ARCHIVUM MATHEMATICUM (BRNO)
Tomus 56 (2020), 301–305

REMARK ON BILINEAR OPERATIONS ON TENSOR FIELDS

Jan Slovák

Abstract. This short note completes the results of [3] by removing the locality assumption on the operators. After providing a quick survey on (infinitesimally) natural operations, we show that all the bilinear operators classified in [3] can be characterized in a completely algebraic way, even without any continuity assumption on the operations.

Bilinear operations transforming two tensor fields into another tensor field are often met in many applications. Let us write $T^{r,s}$ for the functor assigning the tensor bundle $\otimes^r TM \otimes \otimes^s T^*M$ to each m-dimensional manifold, with the obvious action on local diffeomorphisms. The Lie bracket $T^{1,0} \times T^{1,0} \to T^{1,0}$ and its generalizations to the Schouten bracket, the Schouten-Nijenhuis bracket and the Fölicher-Nijenhuis bracket are examples of such bilinear operators.

Such functors are examples of natural bundles and all natural transformations D between the sheaves of germs of smooth sections of natural bundles are called natural operators if they are local, i.e., the values $Ds(x)$ on sections s in point x depend only on the germs of s in x, and they map smooth families of sections into smooth families of values (see [4] for definitions and theory). By the multilinear version of the Peetre theorem, all multilinear natural operators are differential operators of (locally) finite order, see [4].

The aim of this short note is to observe a straightforward extension of the results in [3], where seven special types of operators

$$T^{1,s} \times T^{p,q} \to T^{p,q+s}$$

are completely classified under the assumption they are bilinear natural operators. These are

$$\begin{aligned} &T^{1,0} \times T^{1,0} \to T^{1,0} && T^{1,0} \times T^{0,1} \to T^{0,1} \\ &T^{1,0} \times T^{0,2} \to T^{0,2} && T^{1,1} \times T^{0,1} \to T^{0,2} \\ &T^{1,1} \times T^{1,1} \to T^{1,2} && T^{1,1} \times T^{0,2} \to T^{0,3} \\ &T^{1,2} \times T^{0,1} \to T^{0,3}\,. \end{aligned} \tag{1}$$

If we restrict our operators to sections with compact support, we may clearly consider non-local operators obtained by integration on manifolds. The simplest

2020 *Mathematics Subject Classification*: primary 53A32; secondary 53A55, 58A20.
Key words and phrases: natural operator, tensor field, natural functional, Lie derivative.
Received April 9, 2020. Editor M. Čadek.
DOI: 10.5817/AM2020-5-301

one is the functional mapping each compactly supported top degree form ω on an oriented manifold M to the constant $\int_M \omega$, thus, providing an operator $I\colon T^{0,m} \to T^{0,0}$ (here we understand the constant value as providing the constant section of the trivial line bundle $T^{0,0}M$). The question how to extend the concept of naturality of such operations from the local operators mentioned above to arbitrary operations on sections with compact support was tackled in [2].

First of all we should notice that dealing with global sections instead of their germs does not allow to induce the action of the locally invertible diffeomorphisms to the sheaves. Fortunately, we may resolve this problem by looking at the infinitesimal version of the action. Indeed, the natural bundles F are functors providing smooth families of mappings $F(f_t)\colon FM \to FN$ for smooth families $f_t\colon M \to N$. Thus, applying F to the flows Fl^X_t of vector fields X on M provides flows of vector fields on TM and there is the general concept of the Lie derivative $\mathcal{L}_X s$ of sections s of $FM \to M$ valued in the vertical tangent bundle VFM,

$$\mathcal{L}_X s = \frac{\partial}{\partial t}_{|t=0} \big(F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}_t\big)\,,$$

see [4] for details. Obviously, if F is a natural vector bundle, than $VFM = FM \oplus FM$ and the second component of the above formula provides the usual Lie derivative.

In particular, commuting of the Lie derivatives with operations between sheaves of sections makes sense and, as shown in [1], this is an essentially equivalent concept for the (local) natural operators between natural bundles. We just have to restrict our category to oriented manifolds and the orientation preserving local diffeomorphisms (so there might be even more local operators available, e.g. the vector product in dimension three).

Restricting our attention to the natural vector bundles and multilinear operators $D\colon E_1 \times \cdots \times E_k \to E$, commuting with the Lie derivatives means:

$$\mathcal{L}_X\big(D(s_1,\dots,s_k)\big) = \sum_{n=1}^{k} D(s_1,\dots,\mathcal{L}_X s_n,\dots,s_k)\,. \tag{2}$$

Notice that for the operators of the type of Lie bracket of vector fields, (2) is just the Jacobi identity on vector fields.

In the rest of this short note, we prove the following theorem:

Theorem 1. *All bilinear operators of the seven types listed in* (1) *defined on compactly supported sections and commuting with Lie derivatives are bilinear natural differential operators classified in* [3]*. In particular, no continuity or locality have to be assumed.*

The above operator I is an example of a non-local operator commuting with the Lie derivative. Indeed, the derivatives of the constant functions are always zero, while

$$\int \mathcal{L}_X\omega = \int d\,i_X(\omega) = 0$$

by the Stokes theorem (remember ω has got compact support).

There are many similar non-local operators commuting with the Lie derivatives, introduced in [2] under the name *almost natural operators*. The simplest ones are functionals built with the help of a (local) natural $(k-1)$-linear operator $D_0\colon E_1 \times \cdots \times E_{k-1} \to E_k^* \otimes \Omega^m$,

$$D(s_1, \dots, s_{k-1}, s_k) = \int \langle D_0(s_1, \dots, s_{k-1}), s_k \rangle \tag{3}$$

where Ω^m stays for the top degree forms, $\langle\ ,\ \rangle$ means the usual dual pairing, and the constant value is considered as a constant multiple of the invariant constant function 1, i.e. the invariant section of the trivial line bundle.

In general, we may combine such functionals on subsets of the arguments and leave some arguments for a local operator valued in E. We also admit operators with no arguments, which means we consider invariant sections in the bundles in question. Almost natural operators are than all linear combinations of such operators.

For example, we can consider the operator $\Omega^p \times \Omega^q \to T^{1,1}$, $p + q = m$, the dimension of the manifolds, $D(\omega, \eta) = \left(\int \omega \wedge \eta\right) \cdot \mathbb{I}$, where $\mathbb{I}$ is the invariant section of $T^{1,1}$ corresponding to the identity on TM.

We are interested in bilinear operators only and the almost natural bilinear operators $D\colon E_1 \times E_2 \to E$ are given as linear combinations (over reals) of operators from the following list:

(i) An invariant section in E, multiplied by the value of a bilinear functional λ of the form
$$\lambda(s_1, s_2) = \int \langle D(s_1), s_2 \rangle$$
with a natural differential operator $D\colon E_1 \to E_2^* \otimes \Omega^m$, or such operators with E_1 and E_2 swapped.

(ii) There is an invariant section σ of $E_1^* \otimes \Omega^m$ and a natural (local) operator $D_0 : E_2 \to E$, and the operator is of the form
$$D(s_1, s_2) = \left(\int \langle \sigma, s_1 \rangle \right) \cdot D_0(s_2)\,,$$
or similarly with E_1 and E_2 swapped.

(iii) The operator is a local natural operator, thus a finite order bilinear differential operator commuting with the orientation preserving local diffeomorphisms.

The main result in [2] is the following:

Theorem 2. *Let $E_1, \dots, E_k$, E be natural vector bundles defined on the category of connected oriented m-dimensional manifolds, $m \geq 2$, and orientation preserving local diffeomorphisms. Then any separately continuous k-linear operator $D\colon E_1 \times \cdots \times E_k \to E$ on compactly supported sections which commutes with Lie derivatives is an almost natural operator. In particular any such operator is automatically jointly continuous and the space of such operators is always finite dimensional.*

Moreover, in one of the crucial parts of the proof of this theorem, the description of linear functionals commuting with Lie derivatives, the continuity assumption is not necessary. This implies (see [2, Corollary 6.2])

Corollary 3. *Let E_1, E_2, E be natural vector bundles defined on the category of connected oriented m-dimensional manifolds, $m \geq 2$, and orientation preserving local diffeomorphisms and assume there are no non-zero invariant sections of E. Then every bilinear operator $D\colon E_1 \times E_2 \to E$ on compactly supported sections which commutes with Lie derivatives is an almost natural operator.*

This is a truly remarkable theorem. Let us look at the example of a functional as in (3). First, the theorem says that such a functional has to be defined by an integral operator with a kernel. Next, the kernel must be given by a natural differential operator. Finally, the only option is to have the operator dependent on $k-1$ arguments, valued in $E^*_\ell \otimes \Omega^m$ and algebraically paired with the remaining argument, i.e. it cannot be an arbitrary k-linear differential operator valued in top degree forms (all others would have values integrating to zero). On top of all that, the theorem says that all (separately continuous) operators commuting with the Lie derivatives are built of such blocks.

There is a good reason to believe that even the weak separate continuity assumption in the theorem is not necessary.

So far, this conjecture that all multilinear operators commuting with Lie derivatives are almost natural operators has not been proved.

Proof of Theorem 1. Let us consider bilinear operators

$$D : T^{r,s} \times T^{p,q} \to T^{k,\ell}\,. \tag{4}$$

Just by observing the action of the center of $GL(m,\mathbb{R})$ we immediately see that there are no invariant sections in $T^{k,\ell}$, except the zero section, whenever $k \neq \ell$. Indeed, the homotheties $\tau \cdot \mathrm{id}$ act by multiplying with $\tau^{k-\ell}$ and thus there are no nonzero fixed points in the representation spaces. Similarly, there cannot be any nontrivial invariant sections in $\Omega^m \otimes (T^{r,s})^* = \Omega^m \otimes T^{s,r}$ if $s \neq r + m$. Thus, under the simultaneous assumptions $k \neq \ell$, $s \neq m + r$, $q \neq m + p$, all operators (4) commuting with Lie derivatives are of the type (iii) in the list above, without any further continuity assumptions. Consequently, they must be standard local natural differential operators.

Obviously, all the three assumptions on the tensor spaces are satisfied, if we restrict ourselves to those from the list (1) and the dimension is $m > 2$.

The two exceptions of the tensors $T^{0,2}$ appearing among the arguments on 2-dimensional manifolds would require linear natural differential operators $T^{1,0} \to T^{0,2}$ or $T^{1,1} \to T^{0,3}$. It is well known, that all linear natural differential operators are built exclusively from algebraic decompositions into irreducible components and the exterior differential, see [4, Section 32]. Thus there cannot be natural operators of the requested types.

In particular, we have verified that [3] has fully classified all bilinear operators listed in (1) on manifolds of dimension at least two commuting with Lie derivatives, i.e. satisfying the algebraic property (2), without any further continuity or locality assumptions. □

As an example of a similar non-local operator commuting with the Lie derivatives, notice the operator $D\colon T^{1,1} \times T^{0,2} \to T^{0,1}$ on 2-dimensional manifolds given by the formula

$$D(X \otimes \varphi, \eta) = \Big(\int \operatorname{Alt} \eta \Big) \cdot d(i_X \varphi)\,.$$

Acknowledgement. The author gratefully acknowledges support from the Grant Agency of the Czech Republic, grant Nr. GX19-28628X. The author is also most grateful to the anonymous referee for valuable suggestions.

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Department of Mathematics and Statistics,
Masaryk University, Faculty of Science,
Kotlářská 2, 611 37 Brno, Czech Republic
E-mail: slovak@muni.cz