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ON RIEMANN-POISSON LIE GROUPS

Brahim Alioune[a], Mohamed Boucetta[b], and Ahmed Sid'Ahmed Lessiad[c]

Abstract. A Riemann-Poisson Lie group is a Lie group endowed with a left invariant Riemannian metric and a left invariant Poisson tensor which are compatible in the sense introduced in [4]. We study these Lie groups and we give a characterization of their Lie algebras. We give also a way of building these Lie algebras and we give the list of such Lie algebras up to dimension 5.

1. Introduction

In this paper, we study Lie groups endowed with a left invariant Riemannian metric and a left invariant Poisson tensor satisfying a compatibility condition to be defined below. They constitute a subclass of the class of *Riemann-Poisson manifolds* introduced and studied by the second author (see [2, 3, 4, 5]).

Let $(M,\pi,\langle\ ,\ \rangle)$ be smooth manifold endowed with a Poisson tensor π and a Riemannian metric $\langle\ ,\ \rangle$. We denote by $\langle\ ,\ \rangle^*$ the Euclidean product on T^*M naturally associated to $\langle\ ,\ \rangle$. The Poisson tensor defines a Lie algebroid structure on T^*M where the anchor map is the contraction $\#_\pi\colon T^*M\longrightarrow TM$ given by $\prec\beta,\#_\pi(\alpha)\succ=\pi(\alpha,\beta)$ and the Lie bracket on $\Omega^1(M)$ is the Koszul bracket given by

$$[\alpha,\beta]_\pi=\mathcal{L}_{\#_\pi(\alpha)}\beta-\mathcal{L}_{\#_\pi(\beta)}\alpha-d\pi(\alpha,\beta),\quad \alpha,\beta\in\Omega^1(M)\,. \tag{1}$$

This Lie algebroid structure and the metric $\langle\ ,\ \rangle^*$ define a contravariant connection $\mathcal{D}\colon\Omega^1(M)\times\Omega^1(M)\longrightarrow\Omega^1(M)$ by Koszul formula

$$\begin{aligned}2\langle\mathcal{D}_\alpha\beta,\gamma\rangle^*&=\#_\pi(\alpha).\langle\beta,\gamma\rangle^*+\#_\pi(\beta).\langle\alpha,\gamma\rangle^*-\#_\pi(\gamma).\langle\alpha,\beta\rangle^*\\&\quad+\langle[\alpha,\beta]_\pi,\gamma\rangle^*+\langle[\gamma,\alpha]_\pi,\beta\rangle^*+\langle[\gamma,\beta]_\pi,\alpha\rangle^*,\quad\alpha,\beta,\gamma\in\Omega^1(M)\,.\end{aligned} \tag{2}$$

This is the unique torsionless contravariant connection which is metric, i.e., for any $\alpha,\ \beta,\ \gamma\in\Omega^1(M)$,

$$\mathcal{D}_\alpha\beta-\mathcal{D}_\beta\alpha=[\alpha,\beta]_\pi\quad\text{and}\quad\#_\pi(\alpha).\langle\beta,\gamma\rangle^*=\langle\mathcal{D}_\alpha\beta,\gamma\rangle^*+\langle\beta,\mathcal{D}_\alpha\gamma\rangle^*\,.$$

The notion of contravariant connection was introduced by Vaisman in [13] and studied in more details by Fernandes in the context of Lie algebroids [8]. The

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connection $\mathcal{D}$ defined above is called *contravariant Levi-Civita connection* associated to the couple $(\pi, \langle\ ,\ \rangle)$ and it appeared first in [2].

The triple $(M, \pi, \langle\ ,\ \rangle)$ is called a *Riemannian-Poisson manifold* if $\mathcal{D}\pi = 0$, i.e., for any $\alpha, \beta, \gamma \in \Omega^1(M)$,

$$\mathcal{D}\pi(\alpha, \beta, \gamma) := \#_\pi(\alpha).\pi(\beta, \gamma) - \pi(\mathcal{D}_\alpha\beta, \gamma) + \pi(\beta, \mathcal{D}_\alpha\gamma) = 0\,. \tag{3}$$

This notion was introduced by the second author in [2]. Riemann-Poisson manifolds turned out to have interesting geometric properties (see[2, 3, 4, 5]). Let's mention some of them.

(1) The condition of compatibility (3) is weaker than the condition $\nabla\pi = 0$ where ∇ is the Levi-Civita connection of $\langle\ ,\ \rangle$. Indeed, the condition (3) allows the Poisson tensor to have a variable rank. For instance, linear Poisson structures which are Riemann-Poisson exist and were characterized in [5]. Furthermore, let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and $(X_1, \ldots, X_r)$ a family of commuting Killing vector fields. Put

$$\pi = \sum_{i,j} X_i \wedge X_j\,.$$

Then $(M, \pi, \langle\ ,\ \rangle)$ is a Riemann-Poisson manifold. This example illustrates also the weakness of the condition (3) and, more importantly, it is the local model of the geometry of noncommutative deformations studied by Hawkins (see [10, Theorem 6.6]).

(2) Riemann-Poisson manifolds can be thought of as a generalization of Kähler manifolds. Indeed, let $(M, \pi, \langle\ ,\ \rangle)$ be a Poisson manifold endowed with a Riemannian metric such that π is invertible. Denote by ω the symplectic form inverse of π. Then $(M, \pi, \langle\ ,\ \rangle)$ is Riemann-Poisson manifold if and only if $\nabla\omega = 0$ where ∇ is the Levi-Civita connection of $\langle\ ,\ \rangle$. In this case, if we define $A\colon TM \longrightarrow TM$ by $\omega(u, v) = \langle Au, v\rangle$ then $-A^2$ is symmetric definite positive and hence there exists a unique $Q\colon TM \longrightarrow TM$ symmetric definite positive such that $Q^2 = -A^2$. It follows that $J = AQ^{-1}$ satisfies $J^2 = -\mathrm{Id}_{TM}$, skew-symmetric with respect $\langle\ ,\ \rangle$ and $\nabla J = 0$. Hence $(M, J, \langle\ ,\ \rangle)$ is a Kähler manifold and its Kähler form $\omega_J(u, v) = \langle Ju, v\rangle$ is related to ω by the following formula:

$$\omega(u, v) = -\omega_J\Big(\sqrt{-A^2}u, v\Big)\,, \quad u, v \in TM\,. \tag{4}$$

Having this construction in mind, we will call in this paper a Kähler manifold a triple $(M, \langle\ ,\ \rangle, \omega)$ where $\langle\ ,\ \rangle$ is a Riemannian metric and ω is a nondegenerate 2-form ω such that $\nabla\omega = 0$ where ∇ is the Levi-Civita connection of $\langle\ ,\ \rangle$.

(3) The symplectic foliation of a Riemann-Poisson manifold when π has a constant rank has an important property namely it is both a Riemannian foliation and a Kähler foliation.

Recall that a Riemannian foliation is a foliated manifold $(M, \mathcal{F})$ with a Riemannian metric $\langle\ ,\ \rangle$ such that the orthogonal distribution $T^\perp\mathcal{F}$ is totally geodesic.

Kähler foliations are a generalization of Kähler manifolds (see [6]) and, as for the notion of Kähler manifold, we call in this paper a Kähler foliation a foliated manifold $(M,\mathcal{F})$ endowed with a leafwise metric $\langle\ ,\ \rangle_{\mathcal{F}}\in\Gamma(\otimes^2T^*\mathcal{F})$ and a nondegenerate leafwise differential 2-form $\omega_{\mathcal{F}}\in\Gamma(\otimes^2T^*\mathcal{F})$ such any leaf with the restrictions of $\langle\ ,\ \rangle_{\mathcal{F}}$ and $\omega_{\mathcal{F}}$ is a Kähler manifold.

Theorem 1.1 ([4])**.** *Let $(M,\langle\ ,\ \rangle,\pi)$ be a Riemann-Poisson manifold with π of constant rank. Then its symplectic foliation is both a Riemannian and a Kähler foliation.*

Having in mind these properties particularly Theorem 1.1, it will be interesting to find large classes of examples of Riemann-Poisson manifolds. This paper will describe the rich collection of examples which are obtained by providing an arbitrary Lie group G with a Riemannian metric $\langle\ ,\ \rangle$ and a Poisson tensor π invariant under left translations and such that $(G,\langle\ ,\ \rangle,\pi)$ is Riemann-Poisson. We call $(G,\langle\ ,\ \rangle,\pi)$ a *Riemann-Poisson Lie group.* This class of examples can be enlarged substantially, with no extra work, as follows. If $(G,\langle\ ,\ \rangle,\pi)$ is a Riemann-Poisson Lie group and Γ is any discrete subgroup of G then $\Gamma\backslash G$ carries naturally a structure of Riemann-Poisson manifold.

The paper is organized as follows. In Section 2, we give the material needed in the paper and we describe the infinitesimal counterpart of Riemann-Poisson Lie groups, namely, Riemann-Poisson Lie algebras. In Section 3, we prove our main result which gives an useful description of Riemann-Poisson Lie algebras (see Theorem 3.1). We use this theorem in Section 4 to derive a method for building Riemann-Poisson Lie algebras. We explicit this method by giving the list of Riemann-Poisson Lie algebras up to dimension 5.

2. Riemann-Poisson Lie groups and their infinitesimal characterization

Let G be a Lie group and $(\mathfrak{g}=T_eG,[\ ,\ ])$ its Lie algebra.

(1) A left invariant Poisson tensor π on G is entirely determined by

$$\pi(\alpha,\beta)(a)=r(\mathrm{L}_a^*\alpha,\mathrm{L}_a^*\beta)\,,$$

where $a\in G$, α, $\beta\in T_a^*G$, L_a is the left multiplication by a and $r\in\wedge^2\mathfrak{g}$ satisfies the classical Yang-Baxter equation

$$[r,r]=0\,,\tag{5}$$

where $[r,r]\in\wedge^3\mathfrak{g}$ is given by

$$\begin{aligned}[r,r](\alpha,\beta,\gamma):=&\prec\alpha,[r_\#(\beta),r_\#(\gamma)]\succ+\prec\beta,[r_\#(\gamma),r_\#(\alpha)]\succ\\&+\prec\gamma,[r_\#(\alpha),r_\#(\beta)]\succ,\quad\alpha,\beta,\gamma\in\mathfrak{g}^*,\end{aligned}\tag{6}$$

and $r_\#\colon\mathfrak{g}^*\longrightarrow\mathfrak{g}$ is the contraction associated to r. In this case, the Koszul bracket (1) when restricted to left invariant differential 1-forms induces a Lie bracket on $\mathfrak{g}^*$ given by

$$[\alpha,\beta]_r=\mathrm{ad}^*_{r_\#(\alpha)}\beta-\mathrm{ad}^*_{r_\#(\beta)}\alpha\,,\quad\alpha,\beta\in\mathfrak{g}^*,\tag{7}$$

where $\prec \mathrm{ad}_u^*\alpha, v \succ = - \prec \alpha, [u,v] \succ$. Moreover, $r_\#$ becomes a morphism of Lie algebras, i.e.,

$$r_\#([\alpha,\beta]_r) = [r_\#(\alpha), r_\#(\beta)]\,, \quad \alpha,\beta \in \mathfrak{g}^*. \tag{8}$$

(2) A let invariant Riemannian metric $\langle\ ,\ \rangle$ on G is entirely determined by

$$\langle u,v\rangle(a) = \rho(T_a\mathrm{L}_{a^{-1}}u, T_a\mathrm{L}_{a^{-1}}v),$$

where $a \in G$, u, $v \in T_aG$ and ρ is a scalar product on $\mathfrak{g}$. The Levi-Civita connection of $\langle\ ,\ \rangle$ is left invariant and induces a product $A\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$2\varrho(A_uv,w) = \varrho([u,v],w) + \varrho([w,u],v) + \varrho([w,v],u)\,, \quad u,v,w \in \mathfrak{g}\,. \tag{9}$$

It is the unique product on $\mathfrak{g}$ satisfying

$$A_uv - A_vu = [u,v] \quad \text{and} \quad \varrho(A_uv,w) + \varrho(v,A_uw) = 0\,,$$

for any u, v, $w \in \mathfrak{g}$. We call A the *Levi-Civita product* associated to $(\mathfrak{g},[\ ,\ ],\rho)$.

(3) Let $(G,\langle\ ,\ \rangle,\Omega)$ be a Lie group endowed with a left invariant Riemannian metric and a nondegenerate left invariant 2-form. Then $(G,\langle\ ,\ \rangle,\Omega)$ is a Kähler manifold if and only if, for any $u,v,w\in\mathfrak{g}$,

$$\omega(A_uv,w) + \omega(u,A_uv) = 0\,, \tag{10}$$

where $\omega = \Omega(e)$, $\rho = \langle\ ,\ \rangle(e)$ and A is the Levi-Civita product of $(\mathfrak{g},[\ ,\ ],\rho)$. In this case we call $(\mathfrak{g},[\ ,\ ],\rho,\omega)$ a Kähler Lie algebra.

As all the left invariant structures on Lie groups, Riemann-Poison Lie groups can be characterized at the level of their Lie algebras.

Proposition 2.1. *Let $(G,\pi,\langle\ ,\ \rangle)$ be a Lie group endowed with a left invariant bivector field and a left invariant metric and $(\mathfrak{g},[\ ,\ ])$ its Lie algebra. Put $r=\pi(e)\in \wedge^2\mathfrak{g}$, $\varrho = \langle\ ,\ \rangle_e$ and ϱ^* the associated Euclidean product on $\mathfrak{g}^*$. Then $(G,\pi,\langle\ ,\ \rangle)$ is a Riemann-Poisson Lie group if and only if*

(i) $[r,r]=0$,

(ii) *for any α, β, $\gamma \in \mathfrak{g}^*$, $r(A_\alpha\beta,\gamma) + r(\beta,A_\alpha\gamma) = 0$,*

where A is the Levi-Civita product associated to $(\mathfrak{g}^*,[\ ,\ ]_r,\varrho^*)$.

Proof. For any $u\in\mathfrak{g}$ and $\alpha\in\mathfrak{g}^*$, we denote by u^ℓ and α^ℓ, respectively, the left invariant vector field and the left invariant differential 1-form on G given by

$$u^\ell(a) = T_e\mathrm{L}_a(u) \quad \text{and} \quad \alpha^\ell(a) = T_a^*\mathrm{L}_{a^{-1}}(\alpha)\,, \quad a\in G,\ \mathrm{L}_a(b) = ab\,.$$

Since π and $\langle\ ,\ \rangle$ are left invariant, one can see easily from (1) and (2) that we have, for any $\alpha,\beta,\gamma\in\mathfrak{g}^*$,

$$\begin{cases} [\pi,\pi]_S(\alpha^\ell,\beta^\ell,\gamma^\ell) = [r,r](\alpha,\beta,\gamma),\ \#_\pi(\alpha^\ell) = (r_\#(\alpha))^\ell,\ \mathcal{L}_{\#_\pi(\alpha^\ell)}\beta^\ell = \left(\mathrm{ad}^*_{r_\#(\alpha)}\beta\right)^\ell, \\ [\alpha^\ell,\beta^\ell]_\pi = ([\alpha,\beta]_r)^\ell\,,\ \mathcal{D}_{\alpha^\ell}\beta^\ell = (A_\alpha\beta)^\ell\ . \end{cases}$$

The proposition follows from these formulas, (3) and the fact that $(G,\pi,\langle\ ,\ \rangle)$ is a Riemann-Poisson Lie group if and only if, for any α, β, $\gamma\in\mathfrak{g}^*$,

$$[\pi,\pi]_S(\alpha^\ell,\beta^\ell,\gamma^\ell)=0 \quad\text{and}\quad \mathcal{D}\pi(\alpha^\ell,\beta^\ell,\gamma^\ell)=0\,. \qquad\square$$

Conversely, given a triple $(\mathfrak{g},r,\varrho)$ where $\mathfrak{g}$ is a real Lie algebra, $r\in\wedge^2\mathfrak{g}$ and ϱ a Euclidean product on $\mathfrak{g}$ satisfying the conditions (i) and (ii) in Proposition 2.1 then, for any Lie group G whose Lie algebra is $\mathfrak{g}$, if π and $\langle\ ,\ \rangle$ are the left invariant bivector field and the left invariant metric associated to (r,ϱ) then $(G,\pi,\langle\ ,\ \rangle)$ is a Riemann-Poisson Lie group.

Definition 2.1. A *Riemann-Poisson Lie algebra* is a triple $(\mathfrak{g},r,\varrho)$ where $\mathfrak{g}$ is a real Lie algebra, $r\in\wedge^2\mathfrak{g}$ and ϱ a Euclidean product on $\mathfrak{g}$ satisfying the conditions (i) and (ii) in Proposition 2.1.

To end this section, we give another characterization of the solutions of the classical Yang-Baxter equation (5) which will be useful later.

We observe that $r\in\wedge^2\mathfrak{g}$ is equivalent to the data of a vector subspace $S\subset\mathfrak{g}$ and a nondegenerate 2-form $\omega_r\in\wedge^2S^*$.

Indeed, for $r\in\wedge^2\mathfrak{g}$, we put $S=\mathrm{Im}r_\#$ and $\omega_r(u,v)=r(r_\#^{-1}(u),r_\#^{-1}(v))$ where u, $v\in S$ and $r_\#^{-1}(u)$ is any antecedent of u by $r_\#$.

Conversely, let (S,ω) be a vector subspace of $\mathfrak{g}$ with a non-degenerate 2-form. The 2-form ω defines an isomorphism $\omega^b\colon S\longrightarrow S^*$ by $\omega^b(u)=\omega(u,.)$, we denote by $\#\colon S^*\longrightarrow S$ its inverse and we put $r_\#=\#\circ i^*$ where $i^*\colon\mathfrak{g}^*\longrightarrow S^*$ is the dual of the inclusion $i\colon S\hookrightarrow\mathfrak{g}$.

With this observation in mind, the following proposition gives another description of the solutions of the Yang-Baxter equation.

Proposition 2.2. *Let $r\in\wedge^2\mathfrak{g}$ and (S,ω_r) its associated vector subspace. The following assertions are equivalent:*

(1) $[r,r]=0$.

(2) *S is a subalgebra of $\mathfrak{g}$ and*

$$\delta\omega_r(u,v,w):=\omega_r(u,[v,w])+\omega_r(v,[w,u])+\omega_r(w,[u,v])=0$$

for any u, v, $w\in S$.

Proof. The proposition follows from the following formulas:

$$\prec\gamma,r_\#([\alpha,\beta]_r)-[r_\#(\alpha),r_\#(\beta)]\succ=-[r,r](\alpha,\beta,\gamma)\,,\qquad \alpha,\beta,\gamma\in\mathfrak{g}^*$$

and, if S is a subalgebra,

$$[r,r](\alpha,\beta,\gamma)=-\delta\omega_r(r_\#(\alpha),r_\#(\beta),r_\#(\gamma))\,.\quad\square$$

This proposition shows that there is a correspondence between the set of solutions of the Yang-Baxter equation the set of symplectic subalgebras of $\mathfrak{g}$. We recall that a symplectic algebra is a Lie algebra S endowed with a non-degenerate 2-form ω such that $\delta\omega=0$.

3. A characterization of Riemann-Poisson Lie algebras

In this section, we combine Propositions 2.1 and 2.2 to establish a characterization of Riemann-Poisson Lie algebras which will be used later to build such Lie algebras. We establish first an intermediary result.

Proposition 3.1. *Let $(\mathfrak{g}, r, \varrho)$ be a Lie algebra endowed with $r \in \wedge^2\mathfrak{g}$ and a Euclidean product ϱ. Denote by $\mathcal{I} = \ker r_\#$, $\mathcal{I}^\perp$ its orthogonal with respect to ϱ^* and A the Levi-Civita product associated to $(\mathfrak{g}^*, [\ ,\ ]_r, \varrho^*)$. Then $(\mathfrak{g}, r, \varrho)$ is a Riemann-Poisson Lie algebra if and only if:*

- (c_1) $[r, r] = 0$.
- (c_2) *For all* $\alpha \in \mathcal{I}, A_\alpha = 0$.
- (c_3) *For all* α, β, $\gamma \in \mathcal{I}^\perp$, $A_\alpha\beta \in \mathcal{I}^\perp$ *and*

$$r(A_\alpha\beta, \gamma) + r(\beta, A_\alpha\gamma) = 0\,.$$

Proof. By using the splitting $\mathfrak{g}^* = \mathcal{I} \oplus \mathcal{I}^\perp$, on can see that the conditions (i) and (ii) in Proposition 2.1 are equivalent to

$$\begin{cases} [r, r] = 0\,, \\ r(A_\alpha\beta, \gamma) = 0\,,\ \alpha \in \mathcal{I}\,,\ \beta \in \mathcal{I}\,,\ \gamma \in \mathcal{I}^\perp, \\ r(A_\alpha\beta, \gamma) + r(\beta, A_\alpha\gamma) = 0\,,\ \alpha \in \mathcal{I}\,,\ \beta \in \mathcal{I}^\perp\,,\ \gamma \in \mathcal{I}^\perp, \\ r(A_\alpha\beta, \gamma) = 0\,,\ \alpha \in \mathcal{I}^\perp\,,\ \beta \in \mathcal{I}\,,\ \gamma \in \mathcal{I}^\perp, \\ r(A_\alpha\beta, \gamma) + r(\beta, A_\alpha\gamma) = 0\,,\ \alpha \in \mathcal{I}^\perp,\ \beta \in \mathcal{I}^\perp,\ \gamma \in \mathcal{I}^\perp. \end{cases} \tag{11}$$

Suppose that the conditions (c_1)-(c_3) hold. Then for any $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{I}^\perp$, $A_\beta\alpha = [\beta, \alpha]_r$ and hence $r_\#(A_\beta\alpha) = [r_\#(\beta), r_\#(\alpha)] = 0$ and hence the equations in (11) holds.

Conversely, suppose that (11) holds. Then (c_1) holds obviously.

For any $\alpha, \beta \in \mathcal{I}$, the second equation in (11) is equivalent to $A_\alpha\beta \in \mathcal{I}$ and we have from (7) and (9) $[\alpha, \beta]_r = 0$ and $A_\alpha\beta \in \mathcal{I}^\perp$. Thus $A_\alpha\beta = 0$.

Take now $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{I}^\perp$. For any $\gamma \in \mathcal{I}$, $\varrho^*(A_\alpha\beta, \gamma) = -\varrho^*(\beta, A_\alpha\gamma) = 0$ and hence $A_\alpha\beta \in \mathcal{I}^\perp$. On the other hand,

$$r_\#([\alpha, \beta]_r) = r_\#(A_\alpha\beta) - r_\#(A_\beta\alpha) \overset{(8)}{=} [r_\#(\alpha), r_\#(\beta)] = 0\,.$$

So, for any $\gamma \in \mathcal{I}^\perp$,

$$\prec \gamma, r_\#(A_\alpha\beta) \succ = \prec \gamma, r_\#(A_\beta\alpha) \succ = r(A_\beta\alpha, \gamma) \overset{(11)}{=} 0\,.$$

This shows that $A_\alpha\beta \in \mathcal{I}$ and hence $A_\alpha\beta = 0$. Finally, (c_2) is true. Now, for any $\alpha \in \mathcal{I}^\perp$, the fourth equation in (11) implies that A_α leaves invariant $\mathcal{I}$ and since it is skew-symmetric it leaves invariant $\mathcal{I}^\perp$ and (c_3) follows. This completes the proof. □

Proposition 3.2. *Let $(\mathfrak{g}, \varrho, r)$ be a Lie algebra endowed with a solution of classical Yang-Baxter equation and a bi-invariant Euclidean product, i.e.,*

$$\varrho(\mathrm{ad}_u v, w) + \varrho(v, \mathrm{ad}_u w) = 0\,, \quad u, v, w \in \mathfrak{g}\,.$$

Then $(\mathfrak{g}, \varrho, r)$ *is Riemann-Poisson Lie algebra if and only if* $\mathrm{Im} r_\#$ *is an abelian subalgebra.*

Proof. Since ϱ is bi-invariant, one can see easily that for any $u \in \mathfrak{g}$, ad_u^* is skew-symmetric with respect to ϱ^* and hence the Levi-Civita product A associated to $(\mathfrak{g}^*, [\ ,\ ]_r, \varrho^*)$ is given by $A_\alpha \beta = \mathrm{ad}^*_{r_\#(\alpha)}\beta$. So, $(\mathfrak{g}, \varrho, r)$ is Riemann-Poisson Lie algebra if and only if, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\begin{aligned} 0 &= r(\mathrm{ad}^*_{r_\#(\alpha)}\beta, \gamma) + r(\beta, \mathrm{ad}^*_{r_\#(\alpha)}\gamma) \\ &= \prec \beta, [r_\#(\alpha), r_\#(\gamma)] \succ - \prec \gamma, [r_\#(\alpha), r_\#(\beta)] \succ \\ &\overset{(5)}{=} \prec \alpha, [r_\#(\beta), r_\#(\gamma)] \succ \end{aligned}$$

and the result follows. □

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra, $r \in \wedge^2 \mathfrak{g}$ and ϱ a Euclidean product on $\mathfrak{g}$. Denote by (S, ω_r) the symplectic vector subspace associated to r and by $\#\colon \mathfrak{g}^* \longrightarrow \mathfrak{g}$ the isomorphism given by ϱ. Note that the Euclidean product on $\mathfrak{g}^*$ is given by $\varrho^*(\alpha, \beta) = \varrho(\#(\alpha), \#(\beta))$. We have

$$\mathfrak{g}^* = \mathcal{I} \oplus \mathcal{I}^\perp \quad \text{and} \quad \mathfrak{g} = S \oplus S^\perp ,$$

where $\mathcal{I} = \ker r_\#$. Moreover, $r_\#\colon \mathcal{I}^\perp \longrightarrow S$ is an isomorphism, we denote by $\tau\colon S \longrightarrow \mathcal{I}^\perp$ its inverse. From the relation

$$\varrho\big(\#(\alpha), r_\#(\beta)\big) = \prec \alpha\, ,\ r_\#(\beta) \succ = r(\beta, \alpha)\, ,$$

we deduce that $\#\colon \mathcal{I} \longrightarrow S^\perp$ is an isomorphism and hence $\#\colon \mathcal{I}^\perp \longrightarrow S$ is also an isomorphism.

Consider the isomorphism $J\colon S \longrightarrow S$ linking ω_r to $\varrho_{|S}$, i.e.,

$$\omega_r(u, v) = \rho(Ju, v)\, , \quad u, v \in S\, .$$

On can see easily that $J = -\# \circ \tau$.

Theorem 3.1. *With the notations above,* $(\mathfrak{g}, r, \varrho)$ *is a Riemann-Poisson Lie algebra if and only if the following conditions hold:*

(1) $(S, \varrho_{|S}, \omega_r)$ *is a Kähler Lie subalgebra, i.e., for all* $s_1,\ s_2,\ s_3 \in S$,

$$\omega_r(\nabla_{s_1} s_2, s_3) + \omega_r(s_2, \nabla_{s_1} s_3) = 0\, , \tag{12}$$

where ∇ *is the Levi-Civita product associated to* $(S, [\ ,\ ], \varrho_{|S})$.

(2) *for all* $s \in S$ *and all* $u, v \in S^\perp$,

$$\varrho(\phi_S(s)(u), v) + \varrho(u, \phi_S(s)(v)) = 0\, , \tag{13}$$

where $\phi_S\colon S \longrightarrow \mathrm{End}(S^\perp)$, $u \mapsto \mathrm{pr}_{S^\perp} \circ \mathrm{ad}_u$ *and* $\mathrm{pr}_{S^\perp}\colon \mathfrak{g} \longrightarrow S^\perp$ *is the orthogonal projection.*

(3) *For all* $s_1,\ s_2 \in S$ *and all* $u \in S^\perp$,

$$\omega_r(\phi_{S^\perp}(u)(s_1), s_2) + \omega_r(s_1, \phi_{S^\perp}(u)(s_1)) = 0\, , \tag{14}$$

where $\phi_{S^\perp} : S^\perp \longrightarrow \mathrm{End}(S)$, $u \mapsto \mathrm{pr}_S \circ \mathrm{ad}_u$ *and* $\mathrm{pr}_S\colon \mathfrak{g} \longrightarrow S$ *is the orthogonal projection.*

Proof. Suppose first that $(\mathfrak{g}, r, \varrho)$ is a Riemann-Poisson Lie algebra. According to Propositions 3.1 and 2.2, this is equivalent to

$$\begin{cases} (S,\omega_r) \quad \text{is a symplectic subalgebra,} \\ \forall \alpha\in\mathcal{I},\ A_\alpha = 0\,, \\ \forall\, \alpha,\beta,\gamma\in\mathcal{I}^\perp,\ A_\alpha\beta\in\mathcal{I}^\perp \quad \text{and} \quad r(A_\alpha\beta,\gamma)+r(\beta,A_\alpha\gamma)=0\,, \end{cases} \tag{15}$$

where A is the Levi-Civita product of $(\mathfrak{g}^*, [\ ,\ ]_r, \varrho^*)$.

For $\alpha,\beta\in\mathcal{I}$ and $\gamma\in\mathcal{I}^\perp$,

$$\begin{aligned} 2\varrho^*(A_\alpha\beta,\gamma) &= \varrho^*([\alpha,\beta]_r,\gamma)+\varrho^*([\gamma,\beta]_r,\alpha)+\varrho^*([\gamma,\alpha]_r,\beta) \\ &= \varrho^*(\mathrm{ad}^*_{r_\#(\gamma)}\beta,\alpha)+\varrho^*(\mathrm{ad}^*_{r_\#(\gamma)}\alpha,\beta) \\ &= -\prec \beta,[r_\#(\gamma),\#(\alpha)]\succ - \prec \alpha,[r_\#(\gamma),\#(\beta)]\succ \\ &= -\varrho(\#(\beta),[r_\#(\gamma),\#(\alpha)]) - \varrho(\#(\alpha),[r_\#(\gamma),\#(\beta)])\,. \end{aligned} \tag{16}$$

Since $\#\colon \mathcal{I}\longrightarrow S^\perp$ and $r_\#\colon \mathcal{I}^\perp\longrightarrow S$ are isomorphisms, we deduce from (16) that $A_\alpha\beta=0$ for any α, $\beta\in\mathcal{I}$ is equivalent to (13).

For $\alpha\in\mathcal{I}$ and β, $\gamma\in\mathcal{I}^\perp$,

$$\begin{aligned} 2\varrho^*(A_\alpha\beta,\gamma) &= \varrho^*([\alpha,\beta]_r,\gamma)+\varrho^*([\gamma,\beta]_r,\alpha)+\varrho^*([\gamma,\alpha]_r,\beta) \\ &= -\varrho^*(\mathrm{ad}^*_{r_\#(\beta)}\alpha,\gamma)-\varrho^*(\mathrm{ad}^*_{r_\#(\beta)}\gamma,\alpha)+\varrho^*(\mathrm{ad}^*_{r_\#(\gamma)}\beta,\alpha)+\varrho^*(\mathrm{ad}^*_{r_\#(\gamma)}\alpha,\beta) \\ &= \prec \alpha,[r_\#(\beta),\#(\gamma)]\succ + \prec \gamma,[r_\#(\beta),\#(\alpha)]\succ - \prec \beta,[r_\#(\gamma),\#(\alpha)]\succ \\ &\quad - \prec \alpha,[r_\#(\gamma),\#(\beta)]\succ = \varrho(\#(\gamma),[r_\#(\beta),\#(\alpha)]) \\ &\quad - \varrho(\#(\beta),[r_\#(\gamma),\#(\alpha)]) + \prec\alpha,[r_\#(\beta),\#(\gamma)]\succ - \prec\alpha,[r_\#(\gamma),\#(\beta)]\succ \\ &= -\varrho(J\circ r_\#(\gamma),[r_\#(\beta),\#(\alpha)]) + \varrho(J\circ r_\#(\beta),[r_\#(\gamma),\#(\alpha)]) \\ &\quad + \prec \alpha,[r_\#(\beta),\#(\gamma)]\succ - \prec \alpha,[r_\#(\gamma),\#(\beta)]\succ \\ &= -\omega_r(r_\#(\gamma),\mathrm{pr}_S([r_\#(\beta),\#(\alpha)])) - \omega_r(\mathrm{pr}_S([r_\#(\gamma),\#(\alpha)]),r_\#(\beta)) \\ &\quad + \prec \alpha,[r_\#(\beta),\#(\gamma)]\succ - \prec \alpha,[r_\#(\gamma),\#(\beta)]\succ\ . \end{aligned} \tag{17}$$

Now, $\#(\beta),\#(\gamma)\in S$ and $r_\#(\beta),r_\#(\gamma)\in S$ and since S is a subalgebra we deduce that $[r_\#(\beta),\#(\gamma)],[r_\#(\gamma),\#(\beta)]\in S$ and hence

$$\prec \alpha,[r_\#(\beta),\#(\gamma)]\succ = \prec \alpha,[r_\#(\gamma),\#(\beta)]\succ = 0\,.$$

We have also $\#\colon \mathcal{I}\longrightarrow S^\perp$ and $r_\#\colon \mathcal{I}^\perp\longrightarrow S$ are isomorphisms so that, by virtue of (17), $A_\alpha\beta=0$ for any $\alpha\in\mathcal{I}$ and $\beta\in\mathcal{I}^\perp$ is equivalent to (14).

On the other hand, for any $\alpha,\beta,\gamma\in\mathcal{I}^\perp$, since $\# = -J\circ r_\#$, the relation

$$2\varrho^*(A_\alpha\beta,\gamma) = \varrho^*([\alpha,\beta]_r,\gamma)+\varrho^*([\gamma,\beta]_r,\alpha)+\varrho^*([\gamma,\alpha]_r,\beta)$$

can be written

$$\begin{aligned} 2\varrho\big(J\circ r_\#(A_\alpha\beta), J\circ r_\#(\gamma)\big) &= \varrho\big(J\circ r_\#([\alpha,\beta]_r), J\circ r_\#(\gamma)\big) \\ &\quad + \varrho\big(J\circ r_\#([\gamma,\beta]_r), J\circ r_\#(\alpha)\big) \\ &\quad + \varrho\big(J\circ r_\#([\gamma,\alpha]_r), J\circ r_\#(\beta)\big)\,. \end{aligned}$$

But $r_\#([\alpha,\beta]_r) = [r_\#(\alpha), r_\#(\beta)]$ and hence

$$\begin{aligned}2\langle r_\#(A_\alpha,\beta), r_\#(\gamma)\rangle_J &= \langle [r_\#(\alpha), r_\#(\beta)], r_\#(\gamma)\rangle_J + \langle [r_\#(\mathfrak{g}), r_\#(\beta)], r_\#(\alpha)\rangle_J \\ &\quad + \langle [r_\#(\gamma), r_\#(\alpha)], r_\#(\beta)\rangle_J\,,\end{aligned}$$

where $\langle u,v\rangle_J = \varrho(Ju,Jv)$. This shows that $r_\#(A_\alpha\beta) = \nabla_{r_\#(\alpha)} r_\#(\beta)$ where ∇ is the Levi-Civita product of $(S,[\ ,\ ],\langle\ ,\ \rangle_J)$ and the third relation in (15) is equivalent to

$$\omega_r(\nabla_u v, w) + \omega_r(v, \nabla_u w) = 0\,,\quad u,v,w\in S\,.$$

This is equivalent to $\nabla_u Jv = J\nabla_u v$. Let us show that ∇ is actually the Levi-Civita product of $(S,[\ ,\ ],\varrho)$. Indeed, for any u, v, $w\in S$, $\nabla_u v - \nabla_v u = [u,v]$ and

$$\begin{aligned}\varrho(\nabla_u v, w) + \varrho(\nabla_u w, v) &= \langle J^{-1}\nabla_u v, J^{-1}w\rangle_J + \langle J^{-1}\nabla_u w, J^{-1}v\rangle_J \\ &= \langle \nabla_u J^{-1}v, J^{-1}w\rangle_J + \langle \nabla_u J^{-1}w, J^{-1}v\rangle_J \\ &= 0\,.\end{aligned}$$

So we have shown the direct part of the theorem. The converse can be deduced easily from the relations we established in the proof of the direct part. □

Example 1. Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra and T an even dimensional torus of G. Choose a bi-invariant Riemannian metric $\langle\ ,\ \rangle$ on G, a nondegenerate $\omega\in\wedge^2 S^*$ where S is the Lie algebra of T and put $\varrho = \langle\ ,\ \rangle(e)$. Let $r\in\wedge^2\mathfrak{g}$ be the solution of the classical Yang-Baxter associated to (S,ω). By using either Proposition 3.2 or Theorem 3.1, one can see easily that $(\mathfrak{g},\varrho,r)$ is a Riemann-Poisson Lie algebra and hence $(G,\langle\ ,\ \rangle,\pi)$ is a Riemann-Poisson Lie group where π is the left invariant Poisson tensor associated to r. According to Theorem 1.1, the orbits of the right action of T on G defines a Riemannian and Kähler foliation. For instance, $G = \mathrm{SO}(2n)$, $T = \mathrm{Diagonal}(D_1,\dots,D_n)$ where $D_i = \begin{pmatrix}\cos(\theta_i) & \sin(\theta_i)\\ -\sin(\theta_i) & \cos(\theta_i)\end{pmatrix}$ and $\langle\ ,\ \rangle = -K$ where K is the Killing form.

4. Construction of Riemann-Poisson Lie algebras

In this section, we give a general method for building Riemann-Poisson Lie algebras and we use it to give all Riemann-Poisson Lie algebras up to dimension 5.

According to Theorem 3.1, to build Riemann-Poisson Lie algebras one needs to solve the following problem.

Problem 1. We look for:

(1) A Kähler Lie algebra $(\mathfrak{h},[\ ,\ ]_{\mathfrak{h}},\varrho_{\mathfrak{h}},\omega)$,

(2) a Euclidean vector space $(\mathfrak{p},\varrho_{\mathfrak{p}})$,

(3) a bilinear skew-symmetric map $[\ ,\ ]_{\mathfrak{p}}\colon \mathfrak{p}\times\mathfrak{p}\longrightarrow\mathfrak{p}$,

(4) a bilinear skew-symmetric map $\mu\colon \mathfrak{p}\times\mathfrak{p}\longrightarrow\mathfrak{h}$,

(5) two linear maps $\phi_{\mathfrak{p}}\colon\mathfrak{p}\longrightarrow \mathrm{sp}(\mathfrak{h},\omega)$ and $\phi_{\mathfrak{h}}\colon\mathfrak{h}\longrightarrow\mathrm{so}(\mathfrak{p})$ where $\mathrm{sp}(\mathfrak{h},\omega) = \{J\colon\mathfrak{h}\longrightarrow\mathfrak{h}, J^\omega + J = 0\}$ and $\mathrm{so}(\mathfrak{p}) = \{A\colon\mathfrak{p}\longrightarrow\mathfrak{p}, A^* + A = 0\}$, J^ω is the adjoint with respect to ω and A^* is the adjoint with respect to $\varrho_{\mathfrak{p}}$,

such that the bracket $[\ ,\ ]$ on $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ given, for any $a,\ b\in\mathfrak{p}$ and $u,\ v\in\mathfrak{h}$, by

$$[u,v]=[u,v]_{\mathfrak{h}},\ [a,b]=\mu(a,b)+[a,b]_{\mathfrak{p}},\ [a,u]=-[u,a]=\phi_{\mathfrak{p}}(a)(u)-\phi_{\mathfrak{h}}(u)(a) \tag{18}$$

is a Lie bracket.

In this case, $(\mathfrak{g},[\ ,\ ])$ endowed with $r\in\wedge^2\mathfrak{g}$ associated to $(\mathfrak{h},\omega)$ and the Euclidean product $\varrho=\varrho_{\mathfrak{h}}\oplus\varrho_{\mathfrak{p}}$ becomes, by virtue of Theorem 3.1, a Riemann-Poisson Lie algebra.

Proposition 4.1. *With the data and notations of Problem 1, the bracket given by* (18) *is a Lie bracket if and only if, for any $u,\ v\in\mathfrak{h}$ and $a,\ b,\ c\in\mathfrak{p}$,*
(19)

$$\begin{cases}\phi_{\mathfrak{p}}(a)([u,v]_{\mathfrak{h}})=[u,\phi_{\mathfrak{p}}(a)(v)]_{\mathfrak{h}}+[\phi_{\mathfrak{p}}(a)(u),v]_{\mathfrak{h}}+\phi_{\mathfrak{p}}(\phi_{\mathfrak{h}}(v)(a))(u)-\phi_{\mathfrak{p}}(\phi_{\mathfrak{h}}(u)(a))(v)\,,\\ \phi_{\mathfrak{h}}(u)([a,b]_{\mathfrak{p}})=[a,\phi_{\mathfrak{h}}(u)(b)]_{\mathfrak{p}}+[\phi_{\mathfrak{h}}(u)(a),b]_{\mathfrak{p}}+\phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(b)(u))(a)-\phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(a)(u))(b)\,,\\ \phi_{\mathfrak{h}}([u,v]_{\mathfrak{h}})=[\phi_{\mathfrak{h}}(u),\phi_{\mathfrak{h}}(v)]\,,\\ \phi_{\mathfrak{p}}([a,b]_{\mathfrak{p}})(u)=[\phi_{\mathfrak{p}}(a),\phi_{\mathfrak{p}}(b)](u)+[u,\mu(a,b)]_{\mathfrak{h}}-\mu(a,\phi_{\mathfrak{h}}(u)(b))-\mu(\phi_{\mathfrak{h}}(u)(a),b)\,,\\ \oint[a,[b,c]_{\mathfrak{p}}]_{\mathfrak{p}}=\oint\phi_{\mathfrak{h}}(\mu(b,c))(a)\,,\\ \oint\phi_{\mathfrak{p}}(a)(\mu(b,c))=\oint\mu([b,c]_{\mathfrak{p}},a)\,,\end{cases}$$

where $\oint$ stands for the circular permutation.

Proof. The equations follow from the Jacobi identity applied to (a,u,v), (a,b,u) and (a,b,c). □

We tackle now the task of determining the list of all Riemann-Poisson Lie algebras up to dimension 5. For this purpose, we need to solve Problem 1 in the following four cases: (a) $\dim\mathfrak{p}=1$, (b) $\dim\mathfrak{h}=2$ and $\mathfrak{h}$ non abelian, (c) $\dim\mathfrak{h}=\dim\mathfrak{p}=2$ and $\mathfrak{h}$ abelian, (d) $\dim\mathfrak{h}=2$, $\dim\mathfrak{p}=3$ and $\mathfrak{h}$ abelian.

It is easy to find the solutions of Problem 1 when $\dim\mathfrak{p}=1$ since in this case $\mathrm{so}(\mathfrak{p})=0$ and the three last equations in (19) hold obviously.

Proposition 4.2. *If $\dim\mathfrak{p}=1$ then the solutions of Problem 1 are a Kähler Lie algebra $(\mathfrak{h},\varrho,\omega)$, $\phi_{\mathfrak{h}}=0$, $[\ ,\ ]_{\mathfrak{p}}=0$, $\mu=0$ and $\phi_{\mathfrak{p}}(a)\in\mathrm{sp}(\mathfrak{h},\omega)\cap\mathrm{Der}(\mathfrak{h})$ where a is a generator of $\mathfrak{p}$ and $\mathrm{Der}(\mathfrak{h})$ the Lie algebra of derivations of $\mathfrak{h}$.*

Let us solve Problem 1 when $\mathfrak{h}$ is 2-dimensional non abelian.

Proposition 4.3. *Let $((\mathfrak{h},\omega,\varrho_{\mathfrak{h}}),(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\varrho_{\mathfrak{p}}),\mu,\phi_{\mathfrak{h}},\phi_{\mathfrak{p}})$ be a solution of Problem 1 with $\mathfrak{h}$ is 2-dimensional non abelian. Then there exists an orthonormal basis $\mathbb{B}=(e_1,e_2)$ of $\mathfrak{h}$, $b_0\in\mathfrak{p}$ and two constants $\alpha\neq0$ and $\beta\neq0$ such that:*

(i) $[e_1,e_2]_{\mathfrak{h}}=\alpha e_1$, $\omega=\beta e_1^*\wedge e_2^*$,

(ii) $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\varrho_{\mathfrak{p}})$ *is a Euclidean Lie algebra,*

(iii) $\phi_{\mathfrak{h}}(e_1)=0$, $\phi_{\mathfrak{h}}(e_2)\in\mathrm{Der}(\mathfrak{p})\cap\mathrm{so}(\mathfrak{p})$ *and, for any $a\in\mathfrak{p}$,* $M(\phi_{\mathfrak{p}}(a),\mathbb{B})=\begin{pmatrix}0&\varrho_{\mathfrak{p}}(a,b_0)\\0&0\end{pmatrix}$,

(iv) *for any $a,\ b\in\mathfrak{p}$, $\mu(a,b)=\mu_0(a,b)e_1$ with μ_0 is a 2-cocycle of $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}})$ satisfying*

$$\mu_0(a,\phi_{\mathfrak{h}}(e_2)b)+\mu_0(\phi_{\mathfrak{h}}(e_2)a,b)=-\varrho_{\mathfrak{p}}([a,b]_{\mathfrak{p}},b_0)-\alpha\mu_0(a,b)\,. \tag{20}$$

Proof. Note first that from the third relation in (19) we get that $\phi_{\mathfrak{h}}(\mathfrak{h})$ is a solvable subalgebra of $\mathrm{so}(\mathfrak{p})$ and hence must be abelian. Since $\mathfrak{h}$ is 2-dimensional non abelian then $\dim \phi_{\mathfrak{h}}(\mathfrak{h}) = 1$ and $[\mathfrak{h}, \mathfrak{h}] \subset \ker \phi_{\mathfrak{h}}$. So there exists an orthonormal basis (e_1, e_2) of $\mathfrak{h}$ such that $[e_1, e_2]_{\mathfrak{h}} = \alpha e_1$, $\phi_{\mathfrak{h}}(e_1) = 0$ and $\omega = \beta e_1^* \wedge e_2^*$. If we identify the endomorphisms of $\mathfrak{h}$ with their matrices in the basis (e_1, e_2), we get that $\mathrm{sp}(\mathfrak{h}, \omega) = \mathrm{sl}(2, \mathbb{R})$ and there exists $a_0, b_0, c_0 \in \mathfrak{p}$ such that, for any $a \in \mathfrak{p}$,

$$\phi_{\mathfrak{p}}(a) = \begin{pmatrix} \varrho_{\mathfrak{p}}(a_0, a) & \varrho_{\mathfrak{p}}(b_0, a) \\ \varrho_{\mathfrak{p}}(c_0, a) & -\varrho_{\mathfrak{p}}(a_0, a) \end{pmatrix}.$$

The first equation in (19) is equivalent to

$$\begin{aligned} \alpha\big(\varrho_{\mathfrak{p}}(a_0, a)e_1 + \varrho_{\mathfrak{p}}(c_0, a)e_2\big) &= -\alpha\varrho_{\mathfrak{p}}(a_0, a)e_1 + \alpha\varrho_{\mathfrak{p}}(a_0, a)e_1 \\ &\quad + \varrho_{\mathfrak{p}}\big(a_0, \phi_{\mathfrak{h}}(e_2)(a)\big)e_1 + \varrho_{\mathfrak{p}}\big(c_0, \phi_{\mathfrak{h}}(e_2)(a)\big)e_2\,, \end{aligned}$$

for any $a \in \mathfrak{p}$. Since $\phi_{\mathfrak{h}}(e_2)$ is sekw-symmetric, this is equivalent to

$$\phi_{\mathfrak{h}}(e_2)(a_0) = -\alpha a_0 \quad \text{and} \quad \phi_{\mathfrak{h}}(e_2)(c_0) = -\alpha c_0\,.$$

This implies that $a_0 = c_0 = 0$. The second equation in (19) implies that $\phi_{\mathfrak{h}}(e_2)$ is a derivation of $[\ ,\ ]_{\mathfrak{p}}$. If we take $u = e_1$ in the forth equation in (19), we get that $[e_1, \mu(a, b)] = 0$, for any a, $b \in \mathfrak{p}$ and hence $\mu(a, b) = \mu_0(a, b)e_1$. If we take $u = e_2$ in the forth equation in (19) we get (20). The two last equations are equivalent to $[\ ,\ ]_{\mathfrak{p}}$ is a Lie bracket and μ_0 is 2-cocycle of $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}})$. □

The following proposition gives the solutions of Problem 1 when $\mathfrak{h}$ is 2-dimensional abelian and $\dim \mathfrak{p} = 2$.

Proposition 4.4. *Let $\big((\mathfrak{h}, \omega, \varrho_{\mathfrak{h}}), (\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}}), \mu, \phi_{\mathfrak{h}}, \phi_{\mathfrak{p}}\big)$ be a solution of Problem 1 with $\mathfrak{h}$ is 2-dimensional abelian and $\dim \mathfrak{p} = 2$. Then one of the following situations occurs:*

(1) *$\phi_{\mathfrak{h}} = 0$, $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}})$ is a 2-dimensional Euclidean Lie algebra, there exists $a_0 \in \mathfrak{p}$ and $D \in \mathrm{sp}(\mathfrak{h}, \omega)$ such that, for any $a \in \mathfrak{p}$, $\phi_{\mathfrak{p}}(a) = \varrho_{\mathfrak{p}}(a_0, a)D$ and there is no restriction on μ. Moreover, $a_0 \in [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{p}}^{\perp}$ if $D \neq 0$.*

(2) *$\phi_{\mathfrak{h}} = 0$, $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}})$ is a 2-dimensional non abelian Euclidean Lie algebra, $\phi_{\mathfrak{p}}$ identifies $\mathfrak{p}$ to a two dimensional subalgebra of $\mathrm{sp}(\mathfrak{h}, \omega)$ and there is no restriction on μ.*

(3) *$(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}})$ is a Euclidean abelian Lie algebra and there exists an orthonormal basis $\mathbb{B} = (e_1, e_2)$ of $\mathfrak{h}$ and $b_0 \in \mathfrak{p}$ such that $\omega = \alpha e_1^* \wedge e_2^*$, $\phi_{\mathfrak{h}}(e_1) = 0$, $\phi_{\mathfrak{h}}(e_2) \neq 0$ and, for any $a \in \mathfrak{p}$, $M(\phi_{\mathfrak{p}}(a), \mathbb{B}) = \begin{pmatrix} 0 & \varrho_{\mathfrak{p}}(b_0, a) \\ 0 & 0 \end{pmatrix}$ and there is no restriction on μ.*

Proof. Note first that since $\dim \mathfrak{p} = 2$ the last two equations in (19) hold obviously and $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}})$ is a Lie algebra. We distinguish two cases:

(i) $\phi_{\mathfrak{h}} = 0$. Then (19) is equivalent to $\phi_{\mathfrak{p}}$ is a representation of $\mathfrak{p}$ in $\mathrm{sp}(\mathfrak{h}, \omega) \simeq \mathrm{sl}(2, \mathbb{R})$. Since $\mathrm{sl}(2, \mathbb{R})$ doesn't contain any abelian two dimensional subalgebra, if $\mathfrak{p}$ is an abelian Lie algebra then $\dim \phi_{\mathfrak{p}}(\mathfrak{p}) \leq 1$ and the first situation occurs. If $\mathfrak{p}$ is not abelian then the first or the second situation occurs depending on $\dim \phi_{\mathfrak{p}}(\mathfrak{p})$.

(ii) $\phi_{\mathfrak{h}} \neq 0$. Since $\dim \mathrm{so}(\mathfrak{p}) = 1$ there exists an orthonormal basis $\mathbb{B} = (e_1, e_2)$ of $\mathfrak{h}$ such that $\phi_{\mathfrak{h}}(e_1) = 0$ and $\phi_{\mathfrak{h}}(e_2) \neq 0$. We have $\mathrm{sp}(\mathfrak{h}, \omega) = \mathrm{sl}(2, \mathbb{R})$ and hence, for any $a \in \mathfrak{p}$, $M(\phi_{\mathfrak{p}}(a), \mathbb{B}) = \begin{pmatrix} \varrho_{\mathfrak{p}}(a_0, a) & \varrho_{\mathfrak{p}}(b_0, a) \\ \varrho_{\mathfrak{p}}(c_0, a) & -\varrho_{\mathfrak{p}}(a_0, a) \end{pmatrix}$. Choose an orthonormal basis (a_1, a_2) of $\mathfrak{p}$. Then there exists $\lambda \neq 0$ such that $\phi_{\mathfrak{h}}(e_2)(a_1) = \lambda a_2$ and $\phi_{\mathfrak{h}}(e_2)(a_2) = -\lambda a_1$.

The first equation in (19) is equivalent to

$$\phi_{\mathfrak{p}}(\phi_{\mathfrak{h}}(e_2)(a))(e_1) = 0\,, \quad a \in \mathfrak{p}\,.$$

This is equivalent to

$$\phi_{\mathfrak{p}}(a_1)(e_1) = \phi_{\mathfrak{p}}(a_2)(e_1) = 0\,.$$

Then $a_0 = c_0 = 0$ and hence $\phi_{\mathfrak{p}}(a) = \begin{pmatrix} 0 & \varrho_{\mathfrak{p}}(b_0, a) \\ 0 & 0 \end{pmatrix}$. The second equation in (19) gives

$$\begin{aligned}\phi_{\mathfrak{h}}(e_2)([a_1, a_2]_{\mathfrak{p}}) = {} & [a_1, \phi_{\mathfrak{h}}(e_2)(a_2)]_{\mathfrak{p}} + \phi_{\mathfrak{h}}(e_2)(a_1), a_2]_{\mathfrak{p}} \\ & + \phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(a_2)(e_2))(a_2) - \phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(a_1)(e_2))(a_2)\,,\end{aligned}$$

and hence $\phi_{\mathfrak{h}}(e_2)([a_1, a_2]_{\mathfrak{p}}) = 0$. Thus $[a_1, a_2]_{\mathfrak{p}} = 0$. All the other equations in (19) hold obviously. $\square$

To tackle the last case, we need the determination of 2-dimensional subalgebras of $\mathrm{sl}(2, \mathbb{R})$.

Proposition 4.5. *The 2-dimensional subalgebras of* $\mathrm{sl}(2, \mathbb{R})$ *are*

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}, \quad \mathfrak{g}_2 = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & -\alpha \end{pmatrix}, \ \alpha, \beta \in \mathbb{R} \right\},$$

$$\mathfrak{g}_x = \left\{ \begin{pmatrix} \alpha & \frac{2\beta - \alpha}{x} \\ (\alpha + 2\beta)x & -\alpha \end{pmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$

where $x \in \mathbb{R} \setminus \{0\}$. *Moreover,* $\mathfrak{g}_x = \mathfrak{g}_y$ *if and only if* $x = y$.

Proof. Let $\mathfrak{g}$ be a 2-dimensional subalgebra of $\mathrm{sl}(2, \mathbb{R})$. We consider the basis $\mathbb{B} = (h, e, f)$ of $\mathrm{sl}(2, \mathbb{R})$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$[h, e] = 2e\,, \ [h, f] = -2f \quad \text{and} \quad [e, f] = h\,.$$

If $h \in \mathfrak{g}$ then ad_h leaves $\mathfrak{g}$ invariant. But ad_h has three eigenvalues $(0, 2, -2)$ with the associated eigenvectors (h, e, f) and hence it restriction to $\mathfrak{g}$ has $(0, 2)$ or $(0, -2)$ as eigenvalues. Thus $\mathfrak{g} = \mathfrak{g}_1$ or $\mathfrak{g} = \mathfrak{g}_2$.

Suppose now that $h \notin \mathfrak{g}$. By using the fact that $\mathrm{sl}(2, \mathbb{R})$ is unimodular, i.e., for any $w \in \mathrm{sl}(2, \mathbb{R})$ $\mathrm{tr}(\mathrm{ad}_w) = 0$, we can choose a basis (u, v) of $\mathfrak{g}$ such that (u, v, h) is a basis of $\mathrm{sl}(2, \mathbb{R})$ and

$$[u, v] = u, \ [h, u] = au + v \quad \text{and} \quad [h, v] = du - av - h\,.$$

If (x_1, x_2, x_3) and (y_1, y_2, y_3) are the coordinates of u and v in $\mathbb{B}$, the brackets above gives

$$\begin{cases} -2(x_1y_3 - x_3y_1) - x_1 = 0\,, \\ 2(x_2y_3 - x_3y_2) - x_2 = 0\,, \\ x_1y_2 - x_2y_1 - x_3 = 0\,, \end{cases} \quad \begin{cases} y_1 = (2-a)x_1\,, \\ y_2 = -(a+2)x_2\,, \\ y_3 = -ax_3\,, \end{cases} \quad \text{and} \quad \begin{cases} dx_1 = (a+2)y_1\,, \\ dx_2 = (a-2)y_2\,, \\ dx_3 = ay_3 + 1\,. \end{cases}$$

Note first that if $x_1 = 0$ then $(x_2, x_3) = (0, 0)$ which impossible so we must have $x_1 \neq 0$ and hence $d = 4 - a^2$. If we replace in the third equation in the second system and the last equation, we get $x_3 = \frac{1}{4}$ and $y_3 = -\frac{a}{4}$. The third equation in the first system gives $x_2 = -\frac{1}{16x_1}$ and hence $y_1 = (2-a)x_1$ and $y_2 = \frac{(a+2)}{16x_1}$. Thus

$$\begin{aligned} \mathfrak{g} &= \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{4} & -\frac{1}{16x_1} \\ x_1 & -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{a}{4} & \frac{(a+2)}{16x_1} \\ (2-a)x_1 & \frac{a}{4} \end{pmatrix} \right\} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 1 & -\frac{1}{x} \\ x & -1 \end{pmatrix}, \begin{pmatrix} -a & \frac{(a+2)}{x} \\ (2-a)x & a \end{pmatrix} \right\}; \quad x = 4x_1\,. \end{aligned}$$

But

$$\begin{pmatrix} 0 & \frac{2}{x} \\ 2x & 0 \end{pmatrix} = a \begin{pmatrix} 1 & -\frac{1}{x} \\ x & -1 \end{pmatrix} + \begin{pmatrix} -a & \frac{(a+2)}{x} \\ (2-a)x & a \end{pmatrix}$$

and hence

$$\mathfrak{g} = \operatorname{span}\left\{ \begin{pmatrix} 1 & -\frac{1}{x} \\ x & -1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{2}{x} \\ 2x & 0 \end{pmatrix} \right\} = \mathfrak{g}_x\,.$$

One can check easily that $\mathfrak{g}_x = \mathfrak{g}_y$ if and only if $x = y$. This completes the proof. $\square$

The following two propositions give the solutions of Problem 1 when $\mathfrak{h}$ is 2-dimensional abelian and $\dim \mathfrak{p} = 3$.

Proposition 4.6. *Let* $((\mathfrak{h}, \omega, \varrho_{\mathfrak{h}}), (\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}}), \mu, \phi_{\mathfrak{h}}, \phi_{\mathfrak{p}})$ *be a solution of Problem 1 with* $\mathfrak{h}$ *is 2-dimensional abelian and* $\dim \mathfrak{p} = 3$ *and* $\phi_{\mathfrak{h}} = 0$. *Then one of the following situations occurs:*

(i) $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}})$ *is 3-dimensional Euclidean Lie algebra,* $\phi_{\mathfrak{p}} = 0$ *and* μ *is 2-cocycle for the trivial representation.*

(ii) $\phi_{\mathfrak{p}}$ *is an isomorphism of Lie algebras between* $(\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}})$ *and* $\mathrm{sl}(2, \mathbb{R})$ *and there exists an endomorphism* $L\colon \mathfrak{p} \longrightarrow \mathfrak{h}$ *such that for any* a, $b \in \mathfrak{p}$,

$$\mu(a, b) = \phi_{\mathfrak{p}}(a)(L(b)) - \phi_{\mathfrak{p}}(b)(L(a)) - L([a, b]_{\mathfrak{p}})\,.$$

(iii) *There exists a basis* $\mathbb{B}_{\mathfrak{p}} = (a_1, a_2, a_3)$ *of* $\mathfrak{p}$, $\alpha \neq 0$, $\beta \neq 0$, γ, $\tau \in \mathbb{R}$ *such that* $[\ ,\ ]_{\mathfrak{p}}$ *has one of the two following forms*

$$\begin{cases} [a_1, a_2]_{\mathfrak{p}} = 0\,,\ [a_1, a_3]_{\mathfrak{p}} = \beta a_1\,, \\ [a_2, a_3]_{\mathfrak{p}} = \gamma a_1 + \alpha a_2\,,\ \alpha \neq 0,\ \beta \neq 0 \\ M(\varrho_{\mathfrak{p}}, \mathbb{B}_{\mathfrak{p}}) = \mathrm{I}_3 \end{cases} \quad \textit{or} \quad \begin{cases} [a_1, a_2]_{\mathfrak{p}} = [a_1, a_3]_{\mathfrak{p}} = 0\,, \\ [a_2, a_3]_{\mathfrak{p}} = \alpha a_2\,,\ \alpha \neq 0\,, \\ M(\varrho_{\mathfrak{p}}, \mathbb{B}_{\mathfrak{p}}) = \begin{pmatrix} 1 & \tau & 0 \\ \tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

In both cases, there exists an orthonormal basis $\mathbb{B}_{\mathfrak{h}}=(e_1,e_2)$ of $\mathfrak{h}$, $x\neq 0$, $u\neq 0$ and $v\in\mathbb{R}$ such that $\phi_{\mathfrak{p}}$ has one of the following forms

$$\begin{cases} M(\phi_{\mathfrak{p}}(a_2),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}, \\ \mathcal{M}(\phi_{\mathfrak{p}}(a_3),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}-\frac{\alpha}{2} & v\\ 0 & \frac{\alpha}{2}\end{pmatrix}, \\ \phi_{\mathfrak{p}}(a_1)=0\,, \end{cases} \qquad \begin{cases} M(\phi_{\mathfrak{p}}(a_2),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}0 & 0\\ u & 0\end{pmatrix}, \\ M(\phi_{\mathfrak{p}}(a_3),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}\frac{\alpha}{2} & 0\\ v & -\frac{\alpha}{2}\end{pmatrix}, \\ \phi_{\mathfrak{p}}(a_1)=0\,, \end{cases} \quad or$$

$$\begin{cases} M(\phi_{\mathfrak{p}}(a_2),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}u & -\frac{u}{x}\\ ux & -u\end{pmatrix}, \\ M(\phi_{\mathfrak{p}}(a_3),\mathbb{B}_{\mathfrak{h}})=\begin{pmatrix}v & -\frac{2v+\alpha}{2x}\\ \frac{2v-\alpha}{2}x & -v\end{pmatrix}, \\ \phi_{\mathfrak{p}}(a_1)=0\,. \end{cases}$$

Moreover, μ is a 2-cocycle for $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\phi_{\mathfrak{p}})$.

(iv) *There exists an orthonormal basis $\mathbb{B}=(a_1,a_2,a_3)$ of $\mathfrak{p}$ such that $\phi_{\mathfrak{p}}(a_1)=\phi_{\mathfrak{p}}(a_2)=0$, $\phi_{\mathfrak{p}}(a_3)$ is a non zero element of* $\mathrm{sp}(\mathfrak{h},\omega)$ *and*

$$\begin{cases}[a_1,a_2]_{\mathfrak{p}}=0\,,\ [a_1,a_3]_{\mathfrak{p}}=\beta a_1+\rho a_2\,,\\ [a_2,a_3]_{\mathfrak{p}}=\gamma a_1+\alpha a_2\,,\end{cases} \qquad or \qquad \begin{cases}[a_1,a_2]_{\mathfrak{p}}=\alpha a_2\,,\ [a_1,a_3]_{\mathfrak{p}}=\rho a_2\,,\\ [a_2,a_3]_{\mathfrak{p}}=\gamma a_2,\alpha\neq 0\,.\end{cases}$$

Moreover, μ is a 2-cocycle for $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\phi_{\mathfrak{p}})$.

Proof. In this case, (19) is equivalent to $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}})$ is a Lie algebra and $\phi_{\mathfrak{p}}$ is a representation and μ is a 2-cocycle of $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\phi_{\mathfrak{p}})$.

We distinguish four cases:

(1) $\phi_{\mathfrak{p}}=0$ and the case (i) occurs.

(2) $\dim\phi_{\mathfrak{p}}(\mathfrak{p})=3$ and hence $\mathfrak{p}$ is isomorphic to $\mathrm{sp}(\mathfrak{h},\omega)\simeq\mathrm{sl}(2,\mathbb{R})$ and hence μ is a coboundary. Thus (ii) occurs.

(3) $\dim\phi_{\mathfrak{p}}(\mathfrak{p})=2$ then $\ker\phi_{\mathfrak{p}}$ is a one dimensional ideal of $\mathfrak{p}$. But $\phi_{\mathfrak{p}}(\mathfrak{p})$ is a 2-dimensional subalgebra of $\mathrm{sp}(\mathfrak{h},\omega)\simeq\mathrm{sl}(2,\mathbb{R})$, therefore it is non abelian so $\mathfrak{p}/\ker\mathfrak{p}$ is non abelian.

If $\ker\mathfrak{p}\subset[\mathfrak{p},\mathfrak{p}]_{\mathfrak{p}}$ then $\dim[\mathfrak{p},\mathfrak{p}]_{\mathfrak{p}}=2$ so there exists an orthonormal basis (a_1,a_2,a_3) of $\mathfrak{p}$ such that $a_1\in\ker\mathfrak{p}$ and

$$[a_1,a_2]_{\mathfrak{p}}=\xi a_1,\ [a_1,a_3]_{\mathfrak{p}}=\beta a_1 \quad\text{and}\quad [a_2,a_3]_{\mathfrak{p}}=\gamma a_1+\alpha a_2,\ \alpha\neq 0,\beta\neq 0$$

and we must have $\xi=0$ in order to have the Jacobi identity.

If $\ker\mathfrak{p}\not\subset[\mathfrak{p},\mathfrak{p}]$ then $\ker\mathfrak{p}\subset Z(\mathfrak{p})$ and $\dim[\mathfrak{p},\mathfrak{p}]=1$. Then there exits a basis (a_1,a_2,a_3) of $\mathfrak{p}$ such that $a_1\in\ker\mathfrak{p}$, $a_2\in[\mathfrak{p},\mathfrak{p}]$, $a_3\in\{a_1,a_2\}^{\perp}$ and

$$[a_2,a_3]_{\mathfrak{p}}=\alpha a_2,\ [a_3,a_1]_{\mathfrak{p}}=[a_1,a_2]_{\mathfrak{p}}=0,\ \alpha\neq 0.$$

The matrix of $\varrho_{\mathfrak{p}}$ in (a_1, a_2, a_3) is given by

$$\begin{pmatrix} 1 & \tau & 0 \\ \tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We choose an orthonormal basis (e_1, e_2) of $\mathfrak{h}$ and identify $\mathrm{sp}(\mathfrak{h}, \omega)$ to $\mathrm{sl}(2, \mathbb{R})$. Now $\phi_{\mathfrak{p}}(\mathfrak{p}) = \{\phi_p(a_2), \phi_p(a_3)\}$ is a subalgebra of $\mathrm{sl}(2, \mathbb{R})$ and, according to Proposition 4.5, $\phi_{\mathfrak{p}}(\mathfrak{p}) = \mathfrak{g}_1$, $\mathfrak{g}_2$ or $\mathfrak{g}_x$. But

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathbb{R}e, [\mathfrak{g}_2, \mathfrak{g}_2] = \mathbb{R}f \quad \text{and} \quad [\mathfrak{g}_x, \mathfrak{g}_x] = \left\{ \begin{pmatrix} u & -\frac{u}{x} \\ ux & -u \end{pmatrix} \right\}.$$

So in order for $\phi_{\mathfrak{p}}$ to be a representation we must have

$$\phi_{\mathfrak{p}}(a_2) = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \phi_{\mathfrak{p}}(a_3) = \begin{pmatrix} -\frac{\alpha}{2} & v \\ 0 & \frac{\alpha}{2} \end{pmatrix} \quad \text{and} \quad \phi_{\mathfrak{p}}(a_1) = 0,$$

$$\phi_{\mathfrak{p}}(a_2) = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}, \ \phi_{\mathfrak{p}}(a_3) = \begin{pmatrix} \frac{\alpha}{2} & 0 \\ v & -\frac{\alpha}{2} \end{pmatrix} \quad \text{and} \quad \phi_{\mathfrak{p}}(a_1) = 0,$$

or

$$\phi_{\mathfrak{p}}(a_2) = \begin{pmatrix} u & -\frac{u}{x} \\ ux & -u \end{pmatrix}, \ \phi_{\mathfrak{p}}(a_3) = \begin{pmatrix} p & -\frac{2p+\alpha}{2x} \\ \frac{2p-\alpha}{2}x & -p \end{pmatrix} \quad \text{and} \quad \phi_{\mathfrak{p}}(a_1) = 0.$$

(4) $\dim \phi_{\mathfrak{p}}(\mathfrak{p}) = 1$ then $\ker \phi_{\mathfrak{p}}$ is a two dimensional ideal of $\mathfrak{p}$. Then there exists an orthonormal basis (a_1, a_2, a_3) of $\mathfrak{p}$ such that

$$[a_1, a_2]_{\mathfrak{p}} = \alpha a_2, \ [a_3, a_1]_{\mathfrak{p}} = p a_1 + q a_2 \quad \text{and} \quad [a_3, a_2]_{\mathfrak{p}} = r a_1 + s a_2.$$

The Jacobi identity gives $\alpha = 0$ or $(p, r) = (0, 0)$. We take $\phi_{\mathfrak{p}}(a_1) = \phi_{\mathfrak{p}}(a_2) = 0$ and $\phi_{\mathfrak{p}}(a_3) \in \mathrm{sl}(2, \mathbb{R})$. $\square$

Proposition 4.7. *Let $((\mathfrak{h}, \omega, \varrho_{\mathfrak{h}}), (\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}}), \mu, \phi_{\mathfrak{h}}, \phi_{\mathfrak{p}})$ be a solution of Problem 1 with $\mathfrak{h}$ is 2-dimensional abelian, $\dim \mathfrak{p} = 3$ and $\phi_{\mathfrak{h}} \neq 0$. Then there exists an orthonormal basis (e_1, e_2) of $\mathfrak{h}$, an orthonormal basis (a_1, a_2, a_3) of $\mathfrak{p}$, $\lambda > 0$, $\alpha, p, q, \mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that*

$$\phi_{\mathfrak{h}}(e_1) = 0, \ \phi_{\mathfrak{h}}(e_2)(a_1) = \lambda a_2, \ \phi_{\mathfrak{h}}(e_2)(a_2) = -\lambda a_1 \quad \textit{and} \quad \phi_{\mathfrak{h}}(e_2)(a_3) = 0,$$

$$[a_1, a_2]_{\mathfrak{p}} = \alpha a_3, \ [a_1, a_3]_{\mathfrak{p}} = p a_1 + q a_2, [a_2, a_3]_{\mathfrak{p}} = -q a_1 + p a_2 \quad \textit{and}$$

$$\phi_{\mathfrak{p}}(a_i) = \begin{pmatrix} 0 & \mu_i \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3$$

and one of the following situations occurs:

(i) *$p \neq 0$, $\alpha = 0$ and*

$$\mu(a_1, a_2) = 0, \ \mu(a_2, a_3) = -\lambda^{-1}(p\mu_1 + q\mu_2)e_1 \quad \textit{and} \quad \mu(a_1, a_3) = \lambda^{-1}(-q\mu_1 + p\mu_2)e_1.$$

(ii) *$p = 0$, $\mu_3 \neq 0$, $\alpha = 0$ and*

$$\mu(a_1, a_2) = c e_1, \mu(a_2, a_3) = -\lambda^{-1}(p\mu_1 + q\mu_2)e_1 \quad \textit{and} \quad \mu(a_1, a_3) = \lambda^{-1}(-q\mu_1 + p\mu_2)e_1.$$

(iii) $p=0,\ \mu_3=0$ *and*

$$\mu(a_1,a_2)=c_1e_1+c_2e_2, \mu(a_2,a_3)=-\lambda^{-1}(p\mu_1+q\mu_2)e_1 \quad \textit{and}$$
$$\mu(a_1,a_3)=\lambda^{-1}(-q\mu_1+p\mu_2)e_1\,.$$

Proof. Since $\phi_{\mathfrak{h}}\neq 0$ then $\phi_{\mathfrak{h}}(\mathfrak{h})$ is a non trivial abelian subalgebra of so$(\mathfrak{p})$ and hence it must be one dimensional. Then there exists an orthonormal basis (e_1,e_2) of $\mathfrak{h}$ and an orthonormal basis (a_1,a_2,a_3) of $\mathfrak{p}$ and $\lambda>0$ such that $\phi_{\mathfrak{h}}(e_1)=0$ and

$$\phi_{\mathfrak{h}}(e_2)(a_1)=\lambda a_2,\ \phi_{\mathfrak{h}}(e_2)(a_2)=-\lambda a_1 \quad \text{and} \quad \phi_{\mathfrak{h}}(e_2)(a_3)=0\,.$$

The first equation in (19) is equivalent to

$$\phi_{\mathfrak{p}}(\phi_{\mathfrak{h}}(e_2)(a))(e_1)=0\,, \quad a\in\mathfrak{p}\,.$$

This is equivalent to

$$\phi_{\mathfrak{p}}(a_1)(e_1)=\phi_{\mathfrak{p}}(a_2)(e_1)=0\,.$$

Thus $\phi_{\mathfrak{p}}(a_i)=\begin{pmatrix}0&\mu_i\\0&0\end{pmatrix}$ for $i=1,2$ and $\phi_{\mathfrak{p}}(a_3)=\begin{pmatrix}u&v\\w&-u\end{pmatrix}$. Consider now the second equation in (19)

$$\phi_{\mathfrak{h}}(u)([a,b]_{\mathfrak{p}})=[a,\phi_{\mathfrak{h}}(u)(b)]_{\mathfrak{p}}+[\phi_{\mathfrak{h}}(u)(a),b]_{\mathfrak{p}}+\phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(b)(u))(a)-\phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(a)(u))(b)\,.$$

This equation is obviously true when $u=e_1$ and $(a,b)=(a_1,a_2)$. For $u=e_1$ and $(a,b)=(a_1,a_3)$, we get

$$\phi_{\mathfrak{h}}(\phi_{\mathfrak{p}}(a_3)(e_1))(a_1)=0$$

and hence $w=0$.

For $u=e_2$ and $(a,b)=(a_1,a_2)$, we get $\phi_{\mathfrak{h}}(e_2)([a_1,a_2]_{\mathfrak{p}})=0$ and hence $[a_1,a_2]_{\mathfrak{p}}=\alpha a_3$.

For $u=e_2$ and $(a,b)=(a_1,a_3)$ or $(a,b)=(a_2,a_3)$, we get

$$\phi_{\mathfrak{h}}(e_2)([a_1,a_3]_{\mathfrak{p}})=\lambda[a_2,a_3]_{\mathfrak{p}}-\lambda u a_2 \quad \text{and} \quad \phi_{\mathfrak{h}}(e_2)([a_2,a_3]_{\mathfrak{p}})=-\lambda[a_1,a_3]_{\mathfrak{p}}+\lambda u a_1.$$

This implies that $[a_1,a_3]_{\mathfrak{p}},[a_2,a_3]_{\mathfrak{p}}\in\operatorname{span}\{a_1,a_2\}$ and hence

$$[a_1,a_3]_{\mathfrak{p}}=pa_1+qa_2 \quad \text{and} \quad [a_2,a_3]_{\mathfrak{p}}=ra_1+sa_2.$$

So

$$\begin{cases}\lambda(pa_2-qa_1)=\lambda(ra_1+sa_2-ua_2)\,,\\ \lambda(ra_2-sa_1)=-\lambda(pa_1+qa_2-ua_1)\,.\end{cases}$$

This is equivalent to

$$u=0, p=s \quad \text{and} \quad r=-q\,.$$

To summarize, we get

$$[a_1,a_2]_{\mathfrak{p}}=\alpha a_3,\ [a_1,a_3]_{\mathfrak{p}}=pa_1+qa_2, [a_2,a_3]_{\mathfrak{p}}=-qa_1+pa_2 \quad \text{and}$$
$$\phi_{\mathfrak{p}}(a_i)=\begin{pmatrix}0&\mu_i\\0&0\end{pmatrix}\,.$$

Let consider now the fourth equation in (19)

$$\phi_{\mathfrak{p}}([a,b]_{\mathfrak{p}})(u)=[\phi_{\mathfrak{p}}(a),\phi_{\mathfrak{p}}(b)](u)+[u,\mu(a,b)]_{\mathfrak{h}}-\mu(a,\phi_{\mathfrak{h}}(u)(b))-\mu(\phi_{\mathfrak{h}}(u)(a),b).$$

This equation is obviously true for $u=e_1$.

For $u = e_2$ and $(a, b) = (a_1, a_2)$, $(a, b) = (a_1, a_3)$ or $(a, b) = (a_2, a_3)$, we get

$$\begin{cases} \alpha\mu_3 = 0, \\ (p\mu_1 + q\mu_2)e_1 = -\lambda\mu(a_2, a_3)\,, \\ (-q\mu_1 + p\mu_2)e_1 = \lambda\mu(a_1, a_3)\,. \end{cases}$$

The last two equations are equivalent to

$$\phi_{\mathfrak{p}}(a_3)(\mu(a_1, a_2)) = -2p\mu(a_1, a_2) \quad \text{and} \quad p[a_1, a_2]_{\mathfrak{p}} = 0\,.$$

- $p \neq 0$ then

$$\alpha = 0, \mu(a_1, a_2) = 0,\ \mu(a_2, a_3) = -\lambda^{-1}(p\mu_1 + q\mu_2)e_1 \quad \text{and}$$
$$\mu(a_1, a_3) = \lambda^{-1}(-q\mu_1 + p\mu_2)e_1\,.$$

- $p = 0$ and $\mu_3 \neq 0$ then $\alpha = 0$ and

$$\mu(a_1, a_2) = ce_1, \mu(a_2, a_3) = -\lambda^{-1}(p\mu_1 + q\mu_2)e_1 \quad \text{and}$$
$$\mu(a_1, a_3) = \lambda^{-1}(-q\mu_1 + p\mu_2)e_1\,.$$

- $p = 0$ and $\mu_3 = 0$ then

$$\mu(a_1, a_2) = c_1e_1 + c_2e_2, \mu(a_2, a_3) = -\lambda^{-1}(p\mu_1 + q\mu_2)e_1 \quad \text{and}$$
$$\mu(a_1, a_3) = \lambda^{-1}(-q\mu_1 + p\mu_2)e_1\,.$$

□

By using Propositions 4.2–4.7, we can give all the Riemann-Poisson Lie algebras of dimension 3, 4 or 5.

Let $(\mathfrak{g}, [\ ,\ ], \varrho, r)$ be a Riemann-Poisson Lie algebra of dimension less or equal to 5. According to what above then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and the Lie bracket on $\mathfrak{g}$ is given by (18) and $((\mathfrak{h}, \omega, \varrho_{\mathfrak{h}}), (\mathfrak{p}, [\ ,\ ]_{\mathfrak{p}}, \varrho_{\mathfrak{p}}), \mu, \phi_{\mathfrak{h}}, \phi_{\mathfrak{p}})$ are solutions of Problem 1.

- $\dim \mathfrak{g} = 3$. In this case $\dim \mathfrak{h} = 2$ and $\dim \mathfrak{p} = 1$ and, by applying Proposition 4.2, the Lie bracket of $\mathfrak{g}$, ϱ and r are given in Table 1, where $e^{12} = e_1 \wedge e_2$.

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	Conditions
$[e_1, e_2] = ae_1, [e_3, e_2] = be_1$	αe^{12}	I_3	$a \neq 0, \alpha \neq 0$
$[e_3, e_1] = -be_1 + ce_2, [e_3, e_2] = de_1 + be_2$	αe^{12}	I_3	$\alpha \neq 0$

TAB. 1. Three dimensional Riemann-Poisson Lie algebras

- $\dim \mathfrak{g} = 4$. We have three cases:

($c41$) $\dim \mathfrak{h} = 2$, $\dim \mathfrak{p} = 2$ and $\mathfrak{h}$ is non abelian and we can apply Proposition 4.3 to get the Lie brackets on $\mathfrak{g}$, ϱ and r. They are described in rows 1 and 2 in Table 2.

($c42$) $\dim \mathfrak{h} = 2$, $\dim \mathfrak{p} = 2$ and $\mathfrak{h}$ is abelian and we can apply Propositions 4.4 and 4.5 to get the Lie brackets on $\mathfrak{g}$, ϱ and r. They are described in rows 3 and 8 in Table 2.

($c43$) $\dim \mathfrak{h} = 4$. In this case $\mathfrak{g}$ is a Kähler Lie algebra. We have used [12] to derive all four dimensional Kähler Lie algebra together with their symplectic derivations. The results are given in Table 3. The notation $\mathrm{Der}^s(\mathfrak{h})$ stands for the vector spaces of derivations which are skew-symmetric with respect the symplectic form. The vector space $\mathrm{Der}^s(\mathfrak{h})$ is described by a family of generators and E_{ij} is the matrix with 1 in the i row and j column and 0 elsewhere.

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	Conditions
$[e_1,e_2]=ae_1,\ [e_3,e_2]=be_1+ce_4,$ $[e_4,e_2]=de_1-ce_3$	αe^{12}	I_4	$a\neq 0,\alpha\neq 0$
$[e_1,e_2]=ae_1,\ [e_3,e_2]=be_1,$ $[e_4,e_2]=de_1,[e_3,e_4]=ce_3-a^{-1}cbe_1$	αe^{12}	I_4	$\alpha ac\neq 0,$
$[e_3,e_4]=ae_1+be_2$	αe^{12}	I_4	$\alpha\neq 0$
$[e_3,e_4]=ae_1+be_2+ce_3,\ [e_4,e_1]=xe_1+ye_2,$ $[e_4,e_2]=ze_1-xe_2$	αe^{12}	I_4	$\alpha\neq 0$
$[e_3,e_4]=ae_1+be_2+2e_4,\ [e_3,e_1]=e_1,$ $[e_3,e_2]=-e_2,[e_4,e_2]=e_1$	αe^{12}	$\mathrm{Diag}\left(1,1,\begin{pmatrix}\mu&\nu\\ \nu&\rho\end{pmatrix}\right)$	$\alpha\neq 0,\mu,\rho>0$ $\mu\rho>\nu^2$
$[e_3,e_4]=ae_1+be_2-2e_4,\ [e_3,e_1]=e_1,$ $[e_3,e_2]=-e_2,[e_4,e_1]=e_2$	αe^{12}	$\mathrm{Diag}\left(1,1,\begin{pmatrix}\mu&\nu\\ \nu&\rho\end{pmatrix}\right)$	$\alpha\neq 0,\mu,\rho>0$ $\mu\rho>\nu^2$
$[e_3,e_4]=ae_1+be_2-2e_3,\ [e_3,e_1]=e_1+xe_2,$ $[e_3,e_2]=-\frac{1}{x}e_1-e_2,[e_4,e_1]=xe_2,[e_4,e_2]=\frac{1}{x}e_1$	αe^{12}	$\mathrm{Diag}\left(1,1,\begin{pmatrix}\mu&\nu\\ \nu&\rho\end{pmatrix}\right)$	$\alpha\neq 0,\mu,\rho>0$ $\mu\rho>\nu^2,x\neq 0$
$[e_3,e_4]=ae_1+be_2,\ [e_3,e_2]=xe_1+ye_4,$ $[e_4,e_2]=ze_1-ye_3$	αe^{12}	I_4	$\alpha y\neq 0$

TAB. 2. Four dimensional Riemann-Poisson Lie algebras of rank 2

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	$\mathrm{Der}^s(\mathfrak{h})$
$[e_1,e_2]=e_2,$	$\alpha e^{12}+\beta e^{34}$	$\mathrm{Diag}(a,b,c,d)$	$\{E_{21},E_{33}-E_{44},E_{43},E_{34}\}$
$[e_1,e_2]=-e_3,[e_1,e_3]=e_2,$	$\alpha e^{14}+\beta e^{23}$	$\mathrm{Diag}(a,b,b,c)$	$\{E_{23}-E_{32},E_{41}\}$
$[e_1,e_2]=e_2,[e_3,e_4]=e_4,$	$\alpha e^{12}+\beta e^{34}$	$\mathrm{Diag}(a,b,c,d)$	$\{E_{21},E_{43}\}$
$[e_4,e_1]=e_1,[e_4,e_2]=-\delta e_3,$ $[e_4,e_3]=\delta e_2$	$\alpha e^{14}+\beta e^{23}$	$\mathrm{Diag}(a,b,b,c)$	$\{E_{14},E_{23}-E_{32}\}$
$[e_1,e_2]=e_3,[e_4,e_3]=e_3,$ $[e_4,e_1]=\frac{1}{2}e_1,[e_4,e_2]=\frac{1}{2}e_2,,$	$\alpha(e^{12}-e^{34})$	$\mathrm{Diag}(a,\mu b,\mu a,b)$	$\{E_{34},E_{22}-E_{11},E_{12}+E_{21}\}$
$[e_1,e_2]=e_3,[e_4,e_3]=e_3,$ $[e_4,e_1]=2e_1,[e_4,e_2]=-e_2,$	$\alpha(e^{23}+e^{14})$	$\mathrm{Diag}(a,a,2a,2a)$	$\{2E_{14}-E_{32}\}$
$[e_1,e_2]=e_3,[e_4,e_3]=e_3,$ $[e_4,e_1]=\frac{1}{2}e_1-e_2,$ $[e_4,e_2]=e_1+\frac{1}{2}e_2,$	$\alpha(e^{12}-e^{34})$	$\mathrm{Diag}(a,a,a,a)$	$\{E_{34},E_{12}-E_{21}\}$

TAB. 3. Four-dimensional Kähler Lie algebras and their symplectic derivations, $a,b,c,d>0,\alpha\beta\neq 0$

- $\dim\mathfrak{g}=5$. We have:

($c51$) $\dim\mathfrak{h}=4$ and $\mathfrak{h}$ abelian and hence a symplectic vector space. We can apply Proposition 4.2 and $\mathfrak{g}$ is semi-direct product.

($c52$) $\dim\mathfrak{h}=4$ and $\mathfrak{h}$ non abelian. We can apply Proposition 4.2 and Table 3 to get the Lie brackets on $\mathfrak{g}$, ϱ and r. The result is summarized in Table 4.

($c53$) $\dim\mathfrak{h}=2$ and $\mathfrak{h}$ non abelian. We apply Proposition 4.3. In this case $(\mathfrak{p},[\ ,\ ]_{\mathfrak{p}},\varrho_{\mathfrak{p}})$ is a 3-dimensional Euclidean Lie algebra and one must compute $\mathrm{Der}(\mathfrak{p})\cap\mathrm{so}(\mathfrak{p})$ and solve (20). Three dimensional Euclidean Lie algebras were classified in [9]. For each of them we have computed $\mathrm{Der}(\mathfrak{p})\cap\mathrm{so}(\mathfrak{p})$

and solved (20) by using Maple. The result is summarized in Table 5 when $\mathfrak{p}$ is unimodular and Table 6 when $\mathfrak{p}$ is nonunimodular.

(c54) $\dim \mathfrak{h} = 2$ and $\mathfrak{h}$ abelian and $\phi_{\mathfrak{h}} = 0$. We apply Proposition 4.6 and we perform all the needed computations. We use the classification of 3-dimensional Euclidean Lie algebras given in [9]. The results are given in Tables 7-8.

(c55) $\dim \mathfrak{h} = 2$ and $\mathfrak{h}$ abelian and $\phi_{\mathfrak{h}} \neq 0$. We apply Proposition 4.7 and we perform all the needed computations. The results are given in Table 9.

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	Conditions
$[e_1, e_2] = e_2, [e_5, e_1] = xe_2,$ $[e_5, e_3] = ye_3 + te_4, [e_5, e_4] = ze_3 - ye_4$	$\alpha e^{12} + \beta e^{34}$	$\mathrm{Diag}(a, b, c, d, e)$	$\alpha\beta \neq 0$ $a, b, c, d, e > 0$
$[e_1, e_2] = -e_3, [e_1, e_3] = e_2,$ $[e_5, e_1] = ye_4, [e_5, e_2] = -xe_3, [e_5, e_3] = xe_2$	$\alpha e^{14} + \beta e^{23}$	$\mathrm{Diag}(a, b, b, c, d)$	$\alpha\beta \neq 0$ $a, b, c, d > 0$
$[e_1, e_2] = e_2, [e_3, e_4] = e_4,$ $[e_5, e_1] = xe_2, [e_5, e_3] = ye_4$	$\alpha e^{12} + \beta e^{34}$	$\mathrm{Diag}(a, b, c, d, e)$	$\alpha\beta \neq 0$ $a, b, c, d, e > 0$
$[e_4, e_1] = e_1, [e_4, e_2] = -\delta e_3, [e_4, e_3] = \delta e_2$ $[e_5, e_2] = -ye_3, [e_5, e_3] = ye_2, [e_5, e_4] = xe_1$	$\alpha e^{14} + \beta e^{23}$	$\mathrm{Diag}(a, b, b, c, d)$	$\alpha\beta \neq 0, \delta > 0$ $a, b, c, d > 0$
$[e_1, e_2] = e_3, [e_4, e_3] = e_3, [e_4, e_1] = \frac{1}{2}e_1$ $[e_4, e_2] = \frac{1}{2}e_2, [e_5, e_1] = xe_1 + ye_2,$ $[e_5, e_2] = ye_1 - xe_2, [e_5, e_4] = ze_3$	$\alpha(e^{12} - e^{34})$	$\mathrm{Diag}(a, \mu b, \mu a, b, c)$	$\alpha \neq 0$ $a, b, c, \mu > 0$
$[e_1, e_2] = e_3, [e_4, e_3] = e_3, [e_4, e_1] = 2e_1$ $[e_4, e_2] = -e_2, [e_5, e_2] = xe_3, [e_5, e_4] = -2xe_1$	$\alpha(e^{23} + e^{14})$	$\mathrm{Diag}(a, a, 2a, 2a, b)$	$\alpha \neq 0$ $a, b > 0$
$[e_1, e_2] = e_3, [e_4, e_3] = e_3, [e_4, e_1] = \frac{1}{2}e_1 - e_2$ $[e_4, e_2] = e_1 + \frac{1}{2}e_2, [e_5, e_1] = -xe_2, [e_5, e_2] = xe_1$ $[e_5, e_4] = ye_3$	$\alpha(e^{12} - e^{34})$	$\mathrm{Diag}(a, a, a, a, b)$	$\alpha \neq 0$ $a, b > 0$

TAB. 4. Five-dimensional Riemann-Poisson Lie algebras of rank 4

Non vanishing Lie brackets	r	Matrix of ϱ	Conditions
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1 - ce_4, [e_4, e_2] = d\mu e_1 + ce_3$ $[e_5, e_2] = fe_1, [e_3, e_4] = -fe_1 + e_5$	αe^{12}	$\mathrm{Diag}(1, \rho, \mu, \mu, 1)$	$c\alpha \neq 0$ $\mu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = be_1, [e_4, e_2] = ce_1$ $[e_5, e_2] = d\mu e_1, [e_3, e_5] = be_1 - e_3, [e_4, e_5] = -ce_1 + e_4$	αe^{12}	$\mathrm{Diag}(1, \rho, 1, 1, \mu)$	$\alpha \neq 0$ $\mu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = (b + c)e_1, [e_4, e_2] = (cx + b)e_1$ $[e_5, e_2] = d\mu e_1, [e_3, e_5] = (b + c)e_1 - e_3,$ $[e_4, e_5] = -(xc + b)e_1 + e_4$	αe^{12}	$\mathrm{Diag}\left(1, \rho, \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix}, \mu\right)$	$\alpha \neq 0$ $\mu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = be_1, [e_4, e_2] = c\mu e_1$ $[e_5, e_2] = d\nu e_1, [e_3, e_5] = -\mu ce_1 + e_4, [e_4, e_5] = be_1 - e_3$	αe^{12}	$\mathrm{Diag}(1, \rho, 1, \mu, \nu)$	$\alpha \neq 0$ $\mu, \nu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1, [e_4, e_2] = c\nu e_1$ $[e_5, e_2] = d\rho e_1, [e_3, e_4] = -2\rho de_1 + 2e_5,$ $[e_3, e_5] = 2\nu ce_1 - 2e_4, [e_4, e_5] = 2\mu be_1 - 2e_3$	αe^{12}	$\mathrm{Diag}(1, \xi, \mu, \nu, \rho)$	$\alpha \neq 0, \nu \neq \rho$ $\mu, \nu, \rho, \xi > 0$ $\mu \neq \nu, \mu \neq \rho$
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1, [e_4, e_2] = c\nu e_1 - \lambda e_5$ $[e_5, e_2] = d\nu e_1 + \lambda e_4, [e_3, e_4] = -\frac{2\nu(\lambda c+d)}{1+\lambda^2}e_1 + 2e_5,$ $[e_3, e_5] = \frac{2\nu(c-\lambda d)}{1+\lambda^2}e_1 - 2e_4, [e_4, e_5] = 2\mu be_1 - 2e_3$	αe^{12}	$\mathrm{Diag}(1, \rho, \mu, \nu, \nu)$	$\lambda\alpha \neq 0$ $\mu, \nu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1, [e_4, e_2] = c\nu e_1$ $[e_5, e_2] = d\rho e_1, [e_3, e_4] = -\rho de_1 + e_5,$ $[e_3, e_5] = \nu ce_1 - e_4, [e_4, e_5] = -\mu be_1 + e_3$	αe^{12}	$\mathrm{Diag}(1, \xi, \mu, \nu, \rho)$	$\alpha \neq 0, \nu \neq \rho$ $\mu, \nu, \rho, \xi > 0$ $\mu \neq \nu, \mu \neq \rho$
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1, [e_4, e_2] = c\nu e_1 - \lambda e_5$ $[e_5, e_2] = d\nu e_1 + \lambda e_4, [e_3, e_4] = -\frac{\nu(\lambda c+d)}{1+\lambda^2}e_1 + e_5,$ $[e_3, e_5] = \frac{\nu(c-\lambda d)}{1+\lambda^2}e_1 - e_4, [e_4, e_5] = -\mu be_1 + e_3$	αe^{12}	$\mathrm{Diag}(1, \rho, \mu, \nu, \nu)$	$\lambda\alpha \neq 0$ $\mu, \nu, \rho > 0$
$[e_1, e_2] = e_1, [e_3, e_2] = b\mu e_1 - ue_4 - ve_5,$ $[e_4, e_2] = c\mu e_1 + ue_3 - we_5, [e_5, e_2] = d\mu e_1 + ve_3 + we_4,$ $[e_3, e_4] = xe_1 + e_5, [e_3, e_5] = ye_1 - e_4, [e_4, e_5] = ze_1 + e_3$ $x = -\frac{\mu(buw-cuv+du^2+bv+cw+d)}{1+u^2+v^2+w^2}$ $y = \frac{\mu(-bvw+cv^2-duw+bu-dw+c)}{1+u^2+v^2+w^2}$ $z = -\frac{\mu(bw^2-cvw+duw-cu-dv+b)}{1+u^2+v^2+w^2}$	αe^{12}	$\mathrm{Diag}(1, \rho, \mu, \mu, \mu)$	$\alpha \neq 0$ $\mu, \rho > 0$

TAB. 5. Five-dimensional Riemann-Poisson Lie algebras of rank 2 with non abelian Kähler subalgebra and unimodular complement

Non vanishing Lie brackets	r	Matrix of ϱ
$[e_1,e_2]=e_1,[e_3,e_2]=(f+c\lambda+f\lambda^2)e_1-\lambda e_4,\lambda\neq 0,$ $[e_4,e_2]=ce_1+\lambda e_3,[e_5,e_2]=d\mu e_1,$ $[e_3,e_5]=fe_1-e_3,[e_4,e_5]=(\lambda f+c)e_1-e_4,$	αe^{12}	$\mathrm{Diag}(1,\rho,1,1,\mu)$ $\mu,\rho>0$
$[e_1,e_2]=e_1,[e_3,e_2]=be_1,[e_4,e_2]=c\mu e_1,$ $[e_5,e_2]=d\nu e_1,[e_3,e_5]=\mu ce_1-e_4,$ $[e_4,e_5]=(-fb+2\mu c)e_1+fe_3-2e_4, f=1$ or $f\leq 0$	αe^{12}	$\mathrm{Diag}(1,\rho,1,\mu,\nu)$ $0<\mu<\|f\|,\rho>0$
$[e_1,e_2]=e_1,[e_3,e_2]=(b+c\mu)e_1,[e_4,e_2]=(c+b\mu)e_1,$ $[e_5,e_2]=d\nu e_1,[e_3,e_5]=(\mu b+c)e_1-e_4,$ $[e_4,e_5]=((2-\mu)c+(2\mu-1)b)e_1+e_3-2e_4$	αe^{12}	$\mathrm{Diag}(1,\rho,\begin{pmatrix}1&\mu\\ \mu&1\end{pmatrix},\nu)$ $\mu,\nu,\rho>0$
$[e_1,e_2]=e_1,[e_3,e_2]=(b+c)e_1,[e_4,e_2]=(b+c\mu)e_1,$ $[e_5,e_2]=d\nu e_1,[e_3,e_5]=(b+c\mu)e_1-e_4,$ $[e_4,e_5]=((2-f)b+(2\mu-f)c)e_1+fe_3-2e_4$	αe^{12}	$\mathrm{Diag}(1,\rho,\begin{pmatrix}1&1\\ 1&\mu\end{pmatrix},\nu)$ $\nu,\rho>0,c>\mu>1$
$[e_1,e_2]=e_1,[e_3,e_2]=(b+\frac{1}{2}c)e_1,[e_4,e_2]=(c+\frac{1}{2}b)e_1,$ $[e_5,e_2]=d\nu e_1,[e_3,e_5]=(c+\frac{1}{2}b)e_1-e_4,$ $[e_4,e_5]=(b+2c)e_1-2e_4$	αe^{12}	$\mathrm{Diag}(1,\rho,\begin{pmatrix}1&\frac{1}{2}\\ \frac{1}{2}&1\end{pmatrix},\nu)$ $\rho,\nu>0$
$[e_1,e_2]=e_1,[e_3,e_2]=xe_1,[e_4,e_2]=ye_1,$ $[e_5,e_2]=d\nu e_1,[e_3,e_5]=ze_1-e_4,$ $[e_4,e_5]=te_1+fe_3-2e_4, 0<f<1,$ $x=\frac{((\mu+1)b+(\mu-1)c)f-2b}{2f^2(f-1)},y=z=\frac{(\mu-1)(cf+b)}{2f(f-1)}$ $t=\frac{(1-\mu)cf+((f-2)\mu+f)b}{2f(1-f)}$	αe^{12}	A^tBA $A=\begin{pmatrix}\frac{1+s}{-2fs}&-\frac{1}{2s}&0\\ \frac{1-s}{2fs}&\frac{1}{2s}&\\ 0&0&1\end{pmatrix}$ $B=\mathrm{Diag}(1,\rho,\begin{pmatrix}1&\mu\\ \mu&1\end{pmatrix},\nu)$ $\nu,\rho>0$ $s=\sqrt{1-f},\ 0\leq\mu<1$

TAB. 6. Five-dimensional Riemann-Poisson Lie algebras of rank 2 with non abelian Kähler subalgebra and non unimodular complement. ($\alpha\neq 0$)

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	Conditions
$[e_3,e_4]=ae_1+be_2+e_5,[e_3,e_5]=ce_1+de_2$ $[e_4,e_5]=fe_1+ge_2$	αe^{12}	$\mathrm{Diag}(1,1,\mu,\mu,1)$	$\alpha\neq 0$ $\mu>0$
$[e_3,e_4]=ae_1+be_2,[e_3,e_5]=ce_1+de_2-e_3$ $[e_4,e_5]=fe_1+ge_2+e_4$	αe^{12}	$\mathrm{Diag}(1,1,1,1,\mu)$ $\mathrm{Diag}(1,1,\begin{pmatrix}1&1\\ 1&x\end{pmatrix},\mu)$	$\alpha\neq 0$ $\mu>0$
$[e_3,e_4]=ae_1+be_2,[e_3,e_5]=ce_1+de_2+e_4$ $[e_4,e_5]=fe_1+ge_2-e_3$	αe^{12}	$\mathrm{Diag}(1,1,1,\mu,\nu)$	$\alpha\neq 0$ $\mu,\nu>0$
$[e_3,e_4]=ae_1+be_2+2e_5,[e_3,e_5]=ce_1+de_2-2e_4$ $[e_4,e_5]=fe_1+ge_2-2e_3$	αe^{12}	$\mathrm{Diag}(1,1,\mu,\nu,\rho)$	$\alpha\neq 0$ $\mu,\nu,\rho>0$
$[e_3,e_4]=ae_1+be_2+e_5,[e_3,e_5]=ce_1+de_2-e_4$ $[e_4,e_5]=fe_1+ge_2+e_3$	αe^{12}	$\mathrm{Diag}(1,1,\mu,\nu,\rho)$	$\alpha\neq 0$ $\mu,\nu,\rho>0$
$[e_3,e_5]=ce_1+de_2-e_3$ $[e_4,e_5]=fe_1+ge_2-e_4$	αe^{12}	$\mathrm{Diag}(1,1,1,1,\mu)$	$\alpha\neq 0$ $\mu>0$
$[e_3,e_5]=ce_1+de_2-e_4$ $[e_4,e_5]=fe_1+ge_2+xe_3-2e_4$	αe^{12}	There are many cases See [9]	$\alpha\neq 0$

TAB. 7. Five-dimensional Riemann-Poisson Lie algebras of rank 2 with abelian Kähler subalgebra

Non vanishing Lie brackets	r	Matrix of ϱ
$[e_3,e_1]=-e_2,[e_3,e_2]=e_1,[e_4,e_1]=e_2,[e_4,e_2]=e_1$ $[e_5,e_1]=e_1,[e_5,e_2]=-e_2,$ $[e_3,e_4]=2e_5+(l_{22}-l_{21}-2l_{13})e_1-(l_{12}+l_{11}+2l_{23})e_2$ $[e_3,e_5]=-2e_4+(l_{23}-l_{11}+2l_{12})e_1-(l_{13}-l_{21}-2l_{22})e_2,$ $[e_4,e_5]=-2e_3+(l_{23}-l_{12}+2l_{11})e_1+(l_{13}+l_{22}+2l_{21})e_2$	αe^{12}	$\mathrm{Diag}(1,1,\mu,\nu,\rho)$ $\mu,\nu,\rho>0$
$[e_4,e_2]=ue_1,[e_5,e_1]=-\frac{a}{2}e_1,[e_5,e_2]=ve_1+\frac{a}{2}e_2,$ $[e_3,e_4]=xe_1+ye_2,[e_3,e_5]=be_3+ze_1+te_2,$ $[e_4,e_5]=ce_3+ae_4+re_1+se_2,$ $(a+2b)x-2tu+2yv=0$,$a\neq 0,b\neq 0,(3a+2b)y=0$	αe^{12}	$\mathrm{Diag}(1,1,1,1,1)$
$[e_4,e_2]=ue_1,[e_5,e_1]=-\frac{a}{2}e_1,[e_5,e_2]=ve_1+\frac{a}{2}e_2,$ $[e_3,e_4]=xe_1,[e_3,e_5]=ze_1+te_2,$ $[e_4,e_5]=ae_4+re_1+se_2,a\neq 0,$ $ax-2tu=0$	αe^{12}	$\mathrm{Diag}(1,1,\begin{pmatrix}1&\mu\\\mu&1\end{pmatrix},1)$
$[e_4,e_1]=ue_2,[e_5,e_1]=\frac{a}{2}e_1+ve_2,[e_5,e_2]=-\frac{a}{2}e_2,$ $[e_3,e_4]=xe_1+ye_2,[e_3,e_5]=be_3+ze_1+te_2,$ $[e_4,e_5]=ce_3+ae_4+re_1+se_2,$ $(3a+2b)x=0,a\neq 0,b\neq 0$ $(a+2b)y-2zu+2xv=0$	αe^{12}	$\mathrm{Diag}(1,1,1,1,1)$
$[e_4,e_1]=ue_2,[e_5,e_1]=\frac{a}{2}e_1+ve_2,[e_5,e_2]=-\frac{a}{2}e_2,$ $[e_3,e_4]=ye_2,[e_3,e_5]=ze_1+te_2,$ $[e_4,e_5]=ae_4+re_1+se_2,a\neq 0,ay-2zu=0$	αe^{12}	$\mathrm{Diag}(1,1,\begin{pmatrix}1&\mu\\\mu&1\end{pmatrix},1)$
$[e_4,e_1]=ue_1+upe_2,[e_4,e_2]=-\frac{u}{p}e_1-ue_2,,$ $[e_5,e_1]=ve_1+\frac{(2v-a)p}{2}e_2,[e_5,e_2]=-\frac{(2v+a)}{2p}e_1-ve_2$ $[e_3,e_4]=xe_1+ye_2,[e_3,e_5]=be_3+ze_1+te_2,a\neq 0,b\neq 0$ $[e_4,e_5]=ce_3+ae_4+re_1+se_2,$ $((2a+2b+2v)x-2zu)p-ay+2tu-2yv=0$ $(2xv-ax-2zu)p+(2a+2b-2v)y+2tu=0$	αe^{12}	$\mathrm{Diag}(1,1,1,1,1)$
$[e_4,e_1]=ue_1+upe_2,[e_4,e_2]=-\frac{u}{p}e_1-ue_2,,$ $[e_5,e_1]=ve_1+\frac{(2v-a)p}{2}e_2,[e_5,e_2]=-\frac{(2v+a)}{2p}e_1-ve_2$ $[e_3,e_4]=xe_1+ye_2,[e_3,e_5]=ze_1+te_2,$ $[e_4,e_5]=ae_4+re_1+se_2,a\neq 0,b\neq 0$ $((2a+2v)x-2zu)p-ay+2tu-2yv=0$ $(2xv-ax-2zu)p+(2a-2v)y+2tu=0$	αe^{12}	$\mathrm{Diag}(1,1,\begin{pmatrix}1&\mu\\\mu&1\end{pmatrix},1)$
$[e_5,e_1]=ue_1+ve_2,[e_5,e_2]=we_1-ue_2,$ $[e_3,e_4]=xe_1+ye_2,[e_3,e_5]=ae_3+be_4+ze_1+te_2,$ $[e_4,e_5]=ce_3+de_4+re_1+se_2,$ $(a+d+u)x+yw=0$, $xv+(a+d-u)y=0$	αe^{12}	$\mathrm{Diag}(1,1,1,1,1)$
$[e_5,e_1]=ue_1+ve_2,[e_5,e_2]=we_1-ue_2,$ $[e_3,e_4]=xe_1+ye_2+ae_4,[e_3,e_5]=be_4+ze_1+te_2,$ $[e_4,e_5]=ce_4+re_1+se_2,a\neq 0$ $(c+u)x-ar+yw=0$ $(c-u)y-as+xv=0$	αe^{12}	$\mathrm{Diag}(1,1,1,1,1)$

TAB. 8. Five-dimensional Riemann-Poisson Lie algebras of rank 2 with abelian Kähler subalgebra ($\alpha\neq 0$) (Continued)

Non vanishing Lie brackets	Bivector r	Matrix of ϱ	Conditions
$[e_3,e_2]=xe_1-ae_4, [e_4,e_2]=ye_1+ae_3, [e_5,e_2]=ze_1$ $[e_3,e_5]=pe_3+qe_4+a^{-1}(-qx+py)e_1$, $[e_3,e_5]=-qe_3+pe_4-a^{-1}(px+qy)e_1$	αe^{12}	Diag(1, 1, 1, 1, 1)	$\alpha\neq 0$ $a\neq 0$
$[e_3,e_2]=xe_1-ae_4, [e_4,e_2]=ye_1+ae_3, [e_5,e_2]=ze_1$ $[e_3,e_4]=be_1$ $[e_3,e_5]=qe_4-a^{-1}qxe_1$, $[e_3,e_5]=-qe_3-a^{-1}qye_1$	αe^{12}	Diag(1, 1, 1, 1, 1)	$\alpha\neq 0$ $a\neq 0, z\neq 0$
$[e_3,e_2]=xe_1-ae_4, [e_4,e_2]=ye_1+ae_3$, $[e_3,e_4]=be_1+ce_2$ $[e_3,e_5]=qe_4-a^{-1}qxe_1$, $[e_3,e_5]=-qe_3-a^{-1}qye_1$	αe^{12}	Diag(1, 1, 1, 1, 1)	$\alpha\neq 0$ $a\neq 0$

Tab. 9. Five-dimensional Riemann-Poisson Lie algebras of rank 2 with abelian Kähler subalgebra (Continued)

This theorem unknown to our knowledge can be used to build examples of Riemann-Poisson Lie algebras.

Theorem 4.1. *Let $(G,\langle\ ,\ \rangle)$ be an even dimensional flat Riemannian Lie group. Then there exists a left invariant differential Ω on G such that $(G,\langle\ ,\ \rangle,\Omega)$ is a Kähler Lie group.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\varrho=\langle\ ,\ \rangle(e)$. According to Milnor's Theorem [11, Theorem 1.5] and its improved version [1, Theorem 3.1] the flatness of the metric on G is equivalent to $[\mathfrak{g},\mathfrak{g}]$ is even dimensional abelian, $[\mathfrak{g},\mathfrak{g}]^\perp=\{u\in\mathfrak{g},\mathrm{ad}_u+\mathrm{ad}_u^*=0\}$ is also even dimensional abelian and $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus[\mathfrak{g},\mathfrak{g}]^\perp$. Moreover, the Levi-Civita product is given by

$$\mathrm{L}_a=\begin{cases}\mathrm{ad}_a & \text{if}\quad a\in[\mathfrak{g},\mathfrak{g}]^\perp,\\ 0 & \text{if}\quad a\in[\mathfrak{g},\mathfrak{g}]\end{cases}\tag{21}$$

and there exists a basis $(e_1,f_1,\ldots,e_r,f_r)$ of $[\mathfrak{g},\mathfrak{g}]$ and $\lambda_1,\ldots,\lambda_r\in[\mathfrak{g},\mathfrak{g}]^\perp\setminus\{0\}$ such that for any $a\in[\mathfrak{g},\mathfrak{g}]^\perp$,

$$[a,e_i]=\lambda_i(a)f_i\quad\text{and}\quad[a,f_i]=-\lambda_i(a)e_i\,.$$

We consider a nondegenerate skew-symmetric 2-form ω_0 on $[\mathfrak{g},\mathfrak{g}]^\perp$ and ω_1 the nondegenerate skew-symmetric 2-form on $[\mathfrak{g},\mathfrak{g}]^\perp$ given by $\omega_1=\sum_{i=1}^r e_i^*\wedge f_i^*$. One can sees easily that $\omega=\omega_0\oplus\omega_1$ is a Kähler form on $\mathfrak{g}$. □

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[a]Université de Nouakchott,
Faculté des sciences et techniques
E-mail: pacha.ali86@gmail.com

[b]Université Cadi-Ayyad,
Faculté des sciences et techniques,
BP 549 Marrakech Maroc
E-mail: m.boucetta@uca.ac.ma

[c]Université de Nouakchott,
Faculté des sciences et techniques
E-mail: lessiadahmed@gmail.com