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OSCILLATION CRITERIA FOR FOURTH ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Marko Švec on the occasion of his 100th birthday anniversary

Abstract. Criteria for oscillatory behavior of solutions of fourth order half-linear differential equations of the form

(A) $$\big(|y''|^{\alpha}\operatorname{sgn} y''\big)'' + q(t)|y|^{\alpha}\operatorname{sgn} y = 0, \quad t \ge a > 0,$$

where $\alpha > 0$ is a constant and $q(t)$ is positive continuous function on $[a,\infty)$, are given in terms of an increasing continuously differentiable function $\omega(t)$ from $[a,\infty)$ to $(0,\infty)$ which satisfies $\int_a^{\infty} 1/(t\omega(t))\,dt < \infty$.

1. Introduction

Consider fourth order half-linear differential equations of the form

(A) $$\big(|y''|^{\alpha}\operatorname{sgn} y''\big)'' + q(t)|y|^{\alpha}\operatorname{sgn} y = 0\,, \quad t \ge a > 0\,,$$

where α is a positive constant and $q\colon [a,\infty) \to (0,\infty)$ is a continuous function.

By a *solution* we understand a function $y\colon [T_y,\infty) \to \mathbb{R}$ which is twice continuously differentiable together with $|y''|^{\alpha}\operatorname{sgn} y''$ and satisfies equation (A) for all sufficiently large t. We restrict our consideration to the so called proper solutions of (A), that is, solutions which do not vanish identically in some neighborhood of infinity. Such a solutions is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory.*

This note was motivated by the observation that while oscillation theory of fourth order nonlinear differential equations

(B) $$\big(|y''|^{\alpha}\operatorname{sgn} y''\big)'' + q(t)|y|^{\beta}\operatorname{sgn} y = 0\,, \quad t \ge a > 0\,,$$

where $\alpha \ne \beta$ and its generalization

(C) $$\big(p(t)|y''|^{\alpha}\operatorname{sgn} y''\big)'' + q(t)|y|^{\beta}\operatorname{sgn} y = 0\,, \quad t \ge a > 0\,,$$

has been well developed (see Kamo and Usami [1]–[2], Kusano et al. [4]–[5], Naito and Wu [8]–[9], Tanigawa [12]–[13] and Wu [14]–[15]), we are not aware of any

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oscillation criteria which would be applicable to the case $\alpha = \beta$, i.e. to the half-linear differential equation (A).

In contrast to a lack of effective oscillation criteria for the fourth-order half-linear differential equation (A), the qualitative theory for its particular case with $\alpha = 1$, that is for the linear differential equation

$$y^{(4)} + q(t)y = 0\,, \tag{D}$$

is relatively well-elaborated and since pioneering papers by Švec [10] and Leighton and Nehari [6] many articles have appeared dealing with oscillation, nonoscillation and other asymptotic properties of solutions of linear equation (D) and its generalization (see, for example, Swanson [11, Chapter 3], and the references therein).

It is known that conditions

$$\int_a^\infty t\Big(\int_t^\infty (s-t)q(s)\,ds\Big)^{\frac{1}{\alpha}}\,dt = \infty \tag{1.1}$$

and

$$\int_a^\infty t^{2\alpha+1}q(t)dt = \infty \tag{1.2}$$

are necessary for oscillation of all solutions of equation (A) (see [14]).

Our purpose here is to show that if they are replaced by the conditions

$$\int_a^\infty t^{1-\varepsilon}\Big(\int_t^\infty (s-t)q(s)\,ds\Big)^{\frac{1}{\alpha}}\,dt = \infty \tag{1.3}$$

and

$$\int_a^\infty t^{2\alpha+1-\varepsilon}q(t)\,dt = \infty, \tag{1.4}$$

for some $\varepsilon > 0$, respectively, then any of (1.3)) and (1.4) is also sufficient condition for oscillation of all solutions of equation (A).

2. Main result

To formulate and prove our results, we introduce notations:

$$L_1y(t) = y'(t), \quad L_iy(t) = \big(|y''(t)|^\alpha \operatorname{sgn} y''(t)\big)^{(i-2)}, \quad i = 2,3,4\,.$$

Without loss of generality we may restrict our consideration to eventually positive solutions of (A) since if $y(t)$ is a solution of (A) then so is $-y(t)$.

It is known that if $y(t)$ is one of such positive solutions, then there exists $t_0 \geq a$ such that $y(t)$ satisfies either

$$L_1y(t) > 0\,, \quad L_2y(t) > 0\,, \quad L_3y(t) > 0 \quad \text{for} \quad t \geq t_0\,, \tag{2.1}$$

or

$$L_1y(t) > 0\,, \quad L_2y(t) < 0\,, \quad L_3y(t) > 0 \quad \text{for} \quad t \geq t_0 \tag{2.2}$$

(see [14, Lemma 2.2]).

In what follows we will need the following elementary lemma.

Lemma 2.1. *Let $y(t)$ be a positive solution of* (A) *on* $[t_0, \infty)$.

(i) *If $y(t)$ satisfies* (2.1), *then*

$$(t - t_0) L_3 y(t) \le L_2 y(t)\,, \tag{2.3}$$

$$(t - t_0)\big[L_2 y(t)\big]^{\frac{1}{\alpha}} \le \frac{1+\alpha}{\alpha} L_1 y(t) \tag{2.4}$$

and

$$(t - t_0) L_1 y(t) \le \frac{1+2\alpha}{\alpha} y(t) \tag{2.5}$$

for $t \ge t_0$.

(ii) *If $y(t)$ satisfies* (2.2), *then*

$$(t - t_0) L_1 y(t) \le y(t) \tag{2.6}$$

on $[t_0, \infty)$.

Proof. (i) Let $y(t)$ be a positive solution of (A) satisfying (2.1) on $[T_0, \infty)$. Since $L_3 y(t)$ is decreasing for $t \ge t_0$, we have

$$(t - t_0) L_3 y(t) \le \int_{t_0}^{t} L_3 y(s)\, ds = \int_{t_0}^{t} \big(L_2 y(s)\big)' ds = L_2 y(t) - L_2 y(t_0) \le L_2 y(t)$$

for $t \ge t_0$, which shows that (2.3) is true. Next, we will prove that for a positive solution of (A) satisfying (2.1) the "usual" third derivative $y'''(t)$ exists and is continuous. Indeed, since $\big(L_3 y(t)\big)' < 0$ on $[t_0, \infty)$, one can integrate (A) from t to ∞, obtaining

$$L_3 y(t) = L_3 y(\infty) + \int_t^{\infty} q(s) y(s)^{\alpha}\, ds\,,$$

which, for simplicity, is rewritten as

$$L_3 y(t) = k + \eta(t)\,, \quad \text{where} \quad k = L_3 y(\infty) \ge 0\,, \quad \eta(t) = \int_t^{\infty} q(s) y(s)^{\alpha}\, ds\,. \tag{2.7}$$

Integrating the above on $[t_0, t]$ gives

$$L_2 y(t) = l + k(t - t_0) + \int_{t_0}^{t} \eta(s)\, ds\,, \quad t \ge t_0\,, \tag{2.8}$$

where $l = L_2 y(t_0) > 0$ is a constant. Equality (2.8) implies that

$$y''(t) = \Big[l + k(t - t_0) + \int_{t_0}^{t} \eta(s)\, ds\Big]^{\frac{1}{\alpha}}\,, \quad t \ge t_0\,. \tag{2.9}$$

It is clear that $y''(t)$ given by (2.9) is continuously differentiable, that is, $y'''(t)$ exists and is continuous on $[t_0, \infty)$.

Now, expressing (2.3) explicitly as $(t - t_0)\big(y''(t)^\alpha\big)' \leq y''(t)^\alpha$, we obtain

$$\alpha(t - t_0)y''(t)^{\alpha-1}y'''(t) \leq y''(t)^\alpha\,,$$

or, equivalently,

$$\alpha(t - t_0)y'''(t) \leq y''(t)\,, \quad t \geq t_0\,, \tag{2.10}$$

from which it readily follows that

$$\big((t - t_0)y''(t)\big)' \leq \frac{1+\alpha}{\alpha}y''(t)\,, \quad t \geq t_0\,.$$

Integrating the above on $[t_0, t]$ gives

$$(t - t_0)y''(t) \leq \frac{1+\alpha}{\alpha}\big(y'(t) - y'(t_0)\big) \leq \frac{1+\alpha}{\alpha}y'(t)\,, \quad t \geq t_0\,,$$

verifying the truth of (2.4). Finally, to prove (2.5) it suffices to rewrite (2.4) as

$$\big((t - t_0)y'(t)\big)' \leq \frac{1+2\alpha}{\alpha}y'(t)\,, \quad t \geq t_0\,,$$

and integrate this inequality from t_0 to t.

(ii) Let $y(t)$ be a positive solution of (A) satisfying (2.2) on $[t_0, \infty)$. The assumed negativity of $L_2y(t)$ means that $y'(t)$ is decreasing on $[t_0, \infty)$. Using this fact, we see that

$$(t - t_0)y'(t) \leq \int_{t_0}^{t} y'(s)\,ds = y(t) - y(t_0) \leq y(t)\,, \quad t \geq t_0\,,$$

which is the desired result (2.6). This completes the proof of Lemma 2.1. □

Remark. From the above lemma it follows that if (2.1) holds, then

$$L_2y(t) \leq \Big[\Big(\frac{1+\alpha}{\alpha}\Big)\Big(\frac{1+2\alpha}{\alpha}\Big)\Big]^\alpha \frac{y(t)^\alpha}{(t - t_0)^{2\alpha}}$$

and

$$L_3y(t) \leq \left[\left(\frac{1+\alpha}{\alpha}\right)\left(\frac{1+2\alpha}{\alpha}\right)\right]^\alpha \frac{y(t)^\alpha}{(t - t_0)^{1+2\alpha}}$$

for $t \geq t_0$.

The following theorem contains the main result of this paper.

Theorem 2.1. *All proper solutions of* (A) *are oscillatory if either*

$$\int_a^\infty tq(t)\,dt = \infty \tag{2.11}$$

or if

$$\int_a^\infty tq(t)\,dt < \infty \tag{2.12}$$

and there exists an increasing continuously differentiable function $\omega\colon [a,\infty) \to (0,\infty)$ *such that*

$$\int_a^\infty \frac{dt}{t\omega(t)} < \infty \tag{2.13}$$

and

$$\int_a^\infty \frac{t}{\omega(t)} \Big[\int_t^\infty (s-t)q(s)ds \Big]^{\frac{1}{\alpha}} dt = \infty\,. \tag{2.14}$$

Proof. Suppose that (A) possesses a nonoscillatory solution $y(t)$. We may assume without loss of generality that $y(t)$ is positive on some interval $[t_0,\infty), t_0 \geq a$. It is known that it satisfies either (2.1) or (2.2).

We begin with the case where $y(t)$ satisfies (2.1) on $[t_0,\infty)$. Let $t_1 > t_0$ be fixed. Multiplying (A) by $t/(y(t)^\alpha)$ and integrating it by parts on $[t,\tau]$, $t \geq t_1$, leads to

$$w(\tau) + \alpha \int_t^\tau w(s)\frac{y'(s)}{y(s)}ds + \int_t^\tau sq(s)ds = w(t) + \int_t^\tau \frac{L_3 y(s)}{y(s)^\alpha}\,ds\,, \tag{2.15}$$

where $w(t) = tL_3y(t)/y(t)^\alpha$. This implies that

$$\int_t^\tau sq(s)\,ds \leq w(t) + \int_t^\tau \frac{L_3y(s)}{y(s)^\alpha}\,ds\,. \tag{2.16}$$

Now, rewrite $w(t)$ as

$$w(t) = \frac{v(t)+L_2y(t)}{y(t)^\alpha} \quad \text{where} \quad v(t) = tL_3y(t) - L_2y(t)\,. \tag{2.17}$$

Since $v'(t) \leq 0$ for $t \geq t_0$, $v(t)$ is decreasing on $[t_0,\infty)$. Using this fact and the increasing nature of $L_2y(t)$, we find from (2.17) that there exists a constant $c_1 > 0$ such that

$$w(t) \leq c_1 \frac{L_2y(t)}{y(t)^\alpha}\,, \quad t \geq t_1\,. \tag{2.18}$$

From (2.16) and (2.18) it follows that

$$\int_t^\tau sq(s)ds \leq c_1\frac{L_2y(t)}{y(t)^\alpha} + \int_t^\tau \frac{L_3y(s)}{y(s)^\alpha}\,ds\,, \quad t \geq t_1\,.$$

This, combined with the inequalities

$$\frac{L_2y(t)}{y(t)^\alpha} \leq \frac{k(\alpha)}{(t-t_0)^{2\alpha}}\,, \quad \frac{L_3y(t)}{y(t)^\alpha} \leq \frac{l(\alpha)}{(t-t_0)^{2\alpha+1}}\,, \quad t \geq t_1\,,$$

holding for some positive constants $k(\alpha)$ and $l(\alpha)$ (cf. remark of Lemma 2.1), shows that

$$\begin{aligned} \int_t^\tau sq(s)\,ds &\leq \frac{c_2}{(t-t_0)^{2\alpha}} + c_3\int_t^\tau \frac{ds}{(s-t_0)^{2\alpha+1}} \\ &\leq \frac{c_4}{(t-t_0)^{2\alpha}}\,, \quad \tau \geq t \geq t_1\,, \end{aligned} \tag{2.19}$$

for some positive constants c_2, c_3 and c_4 independent of $[t,\tau]$. Letting $\tau \to \infty$ in (2.19) we get

$$\int_t^\infty sq(s)\,ds \le \frac{c_4}{(t-t_0)^{2\alpha}}\,, \quad t \ge t_1\,, \tag{2.20}$$

where c_2, c_3 and c_4 are positive constants. Using (2.20) and (2.13), we obtain

$$\int_{t_1}^\infty \frac{t}{\omega(t)}\Big[\int_t^\infty sq(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt \le c_4 \int_{t_1}^\infty \frac{t}{\omega(t)(t-t_0)^2}\,dt \le c_5 \int_{t_1}^\infty \frac{dt}{t\omega(t)}\,.$$

for some constant $c_5 > 0$, which clearly implies that

$$\int_a^\infty \frac{t}{\omega(t)}\Big[\int_t^\infty sq(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt < \infty\,. \tag{2.21}$$

Suppose now that (A) has a nonoscillatory solution $y(t)$ which satisfies (2.2). Note that $L_2y(t) < 0$, $L_3y(t) > 0$ and $L_2y(\infty) = L_3y(\infty) = 0$ (see the proof of Lemma K in [13]). Integrating (A) twice from t to ∞, we find

$$-L_2y(t) = \int_t^\infty (s-t)q(s)y(s)^\alpha\,ds \ge y(t)^\alpha \int_t^\infty (s-t)q(s)\,ds\,, \quad t \ge t_1,$$

or

$$-\frac{y''(t)}{y(t)} \ge \Big[\int_t^\infty (s-t)q(s)\,ds\Big]^{\frac{1}{\alpha}}\,, \quad t \ge t_1\,. \tag{2.22}$$

Integrating (2.22) multiplied by $\frac{t}{\omega(t)}$ over $[t_1,t]$ gives

$$\begin{aligned}\int_{t_1}^t \frac{s}{\omega(s)}\Big[\int_s^\infty (r-s)q(r)\,dr\Big]^{\frac{1}{\alpha}}\,ds &\le -u(t)+u(t_1)\\ &\quad - \int_{t_1}^t u(s)\Big[\frac{y'(s)}{y(s)}+\frac{\omega'(s)}{\omega(s)}\Big]^{\frac{1}{\alpha}}\,ds + \int_{t_1}^t \frac{y'(s)}{\omega(s)y(s)}\,ds,\end{aligned} \tag{2.23}$$

for $t \ge t_1$, where $u(t) = ty''(t)/(\omega(t)y(t))$.

Since $y(t) > 0, y'(t) > 0, u(t) \ge 0$ and $\omega'(t) \ge 0$ and since $(t-t_0)y'(t) \le y(t)$ by (ii) of Lemma 2.1, it follows from (2.23) that

$$\begin{aligned}\int_{t_1}^t \frac{s}{\omega(s)}\Big[\int_s^\infty (r-s)q(r)\,dr\Big]^{\frac{1}{\alpha}}\,ds &\le u(t_1) + \int_{t_1}^t \frac{y'(s)}{\omega(s)y(s)}\,ds\\ &\le u(t_1) + \int_{t_1}^t \frac{ds}{(s-t_0)\omega(s)}\end{aligned}$$

for $t \ge t_1$. Letting $t \to \infty$ in the above, we conclude that

$$\int_{t_1}^\infty \frac{s}{\omega(s)}\Big[\int_s^\infty (r-s)q(r)\,dr\Big]^{\frac{1}{\alpha}}\,ds < \infty\,. \tag{2.24}$$

It is clear that in (2.24) t_1 can be replaced with a.

From what is analyzed above it is concluded first that all solutions of (A) are oscillatory if (2.11) holds, and secondly that if (2.11) fails to hold, then the two conditions

$$\int_a^\infty \frac{t}{\omega(t)}\Big[\int_t^\infty sq(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt=\infty\,, \tag{2.25}$$

and

$$\int_a^\infty \frac{t}{\omega(t)}\Big[\int_t^\infty (s-t)q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt=\infty\,, \tag{2.26}$$

with $\omega(t)$ satisfying (2.13) make all solutions of (A) oscillatory. But (2.25) is redundant since

$$\int_t^\infty (s-t)q(s)ds \le \int_t^\infty sq(s)\,ds, \quad t\ge a\,.$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *If either* (2.11) *holds or* $\int_a^\infty tq(t)\,dt<\infty$ *and for some* $\varepsilon>0$

$$\int_a^\infty t^{1-\varepsilon}\Big[\int_t^\infty (s-t)q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt=\infty\,, \tag{2.27}$$

then all proper solutions of (A) *are oscillatory.*

Proof. In Theorem 2.1, take $\omega(t)=t^\varepsilon$, $t\ge a>0$. Then (2.13) is clearly satisfied and condition (2.14) reduces to (2.27). □

Corollary 2.2. *If* $\int_a^\infty tq(t)\,dt<\infty$ *and there exists an* $\varepsilon>1$ *such that*

$$\int_a^\infty \frac{t}{(\log t)^\varepsilon}\Big[\int_t^\infty (s-t)q(s)\,ds\Big]^{\frac{1}{\alpha}}\,dt=\infty\,, \tag{2.28}$$

then all proper solutions of (A) *are oscillatory.*

From Theorem 2.1 we know that if $\int_a^\infty tq(t)\,dt=\infty$, then all proper solutions are oscillatory. Therefore, in what follows we will assume that this condition fails to hold and find another criterion (different from (2.14)) for (A) to have only oscillatory solutions.

Theorem 2.2. *Let* (2.12) *be satisfied and assume that there exists an increasing continuously differentiable function* $\omega\colon [a,\infty)\to(0,\infty)$ *such that* (2.13) *and*

$$\int_a^\infty \frac{t^{2\alpha+1}}{\omega(t)}q(t)\,dt=\infty\,. \tag{2.29}$$

hold. Then all proper solutions of (A) *are oscillatory.*

Proof. Suppose that (A) has a nonoscillatory solution $y(t)$. We may assume that $y(t)$ is positive on $[t_0,\infty)$, $t_0\ge a$.

Let $t_1 > t_0$ be fixed. Multiplying (A) by $t^{2\alpha+1}/(\omega(t)y(t)^\alpha)$ and integrating it by parts on $[t_1, t]$, yields

$$w(t) + \int_{t_1}^{t} w(s)\Big[\frac{\omega'(s)}{\omega(s)} + \alpha\frac{y'(s)}{y(s)}\Big]\,ds + \int_{t_1}^{t} \frac{s^{2\alpha+1}}{\omega(s)}q(s)\,ds$$
$$= w(t_1) + (2\alpha+1)\int_{t_1}^{t} \frac{s^{2\alpha}L_3y(s)}{\omega(s)y(s)^\alpha}\,ds, \quad t \ge t_1,$$

where $w(t) = t^{2\alpha+1}L_3y(t)/(\omega(t)y(t)^\alpha)$. Thus,

$$\int_{t_1}^{t} \frac{s^{2\alpha+1}}{\omega(s)}q(s)\,ds \le w(t_1) + (2\alpha+1)\int_{t_1}^{t} \frac{s^{2\alpha}L_3y(s)}{\omega(s)y(s)^\alpha}\,ds\,, \quad t \ge t_1\,. \tag{2.30}$$

From Remark after Lemma 2.1 we know that if $y(t)$ satisfies (2.1), then there exist a positive constant c_1 and a $t_2 \ge t_1$ such that

$$y(t)^\alpha \ge c_1 t^{2\alpha+1}L_3y(t) \quad \text{for} \quad t \ge t_2\,. \tag{2.31}$$

But from [13, Lemma 2.1] we know that the same is true also for solutions satisfying (2.2) for all large t.

Substituting (2.31) into (2.30) and letting $t \to \infty$, we get

$$\int_{t_1}^{\infty} \frac{s^{2\alpha+1}}{\omega(s)}q(s)\,ds \le w(t_1) + c_1(2\alpha+1)\int_{t_1}^{\infty} \frac{1}{s\omega(s)}\,ds < \infty\,,$$

which contradicts (2.29) and completes the proof. □

Corollary 2.3. *If $\int_a^\infty tq(t)dt < \infty$ and for some $\varepsilon > 0$*

$$\int_a^\infty t^{2\alpha+1-\varepsilon}q(t)\,dt = \infty\,, \tag{2.32}$$

then all proper solutions of (A) *are oscillatory.*

Remark. In the linear case (i.e., if $\alpha = 1$), choice $\varepsilon = 1$ in (2.32) leads to the Leighton-Nehari oscillation criterion

$$\int_a^\infty t^2 q(t)\,dt = \infty \tag{2.33}$$

(see [6, Theorem 11.4]).

Example. Consider the fourth order half-linear differential equation

$$\big(|y''|^\alpha \mathrm{sgn}\; y''\big)'' + kt^\sigma\varphi(t)|y|^\alpha \mathrm{sgn}\; y = 0\,, \tag{2.34}$$

where $k > 0$ and σ are constants and $\varphi(t)$ is a slowly varying function defined on $[a,\infty), a > 0$. (For the definition and basic properties of slowly varying functions the reader is referred to Marić [7, Appendix].) This is the special case of (A) with $q(t) = kt^\sigma\varphi(t)$. We want to apply Theorem 2.1 to this equation. To get the asymptotic relations which appear below, Karamata integration theorem (see [7]) will be used repeatedly.

(i) Let $\sigma > -2$. Then Eq. (2.34) is oscillatory for all $\varphi(t) \in \mathrm{SV}$ by the first part of Theorem 2.1 since

$$\int_a^t sq(s)\,ds = k\int_a^t s^{\sigma+1}\varphi(s)\,ds \sim \frac{kt^{\sigma+2}\varphi(t)}{\sigma+2} \to \infty\,, \quad t \to \infty\,.$$

(ii) Let $\sigma = -2$. The function $\varphi(t)$ satisfies either

$$\int_a^\infty t^{-1}\varphi(t)\,dt = \infty \tag{2.35}$$

or

$$\int_a^\infty t^{-1}\varphi(t)\,dt < \infty\,. \tag{2.36}$$

(a) If (2.35) holds, then since $q(t)$ satisfies

$$\int_a^\infty tq(t)\,dt = k\int_a^\infty t^{-1}\varphi(t)\,dt = \infty\,,$$

Eq. (2.34) is oscillatory by the first part of Theorem 2.1.

(b) Suppose that (2.36) holds. It is clear that

$$\int_a^\infty tq(t)\,dt = k\int_a^\infty t^{-1}\varphi(t)\,dt < \infty\,.$$

On the other hand, it is easy to see that

$$\int_t^\infty q(s)\,ds \sim kt^{-1}\varphi(t)\,,$$
$$\int_t^\infty (s-t)q(s)\,ds \sim k\int_t^\infty s^{-1}\varphi(s)\,ds\,, \quad t \to \infty\,.$$

Put $\Phi(t) = \int_t^\infty s^{-1}\varphi(s)ds$. Clearly $\Phi(t) \in \mathrm{SV}$. Choose $\omega(t) = t^\delta$ with $0 < \delta \le 1$ and compute

$$\int_a^t \frac{s}{\omega(s)}\Big[\int_s^\infty (t-s)q(r)dr\Big]^{\frac{1}{\alpha}}\,ds \sim k^{\frac{1}{\alpha}}\int_a^t s^{1-\delta}\Phi(s)^{\frac{1}{\alpha}}\,ds$$
$$\sim k^{\frac{1}{\alpha}}\frac{t^{2-\delta}\Phi(t)^{\frac{1}{\alpha}}}{2-\delta} \to \infty\,, \quad t \to \infty\,.$$

This result makes it possible to apply the second part of Theorem 2.1 to conclude that Eq. (2.34) is oscillatory if $\sigma = -2$ and (2.36) is satisfied.

(iii) Finally, let $\sigma < -2$. In this case it holds that $\int_a^\infty tq(t)dt < \infty$. Furthermore, since

$$\int_t^\infty q(s)\,ds \sim k\frac{t^{\sigma+1}\varphi(t)}{-(\sigma+1)}\,,$$
$$\int_t^\infty (s-t)q(s)\,ds \sim k\frac{t^{\sigma+2}\varphi(t)}{(\sigma+1)(\sigma+2)}\,, \quad t \to \infty\,,$$

we see that, as $t \to \infty$,

$$\frac{t}{\omega(t)}\Big[k\int_t^\infty (s-t)q(s)\,ds\Big]^{\frac{1}{\alpha}} \sim \frac{t}{\omega(t)}\Big[\frac{t^{\sigma+2}\varphi(t)}{(\sigma+1)(\sigma+2)}\Big]^{\frac{1}{\alpha}}$$
$$\sim k^{\frac{1}{\alpha}}\frac{t^{1+\frac{\sigma+2}{\alpha}}\varphi(t)^{\frac{1}{\alpha}}}{\omega(t)\big((\sigma+1)(\sigma+2)\big)^{\frac{1}{\alpha}}} \tag{2.37}$$

for $t \to \infty$. We limit our attention to those σ satisfying $-2-2\alpha < \sigma < -2$ and define the function $\omega(t)$ by

$$\omega(t) = t^{\varepsilon}\,, \quad \text{where} \quad 0 < \varepsilon < \frac{\sigma+2+2\alpha}{\alpha}\,.$$

Integrating (2.37) with this choice of $\omega(t)$, we obtain

$$\int_a^t \frac{s}{\omega(s)}\Big[\int_s^\infty (r-s)q(r)\,dr\Big]^{\frac{1}{\alpha}} \sim k^{\frac{1}{\alpha}}\int_a^t \frac{s^{1+\frac{\sigma+2}{\alpha}-\varepsilon}\varphi(s)^{\frac{1}{\alpha}}}{\big((\sigma+1)(\sigma+2)\big)^{\frac{1}{\alpha}}}\,ds$$
$$\sim k^{\frac{1}{\alpha}}\frac{t^{2+\frac{\sigma+2}{\alpha}-\varepsilon}\varphi(t)^{\frac{1}{\alpha}}}{\big((\sigma+1)(\sigma+2)\big)^{\frac{1}{\alpha}}} \tag{2.38}$$

as $t \to \infty$. Note that ε is chosen so that $2+\frac{\sigma+2}{\alpha}-\varepsilon > 0$, which implies that the integral in (2.38) tends to ∞ as $t \to \infty$ for all $\varphi(t)$. Thus it is concluded that in this case Eq. (2.36) is oscillatory for all $\varphi(t) \in \mathrm{SV}$.

Summarizing the above computations, we have that if $\sigma > -2-2\alpha$, then equation (2.34) is oscillatory for any slowly varying function $\varphi(t)$.

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