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MODULAR OPERADS WITH CONNECTED SUM AND BARANNIKOV'S THEORY

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Abstract. We introduce the connected sum for modular operads. This gives us a graded commutative associative product, and together with the BV bracket and the BV Laplacian obtained from the operadic composition and self-composition, we construct the full Batalin-Vilkovisky algebra. The BV Laplacian is then used as a perturbation of the special deformation retract of formal functions to construct a minimal model and compute an effective action.

1. Introduction

As was already shown, the algebras considered in the context of string field theory could be understood as algebras over the Feynman transform of some operad. This allows one to apply the operad homotopy theory to study these algebras in order to, for example, find the minimal models.

In the case of a commutative operad, the minimal models of quantum L_∞ algebras could be found via the Homological Perturbation Lemma, as was shown in [5]. This was possible due to the existence of the full Batalin-Vilkovisky algebra structure, including the graded commutative associative product. In this case, the product was given by the symmetric tensor product. However, in the general case (for example quantum A_∞ algebras induced from a modular envelope of the associative operad) such a product has always been missing. This article is an attempt to provide the missing piece.

We "enhance" the modular operads by a connected sum. The elegant geometrical interpretation in terms of moduli spaces of bordered Riemann surfaces from [9] will be still preserved, as well as the already constructed

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generalized BV-algebra from [3], where the BV bracket and the BV Laplacian were constructed as operadic compositions and self-compositions induced by an odd symplectic form. Thanks to the connected sum, we can make sense of the exponential and use it to find the quantum master equation.

The paper is organized as follows. In Section 2 we recall modular operads, introduce the connected sum and present two main examples – Quantum Closed modular operad and Quantum Open modular operad. Besides them, also Endomorphism modular operad with connected sum is presented. In Section 3 we construct the full Batalin-Vilkovisky algebra from operads with connected sum and define the Master Equation. In Section 4 we recall special deformation retracts. Finally, in Section 5 we use the BV-Laplacian as a perturbation in the Homological Perturbation Lemma. We also introduce the construction of the effective action via the exponential.

The main results are Theorem 16, Lemma 18 and Proposition 28. Unfortunately, even the proofs are straightforward, we restrain from them here because of their length. Their full version can be found in [4] that is in preparation.

Conventions and notation. For us, the field $\Bbbk$ is always of characteristic 0. To avoid problems with duals, we assume that all our vector spaces are $\mathbb{Z}$-graded and, unless stated otherwise, degree-wise finite-dimensional. If we consider a dual vector space, we always consider only the graded dual, denoted by V^*. $[n]$ is the set $\{1, 2, \dots n\}$, Σ_n denotes the symmetric group of $[n]$, and $\sqcup$ is the disjoint union. We denote the degree of an element v of a graded vector space as $|v|$ and the cardinality of the set A as $\mathtt{card}(A)$.

2. Modular operads and the connected sum

Definition 1. Denote Cor the **category of stable corollas**: the objects are pairs (C, G) with C a finite set and G a non-negative half-integer such that the stability condition is satisfied: $2(G-1) + \mathtt{card}(C) > 0$. A morphism $(C, G) \to (D, G')$ is defined only if $G = G'$ and it is just a bijection $C \xrightarrow{\sim} D$.

In the following, the pair (C, G) is always understood as an element of this category.

Definition 2. A **modular operad** $\mathcal{P}$ consists of a collection $\{\mathcal{P}(C, G)\}$ of dg vector spaces and three collections

$$\{\mathcal{P}(\rho)\colon \mathcal{P}(C, G) \to \mathcal{P}(D, G) \mid \rho \text{ a morphism in } \mathsf{Cor}\},$$
$$\{{}_a\circ_b\colon \mathcal{P}(C_1 \sqcup \{a\}, G_1) \otimes \mathcal{P}(C_2 \sqcup \{b\}, G_2) \to \mathcal{P}(C_1 \sqcup C_2, G_1 + G_2)\},$$
$$\{\circ_{ab}\colon \mathcal{P}(C \sqcup \{a, b\}, G) \to \mathcal{P}(C, G+1)\}$$

of degree 0 morphisms of dg vector spaces. These data are required to satisfy axioms for Σ-module, equivariance, and associativity of composition maps.

Remark 3. Similarly as in [3], we consider also a special case of *twisted* modular operads, an **odd modular operad**. The twisting is given from Feynman transform by cocycle $Det(Edge(\Gamma))$, where $Edge(\Gamma)$ are edges of the graph Γ. For more details see [1]. The operadic compositions, now denoted as ${}_a\bullet_b$ and $\bullet_{ab}$, are of degree 1. The corresponding axioms 5.–8. of Definition 2 of [3] are changed accordingly. For example, the Axiom 5 $\circ_{ab}\ \circ_{cd} = \circ_{cd}\ \circ_{ab}$, is changed in the odd case to $\bullet_{ab}\ \bullet_{cd} = -\bullet_{cd}\ \bullet_{ab}$. See Definition 4 ibid.

Occasionally, we will also need a **skeletal version** of (odd) modular operad, $\overline{\mathcal{P}}$. The definition can be obtained be restriction of the underlying category Cor to corollas of the form $([n], G)$ (see Section III.D in [3]). We will also write just $\mathcal{P}(n, G)$ (instead of $\mathcal{P}([n], G)$). The explicit formulation of operadic compositions and corresponding axioms could be found ibid.

Definition 4. A modular operad with a connected sum is a modular operad $\mathcal{P}$ equipped with a collection of degree 0 chain maps called connected sum defined on two components as

$$\#_2\colon \mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C', G + G' + 1)\,, \tag{1}$$

and on one component as

$$\#_1\colon \mathcal{P}(C, G) \to \mathcal{P}(C, G + 2)\,, \tag{2}$$

such that

(CS1) $(\sigma \sqcup \sigma')\#_2 = \#_2(\sigma \otimes \sigma')$ for all bijections $\sigma\colon C \to D$, $\sigma'\colon C' \to D'$,
(CS2) $\#_2\,\tau = \#_2$, where τ is monoidal symmetry
(from category of vector spaces),
(CS3) $\#_2(1 \otimes \#_2) = \#_2(\#_2 \otimes 1)$,
(CS4) as maps $\mathcal{P}(C, G) \to \mathcal{P}(C - \{i, j\}, G + 3)$

$$\circ_{ij}\,\#_1 = \#_1\,\circ_{ij},$$

(CS5a) as maps $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C' - \{i, j\}, G + G' + 2)$

$$\circ_{ij}\,\#_2 = \begin{cases} \#_2(\circ_{ij} \otimes 1) & \dots\ i, j \in C\,, \\ \#_2(1 \otimes \circ_{ij}) & \dots\ i, j \in C'\,, \\ \#_1\,{}_i\circ_j & \dots\ i \in C, j \in C'\,, \\ \#_1\,{}_j\circ_i & \dots\ j \in C, i \in C'\,, \end{cases}$$

(CS5b) as maps $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C' - \{i, j\}, G + G' + 2)$,

$${}_i\circ_j(\#_1 \otimes 1) = \#_1\,{}_i\circ_j \qquad \dots\ i \in C, j \in C'\,,$$

(CS6) as maps $\mathcal{P}(C,G)\otimes\mathcal{P}(C',G')\otimes\mathcal{P}(C'',G'')\to\mathcal{P}(C\sqcup C'\sqcup C'' -\{i,j\},G+G'+G''+1)$,

$$ {}_i\circ_j(1\otimes\#_2)=\begin{cases}\#_2({}_i\circ_j\otimes 1) & \dots j\in C'\,,\\ \#_2(1\otimes {}_i\circ_j)(\tau\otimes 1) & \dots j\in C''\,,\end{cases} $$

where $i\in C$ and the map $(\tau\otimes 1)$ switches the first two tensor factors.

Remark 5. A *twisted* modular operad with a connected sum is defined precisely as in the normal, i.e., untwisted case. $\#_2$, $\#_1$ are again degree 0 operations. To make the distinction between twisted and untwisted case more explicit, we write the axiom (CS6) evaluated on elements for the odd modular operad.[1] In untwisted case

$$ p\,{}_a\circ_b(p'\,\#_2\,p'')=\begin{cases}(p\,{}_a\circ_b\,p')\,\#_2\,p'' & \dots\ b\in C'\,,\\ p'\,\#_2\,(p\,{}_a\circ_b\,p'') & \dots\ b\in C''\,,\end{cases} $$

and in the odd case

$$ p\bullet_{ab}(p'\,\#_2\,p'')=\begin{cases}(p\,{}_a\bullet_b\,p')\,\#_2\,p'' & \dots\ b\in C'\,,\\ (-1)^{|p||p'|+|p'|}p'\,\#_2\,(p\,{}_a\bullet_b\,p'') & \dots\ b\in C''\,.\end{cases} $$

The following two examples are taken from [3]. For a fuller treatment, we refer the reader to Sections IV.A and V.A there. Let us here just recall some of the basic properties and the geometrical interpretation.

Example 6. The Quantum Closed modular operad $\mathcal{QC}$. The components are given as one dimensional spaces $\mathcal{QC}(C,G)\equiv\mathrm{Span}_{\Bbbk}\{C^G\}$, where C^G is a symbol of degree 0 and G is such that $g\equiv\frac{G}{2}-\frac{\mathtt{card}(C)}{4}+\frac{1}{2}$ is an integer. The connected sum is defined simply as

$$ \begin{aligned} C_1^{G_1}\,\#_2\,C_2^{G_2}&=(C_1\sqcup C_2)^{G_1+G_2+1}\,,\\ \#_1\ \left(C^G\right)&=C^{G+2}\,. \end{aligned} $$

In geometrical interpretation, each component is a homeomorphism class of a compact surface of genus g and set C of punctures in the interior. The connected sum $\#_2$ corresponds to gluing together interiors of two surfaces using a cylinder. Similarly, the connected sum $\#_1$ corresponds to gluing a new handle to the bulk of one surface.

[1] Similarly, also axiom (CS5a) is changed accordingly.

Example 7. The Quantum Open modular operad $\mathcal{QO}$. A cycle $\mathbf{o}$ in a set O is a cyclic word made of several distinct elements of O.[2] The components $\mathcal{QO}(O, G)$ are then given as

$$\mathrm{Span}_{\Bbbk}\Big\{\{\mathbf{o}_1, \dots \mathbf{o}_b\}^g \mid b, g \in \mathbb{N}_0, \mathbf{o}_i \text{ cycle in } O, \bigsqcup_{i=1}^{b} \mathbf{o}_i = O, G = 2g + b - 1\Big\}.$$

In geometrical interpretation, this is a homeomorphism class of a compact orientable surfaces of genus g with b boundary components, and the set $\mathbf{o}_i$ of punctures on the boundary i. The connected sum is defined in this case as

$$\{\mathbf{o}_1, \dots \mathbf{o}_{b_1}\}^{g_1} \#_2 \{\mathbf{o}'_1, \dots \mathbf{o}'_{b_2}\}^{g_2} = \{\mathbf{o}_1, \dots \mathbf{o}_{b_1}, \mathbf{o}'_1 \dots \mathbf{o}'_{b_2}\}^{g_1+g_2},$$
$$\#_1 (\{\mathbf{o}_1, \dots \mathbf{o}_b\}^g) = \{\mathbf{o}_1, \dots \mathbf{o}_b\}^{g+1}.$$

Remark 8. Similarly, as operad could be defined more abstractly as an algebra over monad, we can also define a modular operad with a connected sum as an algebra over a monad. More details will appear in [4].

Definition 9. For any set C, $\mathtt{card}(C) = n$ and the vector space V we define the **unordered tensor product** as

$$\bigotimes_C V \equiv \bigoplus_{\psi\colon C \to [n]} V^{\otimes n} \Big/ \sim .$$

If we denote as $i_\psi \colon V^{\otimes n} \hookrightarrow \bigoplus_{\psi\colon C\to[n]} V^{\otimes n}$ the canonical inclusion into the ψ-th summand, then the equivalence is given by $i_\psi(v_1 \otimes \dots \otimes v_n) \sim i_{\sigma\psi}(\sigma(v_1 \otimes \dots \otimes v_n))$, where $\sigma \in \Sigma_n$.

Let $I_\psi \colon V^{\otimes n} \to \bigotimes_C V$ denote the inclusion i_ψ followed by the natural projection. For $F \in \bigotimes_C V^* \subseteq (\bigotimes_C V)^*$ we denote by $(F)_\psi = F \circ I_\psi \colon V^{\otimes n} \to \Bbbk^{\otimes n} \cong \Bbbk$.

Definition 10. The **endomorphism odd modular operad** $\mathcal{E}_V$ is a collection of dg vector spaces

$$\mathcal{E}_V(C, G) \equiv \bigotimes_C V^*,$$

where G is again a half-integer such that the stability condition is satisfied, V is degree-wise finite vector space, and V^* is its graded dual.[3]

The Σ-module structure and the differential are defined as usually. The operadic composition and self-composition are given by contraction of the tensors using an odd symplectic form $\omega \colon V \otimes V \to \Bbbk$ of degree -1 compatible with the differential. For definition of (self-)composition see Definition 9 of [3].

[2] We also admit an empty cycle.

[3] Notice that in general $\mathcal{E}_V(C, G)$ as vector space is not degree-wise finite.

Remark 11. In finite dimensional vector spaces the non-degeneracy of ω gives an isomorphism $X\colon V \to V^*$, $a \mapsto \omega(a,\cdot)$. From this isomorphism, it is possible to define $\omega^*\colon V^* \otimes V^* \to \mathbb{K}$, $\omega^*(\alpha,\beta) = \omega(X^{-1}(\alpha), X^{-1}(\beta))$ such that matrix of ω^* is the inverse matrix of ω, i.e., $\omega_{ij} \cdot \omega^{jk} = \delta_i^k$.

In the infinite-dimensional case, this becomes a bit more complicated since ω^* is, in general, an element of $(V \otimes V)^{**}$ – a space that is much "bigger" than $V \otimes V$. But thanks to our assumption, we still have an inverse of ω.

First, let us fix the basis of $V = \bigoplus_i V_i$. Since each V_i is finite-dimensional, we can order the basis of V as $\{\{a_i\}_0, \{a_i\}_1, \{a_i\}_{-1}, \dots \{a_i\}_k, \{a_i\}_{-k}, \dots\}$, where $\{a_i\}_k$ is a basis of V_k and each of these bases can be picked in such a way that ω has a form

$$\begin{pmatrix} 0 & A_1 & 0 & 0 & \dots \\ -A_1^T & 0 & 0 & 0 & \\ 0 & 0 & 0 & A_2 & \\ 0 & 0 & -A_2^T & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

where A_k is the regular matrix corresponding to the (non-degenerate) pairing of elements from V_k with elements from V_{-k+1}.

Therefore, ω^{ij} as the components of the matrix inverse to $\omega_{ij} = \omega(a_i, a_j)$ are well-defined. Hence, we can consider instead of ω^* an element $s \in V \otimes V$ such that $\omega(s) = 1$.

Theorem 12. Let $f \in \mathcal{E}_V(C_1, G_1) \cong \bigotimes_{C_1} V^*$, $g \in \mathcal{E}_V(C_2, G_2) \cong \bigotimes_{C_2} V^*$, $\psi\colon C_1 \sqcup C_2 \to [n_1 + n_2]$ where $n_1 = \texttt{card}(C_1)$, $n_2 = \texttt{card}(C_2)$. Define

$$\#_1 f \equiv f\,, \tag{3}$$

where f on the right-hand side is understood as an element of $\mathcal{E}_V(C_1, G_1+2) \cong \bigotimes_{C_1} V^*$. Further, we define

$$(f \#_2\, g)_\psi \equiv ((f)_{\psi_1} \cdot (g)_{\psi_2})\overline{\psi}\,, \tag{4}$$

where $\overline{\psi} \in Sh(n_1, n_2)$ is defined as $\overline{\psi}\big|_{[n_1]} = \psi|_{C_1}$ and $\overline{\psi}\big|_{n_1+[n_2]} = \psi|_{C_2}$ and ψ_1, ψ_2 as compositions $\psi_1\colon C_1 \xrightarrow{\psi|_{C_1}} \psi(C_1) \xrightarrow{\text{o.p.}} [n_1]$ and $\psi_2\colon C_2 \xrightarrow{\psi|_{C_2}} \psi(C_2) \xrightarrow{\text{o.p.}} [n_2]$ where "o.p." means order-preserving. Then $\mathcal{E}_V$ with the above defined operation $\#_2$ and $\#_1$ which is trivially shifting the G-grading is odd modular operad with connected sum.

3. Connected sum and master equation

Definition 13. Let W be a vector space with a linear action of a finite group H. The space of **invariants** is $W^H \equiv \{w \in W \mid \forall h \in H\colon h \cdot w = w\}$.

Let $\mathcal{P}$ be a dg modular operad and $\mathcal{Q}$ an odd dg modular operad. Let us define the space of invariants under the diagonal action

$$\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G) \equiv (\mathcal{P}(n,G)\otimes\mathcal{Q}(n,G))^{\Sigma_n}\ ,$$
$$\mathrm{Fun}(\mathcal{P},\mathcal{Q}) \equiv \prod_{n\geq 0}\prod_{G\geq 0}\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G)\,.$$

There are operations

$$\begin{aligned} d:&\quad \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G)\to \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G),\\ \Delta:&\quad \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n+2,G)\to \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G+1),\\ \{\cdot,\cdot\}:&\quad \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_1+1,G_1)\otimes \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_2+1,G_2)\to\\ &\quad \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_1+n_2,G_1+G_2),\end{aligned}$$

of degrees 1, 1 and 0, respectively, defined by

$$d \equiv d_{\mathcal{P}}\otimes 1 - 1\otimes d_{\mathcal{Q}}\,, \tag{5}$$

$$\Delta \equiv (\,\circ_{ij}\otimes \bullet_{ij})(\theta\otimes\theta) \tag{6}$$

for arbitrary bijection $\theta\colon [n]\sqcup\{i,j\}\xrightarrow{\sim}[n+2]$, and

$$\begin{aligned}\{X,Y\} \equiv (-1)^{|X|}\cdot 2\sum_{C_1,C_2}(\,{}_i\circ_j\otimes {}_i\bullet_j)(\theta_1\otimes\theta_2\otimes\theta_1\otimes\theta_2\\ \times(1\otimes\tau\otimes 1)(X\otimes Y)\,,\end{aligned} \tag{7}$$

where we sum over all disjoint decompositions $C_1\sqcup C_2=[n_1+n_2]$, such that $\mathtt{card}(C_1)=n_1$, $\mathtt{card}(C_2)=n_2$, the bijections (no summation over those) $\theta_1\colon C_1\sqcup\{i\}\xrightarrow{\sim}[n_1+1]$, $\theta_2\colon C_2\sqcup\{j\}\xrightarrow{\sim}[n_2+1]$ are chosen arbitrarily, and τ is the monoidal symmetry. These operations extend to $\mathrm{Fun}(\mathcal{P},\mathcal{Q})$ in the usual way.

Remark 14. In this article, we restrict ourselves only to the case when the differential $d_{\mathcal{P}}$ equals 0.

In Theorem 20 of [3] (in a slightly different sign convention) were proven the following compatibility properties of d,Δ and $\{\cdot,\cdot\}$

$$d^2=0,\qquad \Delta^2=0,\qquad \Delta\, d + d\,\Delta = 0,$$
$$d\{\cdot,\cdot\}+\{\cdot,\cdot\}(d\otimes 1+1\otimes d)=0,\qquad \Delta\{\cdot,\cdot\}+\{\cdot,\cdot\}(\Delta\otimes 1+1\otimes\Delta)=0,$$

and the Jacobi identity

$$\{X,\{Y,Z\}\} = \{\{X,Y\},Z\} + (-1)^{(|X|+1)(|Y|+1)}\{Y,\{X,Z\}\}\,.$$

We obtain what is referred to as *generalized Batalin-Vilkovisky algebra* in [3]. But to have a "standard" BV algebra, the graded commutative associative product is missing. This missing piece will be filled in the following definition and theorem.

Definition 15. Let $\mathcal{P}$ and $\mathcal{Q}$ be dg modular operads defined as above, both with connected sum. A **product** $\star$:

$$\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_1,G_1)\otimes \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_2,G_2)\to \mathrm{Fun}(\mathcal{P},\mathcal{Q})(n_1+n_2,G_1+G_2+1)$$

is defined as

$$\star \equiv \sum_{C_1,C_2} (\#_2\otimes\#_2)(\theta_1\otimes\theta_2\otimes\theta_1\otimes\theta_2)(1\otimes\tau\otimes 1)\,, \tag{8}$$

where again the sum runs over all disjoint decompositions $C_1\sqcup C_2=[n_1+n_2]$, $\mathtt{card}(C_1)=n_1$, $\mathtt{card}(C_2)=n_2$, the bijections $\theta_1\colon C_1\xrightarrow{\sim}[n_1]$, $\theta_2\colon C_2\xrightarrow{\sim}[n_2]$ are chosen arbitrarily, and τ is the monoidal symmetry.

An **operator** $\sharp$: $\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G)\to\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G+2)$ is defined as

$$\sharp\equiv \#_1\otimes\#_1\,. \tag{9}$$

Theorem 16. If $\mathcal{P}$ is dg modular operad with a connected sum, and $\mathcal{Q}$ an odd dg modular operad with a connected sum, then $\mathrm{Fun}(\mathcal{P},\mathcal{Q})$ with operations $d,\Delta,\{\cdot,\cdot\}$ of degree 1 and $\star$ from definition 15 is a **Batalin-Vilkovisky algebra**, i.e.,

(1) $\star$ is a graded commutative associative product, i.e., on elements:

$$X\star Y=(-1)^{|X|\cdot|Y|}Y\star X\,,\qquad \text{and}\qquad (X\star Y)\star Z=X\star(Y\star Z)\,. \tag{10}$$

(2) $\Delta\star=\star(\Delta\otimes 1)+\star(1\otimes\Delta)+\sharp\{\cdot,\cdot\}$, i.e., on elements:

$$\Delta(X\star Y)=(\Delta X)\star Y+(-1)^{|X|}X\star(\Delta Y)+(-1)^{|X|}\sharp\{X,Y\}\,. \tag{11}$$

(3) $\{\cdot,\cdot\}(1\otimes\star)=\star(\{\cdot,\cdot\}\otimes 1)+\star(1\otimes\{\cdot,\cdot\})(\tau\otimes 1)$, i.e., on elements:

$$\{X,Y\star Z\}=\{X,Y\}\star Z+(-1)^{|X|\cdot|Y|+|Y|}Y\star\{X,Z\}\,. \tag{12}$$

Proof. The calculations are straightforward and given in [4]. □

Definition 17. For $\#_1$ is an injective map on both $\mathcal{P}$ and $\mathcal{Q}$, we introduce a **space of formal functions**

$$\mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})=\prod_{n\geq 0,G\geq 0}(\Bbbk((\varkappa))\otimes\mathrm{Fun}(\mathcal{P},\mathcal{Q})(n,G))\Big/\sim\,, \tag{13}$$

where $((\varkappa))$ are formal Laurent series and the equivalence $\sim$ is given by for any element $f \in \mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ as $\sharp f \sim \varkappa f$. Obviously, $\mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ is again BV-algebra thanks to the compatibility of $\sharp$ with d, Δ, $\{\cdot,\cdot\}$, and $\star$.

Note that we constructed a non-unital BV algebra. Nevertheless, we would like to define the **exponential** of an element $X \in \mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ as

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{X \star \ldots \star X}_{n-\text{times}} = \underbrace{1 \otimes 1}_{\text{in } \Bbbk\otimes\Bbbk} + \underbrace{X + \sum_{n=2}^{\infty} \frac{1}{n!} X \star \ldots \star X}_{\text{in } \mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})},$$

where $\Bbbk \otimes \Bbbk$ is a tensor product of two 1-dimensional vector spaces. The space $\Bbbk \otimes \Bbbk$ is not a subspace of $\mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ but we can formally add an element $1 \otimes 1$, the generator of $\mathrm{Fun}_\kappa(\mathcal{P},\mathcal{Q})(0,-1)$ which will play the role of the **unit**. We can extend the maps d and Δ naturally as $d(1\otimes 1) = 0$ and $\Delta(1 \otimes 1) = 0$.

Moreover, the element $1 \otimes 1$ has a nice geometric interpretation for both closed and open modular operad – it corresponds to the surface without any punctures or boundaries and with the geometric genus $g = 0$, the sphere.

Lemma 18. If S is a degree 0 element of $\mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ which is of order zero in $\varkappa$, then

$$(d + \Delta) \exp\big(\frac{S}{\varkappa}\big) = \frac{1}{\varkappa} \left(dS + \Delta S + \{S, S\}\right) \exp\big(\frac{S}{\varkappa}\big).$$

Proof. The arguments of the proof are the same as in Lemma 1 of [5]. $\square$

Remark 19. Let $\mathcal{P}$ be as a vector space degree-wise finite. In [1], Barannikov observed that every dg operad morphism from Feynman transform of $\mathcal{P}$ to $\mathcal{Q}$, i.e., $\mathbb{F}ey(\mathcal{P}) \to \mathcal{Q}$, is equivalently given by a degree 0 solution $S \in \mathrm{Fun}_\varkappa(\mathcal{P},\mathcal{Q})$ of the *quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0\,.$$

If we look closely at the case $\mathcal{Q} = \mathcal{E}_V$, the equation (4.8) in [1] is (up to sign convention)

$$(d_{\mathcal{P}} + d_V) S^G_n + \Delta S^{G-1}_{n+2} + \frac{1}{2} \sum_{\substack{G_1+G_2=G\\ I_1 \sqcup I_2 = [n]}} \{S^{G_1}_{I_1\sqcup\{i\}}, S^{G_2}_{I_2\sqcup\{j\}}\} = 0\,,$$

where $S = \sum_{\substack{n,G\\ stable}} S^G_n$, $S^G_n \in \mathrm{Fun}(\mathcal{P},\mathcal{E}_V)(n,G)$. Thanks to the modified definition of BV-algebra in (11), where we used the operator $\sharp$, we get from the above lemma exactly the same equation with respect to the G-grading.

Therefore having an algebra over the Feynman transform of $\mathcal{P}$ is equivalent with the condition that $\exp\big(S/\varkappa\big)$ is $(d+\Delta)$-closed in the space $\mathrm{Fun}_{\varkappa}(\mathcal{P}, \mathcal{E}_V)$.

4. Space of formal functions and Special deformation retracts

In the following, we will specialize to the case $\mathcal{Q} = \mathcal{E}_V$ and further we write shortly $\mathrm{Fun}(\mathcal{P}, V)$ instead of $\mathrm{Fun}(\mathcal{P}, \mathcal{E}_V)$.

Remark 20. We can equivalently define the formal functions as coinvariants since there are mutually inverse isomorphisms between the space of invariants and coinvariants. Coinvariants better capture the idea of "commuting" variables. However, operad theory produces invariants so we stick to them.

Definition 21. A **special deformation retract (SDR)** is a pair (V, d) and (W, e) of dg vector spaces, a pair p and i of their morphisms and a homotopy $k\colon V \to V$ between ip and 1_V

$$\begin{tikzcd} (V,d) \arrow[loop left, "k"] \arrow[r, shift left, "p"] & (W,e) \arrow[l, shift left, "i"] \end{tikzcd}$$

that satisfy the following conditions

$$\begin{aligned}
&d^2 = 0, \quad e^2 = 0, \quad |d| = |e| = 1, && \textit{... differentials,}\\
&pd = ep, \quad ie = di, \quad |p| = |i| = 0, && \textit{... chain maps,}\\
&ip - 1_V = kd + dk, \quad |k| = -1, && \textit{... homotopy map,}\\
&pi - 1_W = 0, && \textit{... deformation retract,}\\
&pk = 0, \quad ki = 0, \quad k^2 = 0 && \textit{... special deformation retract.}
\end{aligned}$$

Remark 22. Few comments on this definition.

It is possible to consider only the first three conditions, i.e., chain maps i, p between chain complexes $(V, d), (W, e)$, with homotopy k. In that case, one gets the so-called *standard situation.* When considering also the fourth condition one gets the *deformation retract.* But in the next, we will always consider all of the listed conditions - the Special Deformation Retract.

For any dg vector space (V, d) it is possible to construct an SDR since there is always a decomposition

$$V \cong H(V) \oplus \mathrm{Im}(d) \oplus W\,,$$

where $H(V)$ is a cohomology of vector space V with respect to d. Such decomposition is also known as *harmonious Hodge decomposition.* The homotopy map k is defined as follows

$$k|_{H(V)\oplus W} = 0\,, \qquad\qquad k|_{\mathrm{Im}(d)} = (\,d|_W)^{-1}\,.$$

As was shown in Proposition 2.5 of [2], in case we have a bilinear homogeneous form compatible with the differential d that is either graded symmetric or graded antisymmetric, it is possible to choose the decomposition compatible with this form. Obviously, the symplectic form ω of degree -1 (introduced in definition 10) satisfies these conditions.

By similar arguments as those in [5] we can consider a basis $\{\alpha^k\}$ of $H(V)^*$, a basis $\{\beta^k\}$ of $(\mathrm{Im}\,d)^*$ and a basis $\{\gamma^k\}$ of W^* such that we can split the symplectic form to $\omega' = (a_i, a_j)$, $\omega'' = (b_i, c_j)$. This allows us, similarly as in Lemma 5 ibid, to also decompose the BV Laplacian $\Delta = \Delta_\alpha + \Delta_{\beta\gamma}$ and the BV bracket $\{,\} = \{,\}_\alpha + \{,\}_{\beta\gamma}$ on $\mathrm{Fun}(\mathcal{P}, V)$. Thus we have a BV algebra structure on $\mathrm{Fun}(\mathcal{P}, H(V))$.

The next lemma simply follows from the well-known **tensor trick**: Given two SDRs

$$\text{(14)}\qquad {}^{k_1}\circlearrowright (V_1, d_1) \underset{i_1}{\overset{p_1}{\rightleftarrows}} (W_1, e_1) \qquad {}^{k_2}\circlearrowright (V_2, d_2) \underset{i_2}{\overset{p_2}{\rightleftarrows}} (W_2, e_2)$$

we can construct an SDR on their tensor product $V_1 \otimes V_2, W_1 \otimes W_2$. The differentials, chain maps and the homotopy are given as $d_1 \otimes 1 + 1 \otimes d_2$, $e_1 \otimes 1 + 1 \otimes e_2$, $i_1 \otimes i_2$, $p_1 \otimes p_2$, and $1 \otimes k_2 + k_1 \otimes i_2 p_2$, respectively.

Lemma 23. Let $\mathcal{P}$ be operad with the trivial differential. If

$$ {}^{k}\circlearrowright \ (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(V), 0)$$

is an SDR, then

$$\text{(15)}\qquad {}^{K}\circlearrowright \ (\mathrm{Fun}(\mathcal{P}, V), D) \underset{I}{\overset{P}{\rightleftarrows}} (\mathrm{Fun}(\mathcal{P}, H(V)), 0)$$

is an SDR, where

$$D = \sum_{n\geq 1}\sum_{i=1}^{n} 1_{\mathcal{P}} \otimes (1^{\otimes i-1} \otimes d^* \otimes 1^{\otimes n-i}),$$

$$I = \sum_{n\geq 1} 1_{\mathcal{P}} \otimes (p^*)^{\otimes n}, \qquad\qquad P = \sum_{n\geq 1} 1_{\mathcal{P}} \otimes (i^*)^{\otimes n},$$

$$\text{(16)}\qquad K = \sum_{n\geq 1}\sum_{\sigma\in\Sigma_n}\sum_{i=1}^{n} \frac{\sigma}{n!} 1_{\mathcal{P}} \otimes (1^{\otimes i-1} \otimes k^* \otimes (p^* i^*)^{\otimes n-i})\,.$$

Remark 24. Everything said above is valid also for $\mathrm{Fun}_{\varkappa}(\mathcal{P}, V)$ and $\mathrm{Fun}_{\varkappa}(\mathcal{P}, H(V))$. We only need to set

$$D(1 \otimes 1) = 0\,, \qquad K(1 \otimes 1) = 0\,,$$
$$I(1 \otimes 1) = 1 \otimes 1\,, \qquad P(1 \otimes 1) = 1 \otimes 1\,,$$

for the "artificially" added unit. Obviously, the condition $IP - 1 = KD + DK$ is satisfied also for the element $1 \otimes 1$.

5. Homological perturbation lemma

Definition 25. Let (V, d) be a dg vector space. A **perturbation** $\delta : V \to V$ of the differential d is a linear map of degree 1 such that $(d + \delta)^2 = 0$.

Theorem 26 (Perturbation lemma)**.** Consider an SDR as above:

$$\begin{tikzcd} (V,d) \arrow[loop left, "k"] \arrow[r, shift left, "p"] & (W,e) \arrow[l, shift left, "i"] \end{tikzcd} \tag{17}$$

Let δ be a perturbation of d which is small in the sense that

$$(1 - \delta k)^{-1} \equiv \sum_{i=0}^{\infty} (\delta k)^i$$

is a well defined linear map $V \to V$. Denote $A \equiv (1 - \delta k)^{-1}\delta$ and $d' \equiv d + \delta$, $e' \equiv e + pAi$, $i' \equiv i + kAi$, $p' \equiv p + pAk$, $k' \equiv k + kAk$

$$\begin{tikzcd} (V,d') \arrow[loop left, "k'"] \arrow[r, shift left, "p'"] & (W,e') \arrow[l, shift left, "i'"] \end{tikzcd} \tag{18}$$

Then if (17) is an SDR, then (18) is an SDR.

We now apply this theorem to our situation. Consider the SDR of Lemma 23

$$\begin{tikzcd} (\mathrm{Fun}(\mathcal{P}, V), D) \arrow[loop left, "K"] \arrow[r, shift left, "P"] & (\mathrm{Fun}(\mathcal{P}, H(V)), 0) \arrow[l, shift left, "I"] \end{tikzcd}$$

Obviously perturbation by $\delta = \Delta$ satisfies the assumptions of the theorem.

Definition 27. The **effective action** W is defined by

$$\exp\big(W/\varkappa\big) \equiv P'\big(\exp\big(S/\varkappa\big)\big) = P\Big(1 + \sum_{i=1}^{\infty} (\Delta K)^i\Big)\big(\exp\big(S/\varkappa\big)\big)\,,$$

where $S \in \mathrm{Fun}_{\varkappa}(\mathcal{P}, V)$ is the solution of the quantum master equation of Remark 19.

Proposition 28. The effective action W is a well-defined element of $\mathrm{Fun}_{\varkappa}(\mathcal{P}, H(V))$ of the zeroth order in $\varkappa$. Moreover, W satisfies the master equation on $\mathrm{Fun}_{\varkappa}(\mathcal{P}, H(V))$

$$\Delta_\alpha \exp\big(W/\varkappa\big) = 0\,.$$

Or equivalently,

$$\Delta_\alpha W + \frac{1}{2}\{W, W\}_\alpha = 0\,.$$

The proof of the proposition with all the details could be found in [4].

Remark 29. For the commutative modular operad, i.e., $\mathcal{P} = \mathcal{QC}$, the space of formal functions is just an algebra of symmetric tensor powers of V^*.

In this case, the connected sum corresponds to the symmetric tensor powers. The BV Laplace (6) and the BV bracket (7) give a standard BV formalism, [8]. A reader, interested in an explicit comparison, might find useful section "Skeletal version" of [4].

In remark 19, we recalled that Feynman transform of modular operad is equivalently given as a solution S of the quantum master equation. In [7], this algebraic structure was in the commutative case called *loop homotopy Lie algebra*. The proposition 28 restricted to this case then says that we constructed a *minimal model* of the loop homotopy Lie algebra.

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References

[1] Barannikov, S., *Modular operads and Batalin-Vilkovisky geometry*, Int. Math. Res. Not. IMRN **19** (2007), 31 pp., Art. ID rnm075.

[2] Chuang, J., Lazarev, A., *Abstract Hodge decomposition and minimal models for cyclic algebras*, Lett. Math. Phys. **89** (1) (2009), 33–49.

[3] Doubek, M., Jurčo, B., Münster, K., *Modular operads and the quantum open-closed homotopy algebra*, J. High Energy Phys. **158** (12) (2015), 54 pp., Article ID 158.

[4] Doubek, M., Jurčo, B., Peksová, L., Pulmann, J., *Quantum homotopy algebras*, in preparation.

[5] Doubek, M., Jurčo, B., Pulmann, J., *Quantum L_∞ algebras and the homological perturbation lemma*, Comm. Math. Phys. **367** (1) (2019), 215–240.

[6] Eilenberg, S., MacLane, S., *On the groups $H(\Pi, n)$. I*, Ann. of Math. (2) **58** (1) (1953), 55–106.

[7] Markl, M., *Loop homotopy algebras in closed string field theory*, Comm. Math. Phys. **221** (2) (2001), 367–384.

[8] Schwarz, A., *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (2) (1993), 249–260.

[9] Zwiebach, B., *Oriented open-closed string theory revisited*, Ann. Physics **267** (2) (1998), 193–248.

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