$-derivative property and the $L(0)$-conjugation property. Thus $\delta_m^n(\Phi) \in \widehat{C}_{m-1}^{n+1}(V, W)$ and δ_m^n is a map with image in $\widehat{C}_{m-1}^{n+1}(V, W)$. $\square$

4. Conclusions

The notion of sewn elliptic cohomology for grading-restricted vertex algebras is devoted to enlarge the structure of cohomology of vertex algebras. Taking into account the above definitions and construction, we would like to develop a theory [4] of characteristic classes for vertex algebras.

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5. Appendix: Grading-restricted vertex algebras and modules

In this section, we recall [5] the definitions of grading-restricted vertex algebra and grading-restricted generalized module. The description is over the field $\mathbb{C}$ of complex numbers. A vertex algebra $(V, Y_V, \mathbf{1}_V)$, [6] consists of a $\mathbb{Z}$-graded complex vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)} ,$$

where $\dim V_{(n)} < \infty$ for each $n \in \mathbb{Z}$, a linear map

$$Y_V \colon V \to \mathrm{End}(V)[[z, z^{-1}]] ,$$

for a formal parameter z and a distinguished vector $\mathbf{1}_V$. For each $v \in V$, the image under the map Y_V is the vertex operator

$$Y_V(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} ,$$

with modes $(Y_V)_n = v(n) \in \mathrm{End}(V)$, where $Y_V(v, z)\mathbf{1} = v + O(z)$.

We recall here definitions introduced in [5]. A grading-restricted vertex algebra satisfies the following conditions:

(1) Grading-restriction condition: For $n \in \mathbb{Z}$, $\dim V_{(n)} < \infty$, and when n is sufficiently negative, $V_{(n)} = 0$.

(2) Lower-truncation condition for vertex operators: For $u, v \in V$, $Y_V(u, x)v$ contain only finitely many negative power terms, that is, $Y_V(u, x)v \in V((x))$ (the space of formal Laurent series in x with coefficients in V and with finitely many negative power terms).

(3) Identity property: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$.

(4) Creation property: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{x\to 0} Y_V(u, x)\mathbf{1} = u$.

(5) Duality: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$, the series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle \,,$$

$$\langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle \,,$$

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \,,$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

(6) $L_V(0)$-bracket formula: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, x)] = Y_V(L_V(0)v, x) + x\frac{d}{dx}Y_V(v, x)\,,$$

for $v \in V$.

(7) $L_V(-1)$-derivative property: Let $L_V(-1) : V \to V$ be the operator given by

$$L_V(-1)v = \mathrm{Res}_x x^{-2} Y_V(v, x)\mathbf{1} = Y_{(-2)}(v)\mathbf{1},$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx}Y_V(u, x) = Y_V(L_V(-1)u, x) = [L_V(-1), Y_V(u, x)]\,. \tag{5.1}$$

We denote $(v) = k$ for $v \in V_{(k)}$. One also defines a special operation $o(v) = v_{(wtv-1)}$. One also has

$$Y_V(\mathbf{1}_V, z) = 1, \quad \lim_{z\to 0} Y(u, z)\mathbf{1}_V = u\,.$$

Correspondingly, a grading-restricted generalized V-module is a vector space W equipped with a vertex operator map

$$\begin{aligned} Y_W\colon V\otimes W &\to W[[x, x^{-1}]]\,, \\ u\otimes w &\mapsto Y_W(u, x)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u)wx^{-n-1} \end{aligned}$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

(1) Grading-restriction condition: The vector space W is $\mathbb{C}$-graded, that is, $W = \coprod_{n\in\mathbb{C}} W_{(n)}$, such that $W_{(n)} = 0$ when the real part of n is sufficiently negative.

(2) Lower-truncation condition for vertex operators: For $u \in V$ and $w \in W$, $Y_W(u, x)w$ contain only finitely many negative power terms, that is, $Y_W(u, x)w \in W((x))$.

(3) Identity property: Let 1_W be the identity operator on W. Then $Y_W(\mathbf{1}, x) = 1_W$.

(4) Duality: For $u_1, u_2 \in V$, $w \in W$, $w' \in W' = \coprod_{n\in\mathbb{Z}} W^*_{(n)}$, the series

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle\,,$$
$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle\,,$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

(5) $L_W(0)$-bracket formula: For $v \in V$,

$$[L_W(0), Y_W(v, x)] = Y_W(L(0)v, x) + x\frac{d}{dx}Y_W(v, x)\,.$$

(6) $L_W(0)$-grading property: For $w \in W_{(n)}$, there exists $N \in \mathbb{Z}_+$ such that $(L_W(0) - n)^N w = 0$.

(7) $L_W(-1)$-derivative property: For $v \in V$,

$$\frac{d}{dx}Y_W(u, x) = Y_W(L_V(-1)u, x) = [L_W(-1), Y_W(u, x)]\,.$$

Note also the $L(-1)$-translation property [6] of vertex operators which we will make use later

$$Y_W(u, z) = e^{-\zeta L(-1)}Y_W(u, z + \zeta)e^{\zeta L(-1)}\,. \tag{5.2}$$

where $\zeta \in \mathbb{C}$.

6. Appendix: Properties of matrix elements for grading-restricted vertex algebra

Let us recall some facts about matrix elements for a grading-restricted vertex algebra [5]. For a function of $V^{\otimes n}$ inside the matrix element, $L(-1)$-derivative property means

$$\frac{\partial}{\partial z_i}\langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', (Y(v_1 \otimes \cdots \otimes v_{i-1} \otimes L_V(-1)v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle\,,$$

for $i = 1, \dots, n$, $v_1, \dots, v_n \in V$ and $w' \in W'$ and (ii)

$$\Big(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\Big)\langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', L_W(-1)(Y(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle\,,$$

and $v_1, \dots, v_n \in V$, $w' \in W'$. Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$.

One has [5] the following property. Let Y be a linear map having the $L(-1)$-derivative property. Then for $v_1, \dots, v_n \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1 + z, \dots, z_n + z) \in F_n\mathbb{C}$,

$$\begin{aligned}(6.1)\quad &\langle w', e^{zL_W(-1)}(Y(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle \\ &\qquad = \langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1 + z, \dots, z_n + z)\rangle ,\end{aligned}$$

and for $v_1, \dots, v_n \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ and $1 \le i \le n$ such that

$$(z_1, \dots, z_{i-1}, z_i + z, z_{i+1}, \dots, z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$(6.2)\qquad \langle w', (Y(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_{i-1}, z_i + z, z_{i+1}, \dots, z_n)\rangle ,$$

in z is equal to the power series

$$(6.3)\qquad \langle w', (Y(v_1 \otimes \cdots \otimes v_{i-1} \otimes e^{zL(-1)} v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle ,$$

in z. In particular, the power series (6.3) in z is absolutely convergent to (6.2) in the disk $|z| < \min_{i\neq j}\{|z_i - z_j|\}$.

7. Appendix: Definition of maps composable with vertex operators

Next we give a definition of a map composable [5] with vertex operators. For a V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ and $m \in \mathbb{C}$, let $P_m \colon \overline{W} \to W_{(m)}$ be the projection from $\overline{W}$ to $W_{(m)}$. Let $\Phi\colon V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ be a linear map. For $m \in \mathbb{N}$, Φ is said [5] to be composable with m vertex operators if the following conditions are satisfied:

(1) Let $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = m + n$, $v_1, \dots, v_{m+n} \in V$ and $w' \in W'$. Set

$$\begin{aligned}(7.1)\quad \Psi_i = (&E_V^{(l_i)}(v_{l_1+\cdots+l_{i-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{i-1}+l_i}; \mathbf{1})) \\ &\qquad (z_{l_1+\cdots+l_{i-1}+1} - \zeta_i, \dots, z_{l_1+\cdots+l_{i-1}+l_i} - \zeta_i)\end{aligned}$$

for $i = 1, \dots, n$. Then there exist positive integers $N(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that the series

$$\sum_{r_1,\dots,r_n\in\mathbb{Z}} \langle w', (\Phi(P_{r_1}\Psi_1 \otimes \cdots \otimes P_{r_n}\Psi_n))(\zeta_1, \dots, \zeta_n)\rangle ,$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|$$

for $i, j = 1, \dots, k$, $i \neq j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. and the sum can be analytically extended to a rational function in $z_1, \dots, z_{m+n}$, independent of $\zeta_1, \dots, \zeta_n$, with the only possible poles at $z_i = z_j$ of order less than or equal to $N(v_i, v_j)$ for $i, j = 1, \dots, k$, $i \neq j$.

(2) For $v_1, \ldots, v_{m+n} \in V$, there exist positive integers $N(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \ldots, k$, $i \neq j$ such that for $w' \in W'$,

$$\sum_{q\in\mathbb{C}} \langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m;$$
$$P_q((\Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))(z_{m+1}, \ldots, z_{m+n})))(z_1, \ldots, z_m)\rangle$$

is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_k| > 0$ for $i = 1, \ldots, m$ and $k = m+1, \ldots, m+n$ and the sum can be analytically extended to a rational function in $z_1, \ldots, z_{m+n}$ with the only possible poles at $z_i = z_j$ of orders less than or equal to $N(v_i, v_j)$ for $i, j = 1, \ldots, k$, $i \neq j$.

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