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STRUCTURE OF GEODESICS IN WEAKLY SYMMETRIC FINSLER METRICS ON H-TYPE GROUPS

Zdeněk Dušek

Abstract. Structure of geodesic graphs in special families of invariant weakly symmetric Finsler metrics on modified H-type groups is investigated. Geodesic graphs on modified H-type groups with the center of dimension 1 or 2 are constructed. The new patterns of algebraic complexity of geodesic graphs are observed.

1. Introduction

Let (M, F) be a Finsler manifold. If there is a connected Lie group G which acts transitively on M as a group of isometries, then M is called a *homogeneous manifold*. Homogeneous manifold M can be naturally identified with the *homogeneous space* G/H, where H is the isotropy group of the origin $p \in M$. A homogeneous Finsler space $(G/H, F)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi\colon G \to G/H = M$. Using this natural identification, from the Minkovski norm and its fundamental tensor on T_pM, we obtain the $\mathrm{Ad}(H)$-invariant Minkowski norm and the $\mathrm{Ad}(H)$-invariant fundamental tensor on $\mathfrak{m}$ and we denote these again by F and g.

We further recall that the *slit tangent bundle* TM_0 is defined as $TM_0 = TM \setminus \{0\}$. Using the restriction of the natural projection $\pi\colon TM \to M$ to TM_0, we naturally construct the pullback vector bundle π^*TM over TM_0. The *Chern connection* is the unique linear connection on the vector bundle π^*TM which is torsion free and almost g-compatible, see some monograph, for example [2] by D. Bao, S.-S. Chern and Z. Shen or [5] by S. Deng for details. Using the Chern connection, the derivative along a curve $\gamma(t)$ can be defined. A regular smooth curve γ with tangent vector field T is a *geodesic* if $D_T(\frac{T}{F(T)}) = 0$. In particular, a geodesic of constant speed satisfies $D_TT = 0$. A geodesic $\gamma(s)$ through the point p is *homogeneous* if it is

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an orbit of a one-parameter group of isometries. More explicitly, if there exists a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector X is called a *geodesic vector*. Geodesic vectors are characterized by the following *geodesic lemma.*

Lemma 1 ([14]). *Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. A nonzero vector $Y \in \mathfrak{g}$ is geodesic vector if and only if it holds*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, U]_{\mathfrak{m}}) = 0 \qquad \forall U \in \mathfrak{m}\,, \tag{1}$$

where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.

Definition 2. A homogeneous space $(G/H, F)$ is called a Finsler g.o. space, if each geodesic of $(G/H, F)$ (with respect to the Chern connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}$, $Z \in \mathfrak{g}$, of the group of isometries G.

We remark that a homogeneous manifold (M, F) may admit more presentations as a homogeneous space in the form G/H, corresponding to various transitive isometry groups. In a g.o. space G/H, we investigate some sets of geodesic vectors which generate all geodesics through a fixed point. Those sets which are reasonable in a good sense are called geodesic graphs. The first concept originated from the work of J. Szenthe [18].

Definition 3. Let $(G/H, F)$ be a g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A geodesic graph is an $\mathrm{Ad}(H)$-equivariant map $\xi\colon \mathfrak{m} \to \mathfrak{h}$ such that $X + \xi(X)$ is a geodesic vector for each $o \neq X \in \mathfrak{m}$.

It often happens that the vector $\xi(X)$ is uniquely determined. Then the map ξ is $\mathrm{Ad}(H)$-equivariant and we are interested in the algebraic structure of the mapping ξ. Sometimes, there are more choices for the vector $\xi(X)$. Then we want to choose it in a way that the algebraic structure of the mapping ξ is as simple as possible.

Theory of Riemannian geodesic graphs (canonical and general) was developed and examples of geodesic graphs on compact and noncompact g.o. manifolds and also on g.o. nilmanifolds in dimensions 5,6 and 7 were described by O. Kowalski and S. Nikčević in [12]. Geodesic graph is either linear, which is equivalent with the natural reductivity of the space G/H, or its components are rational functions $\xi_i = P_i/P$, where P_i and P are homogeneous polynomials and $\deg(P_i) = \deg(P)+1$. The *degree of the geodesic graph* ξ is defined as $\deg(\xi) = \deg(P)$. The special situation of geodesic graph of degree 0 corresponds to the linear geodesic graph. If the geodesic graph is unique, its degree is also the *degree of the g.o. space G/H*. If there are more geodesic graphs in G/H, the degree of the g.o. space is the minimum of the degrees of these geodesic graphs.

The degree of the mentioned examples in [12] is 0 (linear geodesic graph) or 2. Further geodesic graphs of degree 0 or 2 on H-type groups were described by the author in [6]. Geodesic graph of degree 4 on the flag manifold $SO(7)/U(3)$ was constructed by the author in [7]. In [10], the author with O. Kowalski constructed the canonical geodesic graph of degree 6 and a general geodesic graph of degree 3 on the H-type group of dimension 13 with 5-dimensional center.

We recall that in dimension ≤ 5, all Riemannian g.o. manifolds (M, g) are naturally reductive, hence they admit a presentation $M = G/H$ in which a linear geodesic graph exist. Equivalently, they admit a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that all vectors from $\mathfrak{m}$ are geodesic. In dimension 6, all g.o. manifolds which are not naturally reductive were classified by O. Kowalski and L. Vanhecke in [13]. Interesting compact Riemannian g.o. manifolds are, for example, the two series of flag manifolds described by D. Alekseevsky, A. Arvanitoyeorgos in [1]. Interesting Riemannian g.o. nilmanifolds are the modified H-type groups, see Section 3 for details. For a more detailed exposition about geodesic graphs in Riemannian g.o. manifolds, some related topics and further references, we refer the reader to the recent survey paper [8] by the author. Another structural approach to Riemannian g.o. manifolds using the Lie theory can be found in the recent papers [11] and [16] by C.S. Gordon and Yu.G. Nikonorov.

In [19], Z. Yan and S. Deng studied Finsler g.o. spaces and their relation with Riemannian g.o. spaces. Some particular results were obtained for Randers g.o. metrics and for weakly symmetric metrics. Nilpotent examples of reversible Finsler g.o. spaces which are neither Berwaldian nor weakly symmetric and examples of invariant Randers g.o. metrics on spheres S^{2n+1} were constructed in this paper.

In [9], the author investigated invariant Randers g.o. metrics on modified H-type groups and constructed Finslerian geodesic graphs on these g.o. manifolds. In all these examples, geodesic graph is unique. The simplest geodesic graph of the Randers metric is a cone. This situation occurs for H-type groups whose underlying Riemannian metric α is naturally reductive. In other cases, geodesic graph of the Randers g.o. metrics arise as the Riemannian geodesic graph with a deformation term in the numerators.

In the present paper, special families of weakly symmetric Finsler metrics will be considered. A Finsler manifold (M, F) is *weakly symmetric* if for any two points x, $y \in M$ there exist an isometry g of M such that $g(x) = y$ and $g(y) = x$. It is well known that weakly symmetric Finsler metrics are g.o. metrics, see [3] for Riemannian metrics and [5] for Finsler metrics. We shall focus on the special weakly symmetric metrics on modified H-type groups which were studied also in [19]. We shall construct geodesic graphs on 5-dimensional and 6-dimensional H-type groups with these Finsler metrics and compare it with Riemannian geodesic graphs constructed in [12] and with Randers geodesic graphs constructed in [9].

2. Modified H-type groups

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be its orthogonal complement. For each vector $Z \in \mathfrak{z}$, define the operator $J_Z \colon \mathfrak{v} \to \mathfrak{v}$ by the formula

$$\langle J_Z(X), Y \rangle = \langle Z, [X, Y] \rangle \qquad \forall X, Y \in \mathfrak{v} .$$

The algebra n is called a *modified H-type algebra* if, for each $o \neq Z \in \mathfrak{z}$, the operator J_Z satisfies the identity

$$(J_Z)^2 = \lambda(Z) \cdot \mathrm{id}_{\mathfrak{v}}$$

for some $\lambda(Z) < 0$. A connected and simply connected Lie group whose Lie algebra is a modified H-type algebra is diffeomorphic to $\mathbb{R}^n$ and it is called a *modified H-type group*. It is endowed with a left-invariant Riemannian metric. The special case of a *generalized Heisenberg algebra (H-type algebra)* and corresponding *generalized Heisenberg group (H-type group)* is obtained for

$$\lambda(Z) = -\langle Z, Z\rangle\,.$$

It was proved by J. Lauret in [15], that modified H-type algebras are just the pairs $(\mathfrak{n}, \langle\,,\,\rangle_S)$, where $(\mathfrak{n}, \langle\,,\,\rangle)$ is an H-type algebra and S is a positive definite symmetric transformation of $\mathfrak{z}$ which determines the inner product $\langle\,,\,\rangle_S$ by the formula

$$\langle X+U, Y+V\rangle_S = \langle X, Y\rangle + \langle S(U), V\rangle \qquad \forall X, Y \in \mathfrak{v}, \quad \forall U, V \in \mathfrak{z}\,.$$

H-type algebras are completely classified, see for example the book [4] by J. Berndt, F. Tricerri and L. Vanhecke. For each dimension of the center $\mathfrak{z}$, there is a series of H-type algebras. Each algebra of the series contains the center $\mathfrak{z}$ and the complement $\mathfrak{v}$ which decomposes into irreducible $\mathfrak{z}$-modules (the operators J_Z make $\mathfrak{v}$ a $\mathfrak{z}$-module). Irreducible $\mathfrak{z}$-modules are all equivalent if $\dim\mathfrak{z} \neq 3 \pmod 4$, otherwise there exist two nonequivalent irreducible modules of the same dimension (called nonisotypic modules). The classification of H-type groups which are Riemannian g.o. manifolds was obtained by C. Riehm in [17]. The refinement of this classification for modified H-type groups was obtained by J. Lauret in [15]: Modified H-type groups with Riemannian g.o. metrics are:
– all infinite series with $\dim\mathfrak{z} \in \{1, 2, 3\}$ or
– isolated low-dimensional cases with $\dim\mathfrak{z} \in \{5, 6, 7\}$.
Further, naturally reductive modified H-type groups are:
– the infinite series with $\dim\mathfrak{z} = 1$ or
– special infinite series with $\dim\mathfrak{z} = 3$.
In particular, modified H-type groups with invariant Riemannian g.o. metrics are weakly symmetric, up to special situations with $\dim\mathfrak{z} \in \{3, 7\}$. See [15] for details of the classification. In the present paper, we examine a class of special weakly symmetric Finsler g.o. metrics on low-dimensional modified H-type groups with $\dim\mathfrak{z} \in \{1, 2\}$. We construct geodesic graphs for these metrics to compare it with geodesic graphs of Riemannian and Randers g.o. metrics.

3. Weakly symmetric metrics on nilpotent groups

We shall consider a special class of weakly symmetric Finsler metrics on 2-step nilpotent Lie groups, which are described for example in [19]. On a 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$, consider coordinates $(x_i)_{i=1}^r$ on $\mathfrak{v}$ and $(z_k)_{k=1}^s$ on $\mathfrak{z}$ (where $r = \dim\mathfrak{v}$, $s = \dim\mathfrak{z}$). For $\varepsilon \geq 0$ and $p \in \mathbb{N}$, define the Finsler function in the form

$$\begin{aligned} F(x_i, z_k) &= \sqrt{\sum_{i=1}^r x_i^2 + \sum_{k=1}^s z_k^2 + \varepsilon \cdot \sqrt[p]{(\sum_{i=1}^r x_i^2)^p + (\sum_{k=1}^s z_k^2)^p}} \\ &= \sqrt{\|x\|^2 + \|z\|^2 + \varepsilon M^{\frac{1}{p}}}\,. \qquad (2) \end{aligned}$$

Here we put, for short,

$$\|x\|^2 = \sum_{i=1}^{r} x_i^2\,, \qquad \|z\|^2 = \sum_{k=1}^{s} z_k^2\,, \qquad M = (\sum_{i=1}^{r} x_i^2)^p + (\sum_{k=1}^{s} z_k^2)^p\,.$$

Obviously, for $\varepsilon = 0$ or for $p = 1$ the above Finsler norm is Euclidean and it gives rise to an invariant Riemannian metric α. In general, it gives rise to an invariant Finsler metric F on N. It was proved in [19] that the isometry group of (M, F) is the same as the isometry group of the underlying Riemannian metric α and (M, F) is weakly symmetric if and only if the Riemannian metric α is weakly symmetric.

For i, $j = 1 \dots r$ and k, $l = 1 \dots s$, we obtain by the direct calculations the components of the funtamental tensor $g(x_i, z_k)$ in the form

$$\begin{aligned}
g_{ii} &= \frac{1}{2}(F^2)_{x_i x_i} = 1 + \varepsilon \frac{(\|x\|^2)^{p-1}M + 2(p-1)x_i^2(\|x\|^2)^{p-2}(\|z\|^2)^p}{M^{2-\frac{1}{p}}}\,,\\
g_{r+k,r+k} &= \frac{1}{2}(F^2)_{z_k z_k} = 1 + \varepsilon \frac{(\|z\|^2)^{p-1}M + 2(p-1)z_i^2(\|z\|^2)^{p-2}(\|x\|^2)^p}{M^{2-\frac{1}{p}}}\,,\\
g_{ij} &= \frac{1}{2}(F^2)_{x_i x_j} = \frac{2\varepsilon(p-1)x_i x_j(\|x\|^2)^{p-2}(\|z\|^2)^p}{M^{2-\frac{1}{p}}}\,,\\
g_{r+k,r+l} &= \frac{1}{2}(F^2)_{z_k z_l} = \frac{2\varepsilon(p-1)z_k z_l(\|x\|^2)^{p}(\|z\|^2)^{p-2}}{M^{2-\frac{1}{p}}}\,,\\
g_{i,r+k} &= \frac{1}{2}(F^2)_{x_i z_k} = \frac{-2\varepsilon(p-1)x_i z_k(\|x\|^2)^{p-1}(\|z\|^2)^{p-1}}{M^{2-\frac{1}{p}}}\,.
\end{aligned}$$

For the later use, we calculate already now the components of the vector obtained by the contraction $g_Y(Y, .)$, which are $\sum_{i=1}^{r} x_i g_{ij} + \sum_{k=1}^{s} z_k g_{r+k,j}$ for $j = 1 \dots r$ and $\sum_{i=1}^{r} x_i g_{i,r+l} + \sum_{k=1}^{s} z_k g_{r+k,r+l}$ for $l = 1 \dots s$, respectively. We obtain the components of this vector in the form

$$\Big(x_j + \frac{\varepsilon x_j(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}, z_l + \frac{\varepsilon z_l(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)\,. \tag{3}$$

Here components for $j = 1, \dots, r$ are followed by the components for $l = 1, \dots, s$. We shall use this formula later in the application of geodesic lemma.

4. Geodesic graphs

4.1. **Dim($\mathfrak{z}$)=1.** We consider first the 5-dimensional modified H-type group with 1-dimensional center and two 2-dimensional $\mathfrak{z}$-modules $\mathfrak{v}_i$. Let us consider the Lie algebra $\mathfrak{n}$ with the scalar product $\langle\, , \rangle$ determined by the orthonormal basis $B = \{E_1, \dots, E_4, Z\}$ and generated by the nontrivial relations

$$[E_1, E_2] = Z\,, \qquad [E_3, E_4] = \mu Z\,, \qquad \mu > 0\,.$$

We obtain the 1-parameter family of modified H-type algebras and corresponding modified H-type groups (N, α) with the invariant Riemannian metric induced by the above scalar product, in the sense of J. Lauret [15]. Let us further denote by A_{ij} the elementary operators on $\mathfrak{n}$ with the action generated by the relations

$$A_{ij}(E_i) = E_j\,, \quad A_{ij}(E_j) = -E_i\,, \quad A_{ij}(E_k) = 0\,, \quad i < j\,,\ i \neq k \neq j\,.$$

It is easy to verify that the operators $D_1 = A_{12}$ and $D_2 = A_{34}$ act as skew-symmetric derivations on $\mathfrak{n}$. Because these two operators commute, it holds $\mathfrak{h} = \operatorname{span}(D_1, D_2) \simeq \mathfrak{so}(2) \times \mathfrak{so}(2)$. We put $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and we express each group N in the form $N = G/H$, where $H = \mathrm{SO}(2) \times \mathrm{SO}(2)$ and $G = N \rtimes H$.

We now introduce a Finsler metric on each group N, by the function F in formula (2) with respect to the basis B. Obviously, for $\varepsilon = 0$, it reduces to the norm determined by the Riemannian metric α above. The isometry group of the Finsler manifold (M, F) is also $G = N \rtimes H$, see for example [19].

We shall describe first the geodesic graph in the reductive decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. We put $Y = X + \xi(X)$, where $X = x_1 E_1 + \cdots + x_4 E_4 + zZ \in \mathfrak{n}$ and $\xi(X) = \xi_1 D_1 + \xi_2 D_2 \in \mathfrak{h}$, where ξ_1, ξ_2 will be determined to satisfy the geodesic lemma. We write down the Lie brackets

$$
\begin{aligned}
[X + \xi(X), E_1]_{\mathfrak{n}} &= -x_2 Z + \xi_1 E_2\,,\\
[X + \xi(X), E_2]_{\mathfrak{n}} &= x_1 Z - \xi_1 E_1\,,\\
[X + \xi(X), E_3]_{\mathfrak{n}} &= -x_4 \mu Z + \xi_2 E_4\,,\\
[X + \xi(X), E_4]_{\mathfrak{n}} &= x_3 \mu Z - \xi_2 E_3\,.
\end{aligned} \tag{4}
$$

Now we use geodesic lemma, where we substitute, step by step, $U = E_1, \dots, E_4, Z$. Using expressions (4) and formula (3), we obtain the system of five equations, which reduces to the system of two equations

$$
\begin{aligned}
\xi_1\Big(1 + \varepsilon \frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big) &= z\Big(1 + \varepsilon \frac{z^{2(p-1)}}{M^{1-\frac{1}{p}}}\Big)\,,\\
\xi_2\Big(1 + \varepsilon \frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big) &= \mu z\Big(1 + \varepsilon \frac{z^{2(p-1)}}{M^{1-\frac{1}{p}}}\Big)\,.
\end{aligned}
$$

As the unique solution, we obtain the compoments of the geodesic graph

$$
\begin{aligned}
\xi_1 &= z \frac{M^{1-\frac{1}{p}} + \varepsilon z^{2(p-1)}}{M^{1-\frac{1}{p}} + \varepsilon (\|x\|^2)^{p-1}}\,,\\
\xi_2 &= \mu z \frac{M^{1-\frac{1}{p}} + \varepsilon z^{2(p-1)}}{M^{1-\frac{1}{p}} + \varepsilon (\|x\|^2)^{p-1}}\,.
\end{aligned} \tag{5}
$$

For $\varepsilon = 0$ or for $p = 1$, our Finsler metric is Riemannian and geodesic graph reduces to the linear map with components $\xi_1 = z$, $\xi_2 = \mu z$. The existence of a linear geodesic graph is equivalent with the natural reductivity of the Riemannian metric. However, the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ above is not the naturally reductive one. We are now going to construct the geodesic graph in the naturally reductive decomposition of the underlying Riemannian metric α. We define $\tilde{Z} = Z + A_{12} + \mu A_{34}$ and we put $\mathfrak{n}' = \operatorname{span}(E_1, \dots, E_4, \tilde{Z})$. The nontrivial Lie brackets with respect to $\mathfrak{n}'$ are

$$
[E_1, E_2]_{\mathfrak{n}'} = \tilde{Z}\,, \qquad [E_3, E_4]_{\mathfrak{n}'} = \mu \tilde{Z}
$$

and

$$[E_1,\ \tilde Z]_{\mathfrak{n}'} = -E_2, \qquad [E_2,\ \tilde Z]_{\mathfrak{n}'} = E_1\,,$$
$$[E_3,\ \tilde Z]_{\mathfrak{n}'} = -\mu E_4, \qquad [E_4,\ \tilde Z]_{\mathfrak{n}'} = \mu E_3\,.$$

Now $\mathfrak{g} = \mathfrak{n}' + \mathfrak{h}$ is the naturally reductive decomposition, as we will see later. We put again $Y = X + \xi(X)$, where $X = x_1E_1 + \cdots + x_4E_4 + \tilde z\tilde Z \in \mathfrak{n}'$ and $\xi(X) = \xi_1D_1 + \xi_2D_2 \in \mathfrak{h}$; ξ_1, ξ_2 to be determined. We write down the Lie brackets

$$\begin{aligned}
&[X+\xi(X), E_1]_{\mathfrak{n}'} = -x_2\tilde Z + \tilde z E_2 + \xi_1 E_2\,,\\
&[X+\xi(X), E_2]_{\mathfrak{n}'} = x_1\tilde Z - \tilde z E_1 - \xi_1 E_1\,,\\
&[X+\xi(X), E_3]_{\mathfrak{n}'} = -x_4\mu\tilde Z + \tilde z\mu E_4 + \xi_2 E_4\,,\\
&[X+\xi(X), E_4]_{\mathfrak{n}'} = x_3\mu\tilde Z - \tilde z\mu E_3 - \xi_2 E_3\,.
\end{aligned}\tag{6}$$

Now we use again geodesic lemma, where we substitute, step by step, $U = E_1, \dots, E_4, \tilde Z$. Using expressions (6) and formula (3), we obtain the system of five equations

$$\begin{aligned}
\xi_1 x_2\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= x_2\tilde z\Big(1+\varepsilon\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}}\Big) - \tilde z x_2\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,,\\
\xi_1 x_1\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= x_1\tilde z\Big(1+\varepsilon\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}}\Big) - \tilde z x_1\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,,\\
\xi_2 x_4\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= \mu x_4\tilde z\Big(1+\varepsilon\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}}\Big) - \mu\tilde z x_4\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,,\\
\xi_2 x_3\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= \mu x_3\tilde z\Big(1+\varepsilon\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}}\Big) - \mu\tilde z x_3\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,,
\end{aligned}$$

the last equation is satisfied identically. Here $\widetilde M = \big(\sum_{i=1}^r x_i^2\big)^p + \tilde z^{2p}$. This system simipifies into the system of two equations

$$\begin{aligned}
\xi_1\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= \varepsilon\tilde z\Big(\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}} - \frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,,\\
\xi_2\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big) &= \varepsilon\mu\tilde z\Big(\frac{\tilde z^{2(p-1)}}{\widetilde M^{1-\frac1p}} - \frac{(\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p}}\Big)\,.
\end{aligned}$$

The components of the geodesic graph are obtained as the unique solution of this system, which is

$$\begin{aligned}
\xi_1 &= \varepsilon\tilde z\frac{\tilde z^{2(p-1)} - (\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p} + \varepsilon(\|x\|^2)^{p-1}}\,,\\
\xi_2 &= \varepsilon\mu\tilde z\frac{\tilde z^{2(p-1)} - (\|x\|^2)^{p-1}}{\widetilde M^{1-\frac1p} + \varepsilon(\|x\|^2)^{p-1}}\,.
\end{aligned}\tag{7}$$

For $\varepsilon = 0$ or for $p = 1$, we obtain the zero map ξ, which is always the case in the naturally reductive decomposition of any naturally reductive Riemannian metric.

Comparing formulas (7) with formulas (5), we conclude that the naturally reductive decomposition of the underlying Riemannian metric probably does not have any distinguished properties in Finsler geometry, in contrast to Riemannian geometry. From the algebraic point of view, the complexity formulas (7) and (5) is comparable, for general values of parameters $\varepsilon \neq 0$ and $p > 1$.

4.2. **Dim($\mathfrak{z}$)=2.** Let us consider the Lie algebra $\mathfrak{n}$ with the scalar product $\langle\,,\,\rangle$ determined by the orthonormal basis $\{E_1, \dots, E_4, Z_1, Z_2\}$ and generated by the nontrivial relations

$$
\begin{aligned}
&[E_1, E_2] = 0\,, && [E_2, E_3] = bZ_1 + cZ_2\,,\\
&[E_1, E_3] = aZ_1\,, && [E_2, E_4] = -aZ_1\,,\\
&[E_1, E_4] = bZ_1 + cZ_2\,, && [E_3, E_4] = 0
\end{aligned}
$$

for arbitrary parameters a, b, $c \in \mathbb{R}$. We have the 3-parameter family of modified H-type algebras in the sense of J. Lauret [15]. Some of these modified H-type algebras are isometric, because in [15], the modified H-type metrics in this case form a 2-parameter family. However, we keep this notation from [12] to keep the possibility to compare formulas for our new geodesic graphs with formulas for Riemannian metrics in [12] and formulas for the Randers metrics in [9]. We denote by (N, α) the modified H-type groups with the invariant Riemannian metric corresponding to Lie algebras $\mathfrak{n}$ with the scalar product $\langle\,,\,\rangle$. The skew-symmetric derivations on $\mathfrak{n}$ are

$$
\begin{aligned}
D_1 &= A_{12} - A_{34}\,,\\
D_2 &= A_{13} + A_{24}\,,\\
D_3 &= A_{14} - A_{23}\,.
\end{aligned}
$$

If $a^2 = c^2$ and $b = 0$, then also the operator $D_4 = 2B_{12} + A_{12} + A_{34}$ is the derivation on $\mathfrak{n}$. It can be shown that a component of any geodesic graph to this operator is zero. Hence put $\mathfrak{h} = \operatorname{span}(D_1, \dots, D_3)$ for all groups N. If we write down the commutator relations for these operators, we easily verify that $\mathfrak{h} \simeq \mathfrak{su}(2)$. We put $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ and we consider the homogeneous space $N = G/H$, where $H = \mathrm{SU}(2)$ and $G = N \rtimes H$.

We now introduce a Finsler metric on each group N, again by the function F in formula (2) with respect to the basis B. Obviously, for $\varepsilon = 0$, it reduces to the norm determined by the Riemannian metric α above. The isometry group of any Finsler manifold (M, F) is again $G = N \rtimes H$.

We construct the geodesic graph now. The homogeneous space $N = G/H$ with the Riemannian metric α is not naturally reductive and we shall work in the reductive decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. We put again $Y = X + \xi(X)$, where $X = x_1E_1 + \dots + x_4E_4 + z_1Z_1 + z_2Z_2 \in \mathfrak{n}$ and $\xi(X) = \xi_1D_1 + \dots + \xi_3D_3 \in \mathfrak{h}$. The Lie brackets are now

$$
\begin{aligned}
[X + \xi(X), E_1] &= -x_3aZ_1 - x_4(bZ_1 + cZ_2) + \xi_1E_2 + \xi_2E_3 + \xi_3E_4\,,\\
[X + \xi(X), E_2] &= -x_3(bZ_1 + cZ_2) + x_4aZ_1 - \xi_1E_1 + \xi_2E_4 - \xi_3E_3\,,
\end{aligned}
$$

$$
\begin{aligned}
&[X+\xi(X),E_3] = x_1 aZ_1 + x_2(bZ_1+cZ_2) - \xi_1 E_4 - \xi_2 E_1 + \xi_3 E_2\,,\\
(8)\qquad &[X+\xi(X),E_2] = x_1(bZ_1+cZ_2) - x_2 aZ_1 + \xi_1 E_3 - \xi_2 E_2 - \xi_3 E_1\,.
\end{aligned}
$$

We use again geodesic lemma for $U = E_1, \dots, E_4, Z_1, Z_2$, expressions (8) and formula (3). We obtain the system of equations

$$
\begin{aligned}
\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[-\xi_1x_2-\xi_2x_3-\xi_3x_4] &= \Big(1+\varepsilon\frac{(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[-(x_3a+x_4b)z_1-x_4cz_2]\,,\\
\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[\xi_1x_1-\xi_2x_4+\xi_3x_3] &= \Big(1+\varepsilon\frac{(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[(x_4a-x_3b)z_1-x_3cz_2]\,,\\
(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}})[\xi_1x_4+\xi_2x_1-\xi_3x_2] &= \Big(1+\varepsilon\frac{(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[(x_1a+x_2b)z_1+x_2cz_2]\,,\\
\Big(1+\varepsilon\frac{(\|x\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[-\xi_1x_3+\xi_2x_2+\xi_3x_1] &= \Big(1+\varepsilon\frac{(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}}\Big)[(-x_2a+x_1b)z_1+x_1cz_2]\,.
\end{aligned}
$$

The last two equations, for $U = Z_i$, are satisfied identically. The rank of this system is equal to 3 and using the Cramer's rule, we obtain the unique solution

$$
\begin{aligned}
\xi_1 &= \frac{M^{1-\frac{1}{p}}+\varepsilon(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}+\varepsilon(\|x\|^2)^{p-1}}\cdot\frac{2az_1(x_1x_4+x_2x_3)+2\Big[bz_1+cz_2\Big](x_2x_4-x_1x_3)}{x_1^2+\cdots+x_4^2}\,,\\
\xi_2 &= \frac{M^{1-\frac{1}{p}}+\varepsilon(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}+\varepsilon(\|x\|^2)^{p-1}}\cdot\frac{az_1(x_1^2-x_2^2+x_3^2-x_4^2)+2\Big[bz_1+cz_2\Big](x_1x_2+x_3x_4)}{x_1^2+\cdots+x_4^2}\,,\\
(9)\qquad \xi_3 &= \frac{M^{1-\frac{1}{p}}+\varepsilon(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}+\varepsilon(\|x\|^2)^{p-1}}\cdot\frac{2az_1(x_3x_4-x_1x_2)+\Big[bz_1+cz_2\Big](x_1^2-x_2^2-x_3^2+x_4^2)}{x_1^2+\cdots+x_4^2}\,.
\end{aligned}
$$

These formulas determine the geodesic graph. For $\varepsilon = 0$, the factor

$$
t = \frac{M^{1-\frac{1}{p}}+\varepsilon(\|z\|^2)^{p-1}}{M^{1-\frac{1}{p}}+\varepsilon(\|x\|^2)^{p-1}} = \frac{1+\varepsilon(\|z\|^2)^{p-1}M^{\frac{1}{p}-1}}{1+\varepsilon(\|x\|^2)^{p-1}M^{\frac{1}{p}-1}}
$$

in these formulas is equal to 1 and the geodesic graph reduces to the Riemannian geodesic graph as in [12].

5. Conclusions

From the viewpoint of homogeneous geodesics in Riemannian geometry, the naturally reductive decomposition for the Riemannian metric is a distinguished one. The complexity of geodesic graphs given by formulas (5) and (7) above illustrate that probably there is no hope for such a nice property for general Finsler g.o. metrics, even if the underlying Riemannian metric is naturally reductive. The reader can also check geodesic graphs for Randers g.o. metrics on these H-type groups in [9], which also confirm this conclusion.

All geodesic graphs presented in this paper illustrate the fact that a vector $X + D$, where $X \in \mathfrak{n}$ and $D \in \mathfrak{h}$, is geodesic vector of the Riemannian metric α if and only if $X + tD$ is geodesic vector for the Finsler metric F given by formulas (2). This fact was observed in [19], see Lemma 5.10, Theorem 5.11 and Corollary 5.12

there. We point a mistake in Lemma 5.10, where η and δ should be exchanged and the correct formula is $t = \delta/\eta$ and not $t = \eta/\delta$, in the notation of [19].

On the other hand, the factor t in formula (9) above is algebraically rather complicated function. Comparing geodesic graphs (5), (7) and (9) here and geodesic graphs of Randers g.o. metrics in [9], we conclude that there is probably not a straightforward way how to generalize the concept of the degree of the geodesic graph from Riemannian geometry to Finsler geometry, to fit well at least for these two special situations. For the moment, we can at least observe certain patterns which appear in the special situations of Randers metrics, or weakly symmetric metrics, respectively.

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Institute of Technology and Business in České Budějovice
Okružní 517/10, 370 01 České Budějovice, Czech Republic
E-mail: zdusek@mail.vstecb.cz