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FRACTIONAL q-DIFFERENCE EQUATIONS ON THE HALF LINE

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Abstract. This article deals with some results about the existence of solutions and bounded solutions and the attractivity for a class of fractional q-difference equations. Some applications are made of Schauder fixed point theorem in Banach spaces and Darbo fixed point theorem in Fréchet spaces. We use some technics associated with the concept of measure of noncompactness and the diagonalization process. Some illustrative examples are given in the last section.

1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences [4, 6, 7, 24, 31, 29, 30, 32] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivative; [6, 23].

Fractional q-difference equations initiated in the beginning of the 19th century [8, 15], and received significant attention in recent years. Some interesting details about initial and boundary value problems of q-difference and fractional q-difference equations can be found in [10, 11, 19, 20] and references therein.

In [1, 2, 3, 5, 6], Abbas et al. presented some results on the local and global attractivity of solutions for some classes of fractional differential equations involving both the Riemann-Liouville and the Caputo fractional derivatives by employing some fixed point theorems. Motivated by the above papers, in this article we discuss the existence and the attractivity of solutions for the following functional fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f\big(t, u(t)\big)\,; \quad t \in \mathbb{R}_+ := [0, +\infty)\,, \tag{1}$$

with the initial condition

$$u(0) = u_0\,, \tag{2}$$

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where $q \in (0,1)$, $\alpha \in (0,1]$, $u_0 \in \mathbb{R}$, $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and ${}^cD_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

Next, by using a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness, we discuss the existence of solutions for the problem (1)–(2) in Fréchet spaces, where $u_0 \in E$, $f\colon \mathbb{R}_+ \times E \to E$ is a given continuous function, and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

Finally, we discuss the existence of bounded solutions for the problem (1)–(2) on $\mathbb{R}_+$, by applying Schauder's fixed point theorem associated with the diagonalization process.

This paper initiates the study of Caputo fractional q-difference equations in Fréchet spaces, the attractivity and the boundedness of the solutions of fractional q-difference equations on the half line.

2. Preliminaries

Let $I := [0,T]$; $T > 0$. Consider the Banach space $C(I) := C(I,\mathbb{R})$ of continuous functions from I into $\mathbb{R}$ equipped with the usual supremum (uniform) norm

$$\|u\|_\infty := \sup_{t\in I} |u(t)| \,.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v\colon I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_0^T |v(t)|\,dt \,.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q} \,.$$

The q analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1 \,, \quad (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a-bq^k) \,; \quad a,b \in \mathbb{R} \,, \quad n \in \mathbb{N} \,.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty}\Big(\frac{a-bq^k}{a-bq^{k+\alpha}}\Big) \,; \quad a,b,\alpha \in \mathbb{R} \,.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

Definition 2.1 ([22])**.** The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}} \,; \quad \xi \in \mathbb{R} \setminus \{0,-1,-2,\ldots\} \,.$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 2.2 ([22])**.** The q-derivative of order $n \in \mathbb{N}$ of a function $u\colon I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t)-u(qt)}{(1-q)t} \,; \quad t \neq 0, \quad (D_q u)(0) = \lim_{t\to 0}(D_q u)(t) \,,$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t)\,; \quad t \in I, \quad n \in \{1, 2, \ldots\}\,.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 ([22])**.** The q-integral of a function $u\colon I_t \to \mathbb{R}$ is defined by

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n u(tq^n)\,,$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0)\,.$$

Definition 2.4 ([9])**.** The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u\colon I \to \mathbb{R}$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)} u(s) d_q s\,; \quad t \in I\,.$$

Note that for $\alpha = 1$, we have $(I_q^1 u)(t) = (I_q u)(t)$.

Lemma 2.5 ([27])**.** *For $\alpha \in \mathbb{R}_+$ and $\lambda \in (-1, \infty)$ we have*

$$(I_q^\alpha (t-a)^{(\lambda)}) = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda+\alpha)} (t-a)^{(\lambda+\alpha)}\,; \quad 0 < a < t < T\,.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}\,.$$

Definition 2.6 ([28])**.** The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u\colon I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{\lceil \alpha \rceil} I_q^{\lceil \alpha \rceil - \alpha} u)(t)\,; \quad t \in I\,,$$

where $\lceil \alpha \rceil$ is the smallest integer greater or equal to α.

Definition 2.7 ([28])**.** The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u\colon I \to \mathbb{R}$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$({}^C D_q^\alpha u)(t) = (I_q^{\lceil \alpha \rceil - \alpha} D_q^{\lceil \alpha \rceil} u)(t)\,; \quad t \in I\,.$$

Lemma 2.8 ([28])**.** *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0)\,.$$

In particular, if $\alpha \in (0, 1)$, then

$$(I_q^\alpha\, {}^C D_q^\alpha u)(t) = u(t) - u(0)\,.$$

From the above lemma, and in order to define the solution for the problem (1)–(2), we conclude with the following lemma.

Lemma 2.9. *Let $f\colon I \times \mathbb{R} \to \mathbb{R}$ be continuous. Then the problem* (1)–(2) *is equivalent to the problem of obtaining the solutions of the integral equation*

$$u(t) = u_0 + (I_q^\alpha f(\cdot, u(\cdot)))(t)\,.$$

3. Existence and attractivity results

By BC we denote the Banach space of all bounded and continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ equipped with the norm

$$\|u\|_{BC} := \sup_{t\in\mathbb{R}_+} |u(t)|\,.$$

Let $\emptyset \neq \Omega \subset BC$, and let $G\colon \Omega \to \Omega$, and consider the solutions of the equation

$$(Gu)(t) = u(t)\,. \tag{3}$$

We introduce the following concept of attractivity of solutions for equation (3).

Definition 3.1. Solutions of equation (3) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t)$ and $w = w(t)$ of equation (3) belonging to $B(u_0, \eta) \cap \Omega$, we have

$$\lim_{t\to\infty} (v(t) - w(t)) = 0\,. \tag{4}$$

When the limit (4) is uniform with respect to $B(u_0, \eta) \cap \Omega$, solutions of equation (3) are said to be uniformly locally attractive (or equivalently that solutions of (3) are locally asymptotically stable).

Lemma 3.2 ([16, p. 62])**.** *Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) *D is uniformly bounded in BC,*

(b) *The functions belonging to D are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact subset of $\mathbb{R}_+$,*

(c) *The functions from D are equiconvergent, that is, given $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to\infty} u(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $u \in D$.*

In the sequel we will make use of the following fixed point theorems.

Theorem 3.3 (Schauder fixed point theorem, [21])**.** *Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E, and let $N\colon Q \to Q$ be a compact and continuous map. Then N has at least one fixed point in Q.*

In this section, we are concerned with the existence and the attractivity of solutions of the problem (1)–(2).

Definition 3.4. By a solution of the problem (1)–(2) we mean a function $u \in BC$ that satisfies the equation (1) on I and the initial condition (2).

The following hypotheses will be used in the sequel.

(H_1) The function $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists a continuous function $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,u)| \le p(t)\,, \text{ for } t \in \mathbb{R}_+\,, \text{ and each } u \in \mathbb{R}\,,$$

and

$$\lim_{t\to\infty} (I_q^\alpha p)(t) = 0\,.$$

Set

$$p^* = \sup_{t\in\mathbb{R}_+} (I_q^\alpha p)(t)\,.$$

Now, we present a theorem concerning the existence and the attractivity of solutions of our problem (1)–(2).

Theorem 3.5. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem* (1)–(2) *has at least one solution defined on* $\mathbb{R}_+$. *Moreover, solutions of problem* (1)–(2) *are uniformly locally attractive.*

Proof. Consider the operator N such that, for any $u \in BC$,

$$(Nu)(t) = u_0 + \big(I_q^\alpha f\big(\cdot, u(\cdot)\big)\big)(t)\,. \tag{5}$$

The operator N maps BC into BC Indeed the map $N(u)$ is continuous on $\mathbb{R}_+$ for any $u \in BC$, and for each $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
|(Nu)(t)| &\le |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f\big(s,u(s)\big)| d_q s \\
&\le |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\le |u_0| + p^* \\
&= R\,.
\end{aligned}$$

Thus

$$\|N(u)\|_{BC} \le R\,. \tag{6}$$

Hence, $N(u) \in BC$, and the operator N maps the ball

$$B_R := B(0,R) = \{w \in BC : \|w\|_{BC} \le R\}$$

into itself.

From Lemma 2.9, the solutions of the problem (1)–(2) are the fixed points of the operator N. We shall show that the operator $N\colon B_R \to B_R$ satisfies all the assumptions of Theorem 3.3. The proof will be given in several steps.

Step 1. N *is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in \mathbb{R}_+$, we have

$$|(Nu_n)(t) - (Nu)(t)| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f\big(s,u_n(s)\big) - f\big(s,u(s)\big)| d_q s\,. \tag{7}$$

Case 1. If $t \in [0,T]$, $T > 0$, then, since $u_n \to u$ as $n \to \infty$ and f is continuous, by the Lebesgue dominated convergence theorem, equation (7) implies

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Case 2. If $t \in (T, \infty), T > 0$, then from the hypotheses and (7), we get

$$|(Nu_n)(t) - (Nu)(t)| \leq 2 \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \,. \tag{8}$$

Since $u_n \to u$ as $n \to \infty$ and $(I_q^\alpha p)(t) \to 0$ as $t \to \infty$, then (8) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \quad \text{as} \quad n \to \infty \,.$$

Step 2. *$N(B_R)$ is uniformly bounded.*
This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. *$N(B_R)$ is equicontinuous on every compact subset $[0,T]$ of $\mathbb{R}_+$; $T > 0$.*
Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$, and let $u \in B_R$. Set $\tilde{p}_* = \sup\limits_{t \in [0,T]} p(t)$. Then we have

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| \leq & \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f\big(s, u(s)\big)| d_q s \\
& + \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f\big(s, u(s)\big)| d_q s \\
\leq & \, \tilde{p}_* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
& + \tilde{p}_* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \,.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 4. *$N(B_R)$ is equiconvergent.*
Let $t \in \mathbb{R}_+$ and $u \in B_R$. Then we have

$$\begin{aligned}
|(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s))| d_q s \\
&\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\leq |u_0| + (I_q^\alpha p)(t).
\end{aligned}$$

Since $(I_q^\alpha p)(t) \to 0$, as $t \to +\infty$, we get

$$|(Nu)(t)| \to |u_0|, \quad \text{as} \quad t \to +\infty \,.$$

Hence,

$$|(Nu)(t) - (Nu)(+\infty)| \to 0 \,, \text{ as } \ t \to +\infty \,.$$

As a consequence of Steps 1 to 4, together with the Lemma 3.2, we can conclude that $N \colon B_R \to B_R$ is continuous and compact. From an application of Schauder's theorem (Theorem 3.3), we deduce that N has a fixed point u which is a solution of the problem (1)–(2) on $\mathbb{R}_+$.

Step 5. *The uniform local attractivity of solutions.*
Let us assume that u_1 is a solution of problem (1)–(2) with the conditions of this

theorem. Taking $u \in B(u_1, 2p^*)$, we have

$$\begin{aligned}|(Nu)(t) - u_1(t)| &= |(Nu)(t) - (Nu_1)(t)| \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s)) - f(s,u_1(s))| d_qs \\ &\leq 2 \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_qs \\ &\leq 2p^* .\end{aligned}$$

Thus, we get

$$\|N(u) - u_1\|_{BC} \leq 2p^* .$$

Hence, we obtain that N is a continuous function such that

$$N(B(u_1, 2p^*)) \subset B(u_1, 2p^*) .$$

Moreover, if u is a solution of problem (1)–(2), then

$$\begin{aligned}|u(t) - u_1(t)| &= |(Nu)(t) - (Nu_1)(t)| \\ &\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s)) - f(s,u_1(s))|\, ds \\ &\leq 2(I_q^\alpha p)(t) .\end{aligned}$$

Thus

$$|u(t) - u_1(t)| \leq 2(I_q^\alpha p)(t) \to 0 \quad \text{as} \quad t \to \infty .$$

Consequently, all solutions of problem (1)–(2) are uniformly locally attractive. □

4. Existence results in Fréchet spaces

Let $X := C(\mathbb{R}_+, E)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into a Banach space $(E, \|\cdot\|)$, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\| ; \quad n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\} ,$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1 + \|u-v\|_n} ; \quad u, v \in X .$$

Definition 4.1. A nonempty subset $B \subset X$ is said to be bounded if

$$\sup_{v\in B} \|v\|_n < \infty ; \quad \text{for} \quad n \in \mathbb{N}^* .$$

We recall the following definition of the notion of a sequence of measures of noncompactness [17, 18].

Definition 4.2. Let $\mathcal{M}_F$ be the family of all nonempty and bounded subsets of a Fréchet space F. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n \colon \mathcal{M}_F \to [0, \infty)$ is said to be a family of measures of noncompactness in the real Fréchet space F if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_F$:

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B)=0$ for $n\in\mathbb{N}$ if and only if B is precompact,
(b) $\mu_n(B_1)\le\mu_n(B_2)$ for $B_1\subset B_2$ and $n\in\mathbb{N}$,
(c) $\mu_n(\operatorname{Conv} B)=\mu_n(B)$ for $n\in\mathbb{N}$,
(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_F$ such that $B_{i+1}\subset B_i$; $i=1,\ldots$ and if $\lim\limits_{i\to\infty}\mu_n(B_i)=0$, for each $n\in\mathbb{N}$, then the intersection set $B_\infty:=\cap_{i=1}^{\infty}B_i$ is nonempty.

Example 4.3 ([17, 26])**.** For $B\in\mathcal{M}_X$, $x\in B$, $n\in\mathbb{N}$ and $\epsilon>0$, let us denote by $\omega^n(x,\epsilon)$ the modulus of continuity of the function x on the interval $[0,n]$; that is,

$$\omega^n(x,\epsilon)=\sup\{\|x(t)-x(s)\|:t,s\in[0,n],|t-s|\le\epsilon\}\,.$$

Further, let us put

$$\begin{aligned}\omega^n(B,\epsilon)&=\sup\{\omega^n(x,\epsilon):x\in B\}\,,\\ \omega_0^n(B)&=\lim_{\epsilon\to0^+}\omega^n(B,\epsilon)\,,\end{aligned}$$

and

$$\mu_n(B)=\omega_0^n(B)+\sup_{t\in[0,n]}\mu\big(B(t)\big)\,,$$

where μ is the Kuratowski measure of noncompactness on the space X.

The family of mappings $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n\colon\mathcal{M}_X\to[0,\infty)$, satisfies the conditions (a)–(d) from Definition 4.2.

Lemma 4.4 ([14])**.** *If Y is a bounded subset of a Banach space F, then for each $\epsilon>0$, there is a sequence $\{y_k\}_{k=1}^{\infty}\subset Y$ such that*

$$\mu(Y)\le2\mu(\{y_k\}_{k=1}^{\infty})+\epsilon\,,$$

where μ is the Kuratowski measure of noncompactness on F.

Lemma 4.5 ([25])**.** *Let E be a Banach space, and $\{u_k\}_{k=0}^{\infty}\subset L^1([0,n],E)$ be a uniformly integrable sequence, then $\mu(\{u_k\}_{k=1}^{\infty})$ is measurable, and*

$$\mu\Big(\Big\{\int_0^t u_k(s)ds\Big\}_{k=1}^{\infty}\Big)\le2\int_0^t\mu(\{u_k(s)\}_{k=1}^{\infty})\,ds\,,\quad \textit{for each}\ \ t\in[0,n]\,,$$

where μ is the Kuratowski measure of noncompactness on E.

Definition 4.6. Let Ω be a nonempty subset of a Fréchet space F, and let $A\colon\Omega\to F$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $\{k_n\}_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if

$$\mu_n\big(A(B)\big)\le k_n\mu_n(B)$$

for each bounded set $B\subset\Omega$ and $n\in\mathbb{N}$.
If $k_n<1;\ n\in\mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 4.7 ([17, 18]). *Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V\colon \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

Definition 4.8. Let $u_0 \in E$, and $f\colon \mathbb{R}_+ \times E \to E$ be a continuous function. By a solution of the problem (1)–(2) we mean a continuous function $u \in X$ that satisfies the equation (1) on $\mathbb{R}_+$ and the initial condition (2).

The following hypotheses will be used in the sequel.

(H_{01}) The function $t \mapsto f(t,u)$ is measurable on I for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$,

(H_{02}) There exists a continuous function $p\colon I \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le p(t)(1+\|u\|)\,; \quad \text{for a.e. } t\in I\,, \text{ and each } u \in E\,,$$

(H_{03}) For each bounded and measurable set $B \subset E$, and for each $t \in \mathbb{R}_+$, we have

$$\mu\big(f(t,B)\big) \le p(t)\mu(B)\,,$$

where μ is a measure of noncompactness on the Banach space E.

For $n \in \mathbb{N}^*$, let

$$p_n^* = \sup_{t\in[0,n]} p(t)\,,$$

and consider the family of measure noncompactness X as in Example 4.3.

Theorem 4.9. *Assume that hypotheses $(H_{01}) - -(H_{03})$ hold. If*

$$\frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)} < 1\,; \tag{9}$$

for each $n \in \mathbb{N}^*$, then the problem (1)–(2) has at least one solution in X.

Proof. Consider the operator $N\colon X \to X$ defined by (5). Clearly, the fixed points of the operator N are solution of the problem (1)–(2).

For any $n \in \mathbb{N}^*$, we set

$$R_n \ge \frac{\|u_0\|\Gamma_q(1+\alpha) + p_n^* n^\alpha}{\Gamma_q(1+\alpha) - p_n^* n^\alpha}\,,$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{w \in X : \|w\|_n \le R_n\}\,.$$

For any $n \in \mathbb{N}^*$, and each $u \in B_{R_n}$ and $t \in [0,n]$ we have

$$\begin{aligned}
\|(Nu)(t)\| &\le \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\|f(s,u(s))\|d_qs \\
&\le \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)(1+\|u(s)\|)d_qs \\
&\le \|u_0\| + p_n^*(1+R_n)\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs
\end{aligned}$$

$$\begin{aligned}&\le \|u_0\| + \frac{n^\alpha p_n^*}{\Gamma_q(1+\alpha)}(1+R_n)\\ &\le R_n\,.\end{aligned}$$

Thus

$$\|N(u)\|_n \le R_n\,. \tag{10}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N\colon B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 4.7. The proof will be given in several steps.

Step 1. *$N\colon B_{R_n} \to B_{R_n}$ is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$\|(Nu_k)(t) - (Nu)(t)\| \le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|f(s,u_k(s)) - f(s,u(s))\| d_q s\,.$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as} \quad k \to \infty\,.$$

Step 2. *$N(B_{R_n})$ is bounded.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each bounded and equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) \le \ell_n \mu_n(D)$.*
From Lemmas 4.4 and 4.5, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^\infty \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned}\mu\big((ND)(t)\big) &= \mu\Big(\Big\{u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u(s)) d_q s; \quad u \in D\Big\}\Big)\\ &\le 2\mu\Big(\Big\{\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,u_k(s)) d_q s\Big\}_{k=1}^\infty\Big) + \epsilon\\ &\le 4\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mu(\{f(s,u_k(s))\}_{k=0}^\infty) d_q s + \epsilon\\ &\le 4\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s)\mu(\{u_k(s)\}_{k=1}^\infty) d_q s + \epsilon\\ &\le \frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D) + \epsilon\,.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu\big((ND)(t)\big) \le \frac{4n^\alpha p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D)\,.$$

Thus

$$\mu_n\big(N(D)\big) \leq \frac{4n^{\alpha}p_n^*}{\Gamma_q(1+\alpha)}\mu_n(D)\,.$$

As a consequence of Steps 1 to 3 and inequality (9) together with Theorem 4.7, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (1)–(2). □

5. Existence of bounded solutions

In this section, we are concerned with the existence of bounded solutions of our problem

$$\begin{cases} ({}^cD_q^{\alpha}u)(t) = f(t,u(t))\,; & t \in \mathbb{R}_+\,,\\ u(0) = u_0 \in \mathbb{R}\,, & u \text{ is bounded on } \mathbb{R}_+\,, \end{cases} \tag{11}$$

Definition 5.1. By a bounded solution of the problem (11) we mean a measurable and bounded function u on $\mathbb{R}_+$ such that $u(0) = u_0$, and u satisfies the fractional q-difference equation $({}^cD_q^{\alpha}u)(t) = f\big(t,u(t)\big)$ on $\mathbb{R}_+$.

The following hypotheses will be used in the sequel.

(H_{11}) The function $t \mapsto f(t,u)$ is measurable on $I_n := [0,n]$; $n \in \mathbb{N}$ for each $u \in \mathbb{R}$, and the function $u \mapsto f(t,u)$ is continuous for a.e. $t \in I_n$,

(H_{12}) There exists a continuous function $p_n : I_n \to \mathbb{R}_+$ such that

$$|f(t,u)| \leq p_n(t)\,, \quad \text{for a.e.} \quad t \in I_n\,, \quad \text{and each} \quad u \in \mathbb{R}\,.$$

Set

$$p_n^* = \sup_{t\in I_n} p_n(t)\,.$$

Theorem 5.2. *Assume that the hypotheses (H_{11}) and (H_{12}) hold. Then the problem (11) has at least one bounded solution defined on $\mathbb{R}_+$.*

Proof. The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$\begin{cases} ({}^CD_q^{\alpha}u)(t) = f\big(t,u(t)\big)\,; & t \in I_n\,,\\ u(0) = u_0\,. \end{cases} \tag{12}$$

Part 1. We begin by showing that (12) has a solution $u_n \in C(I_n)$ with

$$\|u_n\|_{\infty} \leq R_n := \frac{n^{\alpha}p_n^*}{\Gamma_q(1+\alpha)}\,.$$

Consider the operator $N\colon C(I_n) \to C(I_n)$ defined by (5) Clearly, the fixed points of the operator N are solution of the problem (12).
For any $u \in C(I_n)$, and each $t \in I_n$ we have

$$\begin{aligned} |(Nu)(t)| &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs\\ &\leq |u_0| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p_n(s)d_qs \end{aligned}$$

$$\begin{aligned}
&\leq |u_0| + p_n^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\leq \frac{n^\alpha p_n^*}{\Gamma_q(1+\alpha)} .
\end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq R_n . \tag{13}$$

This proves that N transforms the ball $B_{R_n} := B(0, R_n) = \{w \in C(I_n) : \|w\|_\infty \leq R_n\}$ into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 3.3. The proof will be given in several steps.

Step 1. $N\colon B_{R_n} \to B_{R_n}$ *is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in I_n$, we have

$$\begin{aligned}
&|(Nu_k)(t) - (Nu)(t)| \\
&\leq \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u_k(s)) - f(s, u(s))| d_q s .
\end{aligned} \tag{14}$$

Since $u_k \to u$ as $k \to \infty$ and (H_{11}), then by the Lebesgue dominated convergence theorem, equation (14) implies

$$\|N(u_k) - N(u)\|_\infty \to 0 \quad \text{as} \quad k \to \infty .$$

Step 2. $N(B_{R_n})$ *is uniformly bounded.*
This is clear since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded.

Step 3. $N(B_{R_n})$ *is equicontinuous.*
Let t_1, $t_2 \in I_n$, $t_1 < t_2$ and let $u \in B_{R_n}$. Thus we have

$$\begin{aligned}
&|(Nu)(t_2) - (Nu)(t_1)| \\
&\leq \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s, u(s))| d_q s \\
&\quad + \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f(s, u(s))| d_q s \\
&\leq p_n^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s \\
&\quad + p_n^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s .
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Schauder's theorem (Theorem 3.3), we deduce that N has a fixed point u which is a solution of the problem (12).

Part 2. The diagonalization process.
Now, we use the following diagonalization process. For $k \in \mathbb{N}$ let

$$\begin{cases} w_k(t) = u_{n_k}(t)\,; & t \in [0, n_k]\,, \\ w_k(t) = u_{n_k}(n_k)\,; & t \in [n_k, \infty)\,. \end{cases}$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \ldots n_k < \ldots \uparrow \infty\,.$$

Let $S = \{w_k\}_{k=1}^{\infty}$. Notice that

$$|w_{n_k}(t)| \le R_n \quad \text{for} \quad t \in [0, n_1]\,,\ k \in \mathbb{N}\,.$$

Also, if $k \in \mathbb{N}$ and $t \in [0, n_1]$, we have

$$w_{n_k}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\big(s, w_{n_k}(s)\big) d_q s\,.$$

Thus, for $k \in \mathbb{N}$ and t, $x \in [0, n_1]$, we have

$$|w_{n_k}(t) - w_{n_k}(x)| \le \int_0^{n_1} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} |f\big(s, w_{n_k}(s)\big)| d_q s\,.$$

Hence

$$|w_{n_k}(t) - w_{n_k}(x)| \le p_1^* \int_0^{n_1} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s\,.$$

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a function $z_1 \in C([0, n_1], \mathbb{R})$ with $u_{n_k} \to z_1$ as $k \to \infty$ in $C([0, n_1], \mathbb{R})$ through $\mathbb{N}_1^*$. Let $\mathbb{N}_1 = \mathbb{N}_1^* \setminus \{1\}$.
Notice that

$$|w_{n_k}(t)| \le R_n \quad \text{for} \quad t \in [0, n_2],\ k \in \mathbb{N}\,.$$

Also, if $k \in \mathbb{N}$ and t, $x \in [0, n_2]$, we have

$$|w_{n_k}(t) - w_{n_k}(x)| \le p_2^* \int_0^{n_2} \frac{|(t-qs)^{(\alpha-1)} - (x-qs)^{(\alpha-1)}|}{\Gamma_q(\alpha)} d_q s\,.$$

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2 \in C([0, n_2], \mathbb{R})$ with $u_{n_k} \to z_2$ as $k \to \infty$ in $C([0, n_2], \mathbb{R})$ through $\mathbb{N}_2^*$. Note that $z_1 = z_2$ on $[0, n_1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* \setminus \{2\}$. Proceed inductively to obtain for $m = 3, 4, \ldots$ a subsequence $\mathbb{N}_m^*$ of $\mathbb{N}_{m-1}$ and a function $z_m \in C([0, n_m], \mathbb{R})$ with $u_{n_k} \to z_m$ as $k \to \infty$ in $C([0, n_m], \mathbb{R})$ through $\mathbb{N}_m^*$. Let $\mathbb{N}_m = \mathbb{N}_m^* \setminus \{m\}$.

Define a function y as follows. Fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $t \le n_m$. Then define $u(t) = z_m(t)$. Thus $u \in C([0, \infty), \mathbb{R}))$, $u(0) = u_0$ and $|u(t)| \le R_n$

for $t \in [0, \infty)$.
Again fix $t \in [0, \infty)$ and let $m \in \mathbb{N}$ with $t \le n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{n_k}(t) = u_0 + \int_0^{n_m} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\big(s, w_{n_k}(s)\big) d_q s\,.$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_m(t) = u_0 + \int_0^{n_m} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\big(s, z_m(s)\big) d_q s\,.$$

We can use this method for each $t \in [0, n_m]$ and for each $m \in \mathbb{N}$. Thus

$$({}^C D_q^\alpha u)(t) = f\big(t, u(t)\big); \quad \text{for} \quad t \in [0, n_m]$$

for each $m \in \mathbb{N}$ and the constructed function u is a solution of problem (11).

6. Some examples

Example 1. Consider the following problem of fractional $\frac{1}{4}$-difference equations

$$\begin{cases} ({}^c D_{\frac{1}{4}}^{\frac{1}{2}} u)(t) = f\big(t, u(t)\big)\,; & t \in \mathbb{R}_+\,, \\ u(0) = 1\,, \end{cases} \tag{15}$$

where

$$\begin{cases} f(t,u) = \frac{t^{\frac{-1}{4}} \sin t}{(1+\sqrt{t})(1+|u|)}\,; & t \in (0,\infty),\ u \in \mathbb{R}\,, \\ f(0,u) = 0\,; & u \in \mathbb{R}\,. \end{cases}$$

Clearly, the function f is continuous.
The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \frac{t^{\frac{-1}{4}} |\sin t|}{1+\sqrt{t}}\,; & t \in (0,\infty)\,, \\ p(0) = 0\,. \end{cases}$$

All conditions of Theorem 3.5 are satisfied. Hence, the problem (15) has at least one solution defined on $\mathbb{R}_+$, and solutions of this problem are uniformly locally attractive.

Example 2. Let

$$l^1 = \Big\{ u = (u_1, u_2, \dots, u_k, \dots) : \sum_{k=1}^{\infty} |u_k| < \infty \Big\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{k=1}^{\infty} |u_k|\,,$$

and $F := C(\mathbb{R}_+, l^1)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into l^1, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|_{l^1}\,;\ n \in \mathbb{N}^*\,.$$

Consider the following problem of fractional $\frac{1}{4}-$difference equations

$$(16)\qquad \begin{cases} ({}^{c}D^{\frac{1}{2}}_{\frac{1}{4}}u_k)(t) = f_k\big(t, u(t)\big)\,; & t \in \mathbb{R}_+\,, \\ u_k(0) = 0\,; & t \in \mathbb{R}_+,\ k \in \mathbb{N}\,, \end{cases}$$

where

$$f_k(t,u) = \frac{c_n(2^{-k} + u_k)t^{\frac{5}{4}}\sin t}{64(1+\sqrt{t})}\,; \quad u \in l^1,$$

for each $t \in [0,n]$; $n \in \mathbb{N}^*$, with

$$c_n = n^{-\frac{7}{4}}\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)\,; \quad n \in \mathbb{N}^*\,,$$

$$f = (f_1, f_2, \ldots, f_k, \ldots)\,, \qquad u = (u_1, u_2, \ldots, u_k, \ldots)\,.$$

Since

$$\|f(t,u)\|_{l^1} = \sum_{k=1}^{\infty}|f_k(s,u)| \le \frac{t^{\frac{5}{4}}c_n}{64}(1 + \|u\|_{l^1})\,; \quad t \in [0,n],\ n \in \mathbb{N}^*\,,$$

then hypothesis (H_{02}) is satisfied with

$$p(t) = \frac{t^{\frac{5}{4}}c_n}{64}\,; \quad t \in [0,n],\ n \in \mathbb{N}^*\,.$$

So, for any $n \in \mathbb{N}^*$, we have

$$p_n^* = \frac{n^{\frac{5}{4}}c_n}{64}\,.$$

The condition (9) is satisfied. Indeed;

$$\frac{4n^{\frac{1}{2}}p_n^*}{\Gamma_q(1+\alpha)} = n^{-\frac{7}{4}}\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)\frac{n^{\frac{5}{4}}}{64}\ \frac{4n^{\frac{1}{2}}}{\Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right)} = \frac{1}{16} < 1\,.$$

Therefore all conditions of Theorem 4.9 are satisfied. Hence, the problem (16) has at least one solution defined on $\mathbb{R}_+$.

Example 3. Consider the following problem of fractional $\frac{1}{4}$-difference equations

$$(17)\qquad \begin{cases} ({}^{C}D^{\frac{1}{2}}_{\frac{1}{4}}u)(t) = f(t, u(t))\,; & t \in \mathbb{R}_+, \\ u(0) = 2\,, & u \text{ is bounded on } \mathbb{R}_+\,, \end{cases}$$

where

$$f(t,u) = \frac{e^{t+1}}{1+|u|}(1+u)\,; \quad t \in \mathbb{R}_+\,.$$

The hypothesis (H_{12}) is satisfied with $p_n(t) = e^{t+1}$. So, $p_n^* = e^{n+1}$. Simple computations show that all conditions of Theorem 5.2 are satisfied. It follows that the problem (17) has at least one bounded solution defined on $\mathbb{R}_+$.

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