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ASYMPTOTIC FORMS OF SOLUTIONS OF PERTURBED HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Asymptotic forms of solutions of half-linear ordinary differential equation $\left(|u'|^{\alpha-1}u'\right)' = \alpha\left(1+b(t)\right)|u|^{\alpha-1}u$ are investigated under a smallness condition and some signum conditions on $b(t)$. When $\alpha = 1$, our results reduce to well-known ones for linear ordinary differential equations.

1. Introduction

Let us consider the following quasilinear ordinary differential equation near $+\infty$:

$$(|u'|^{\alpha-1}u')' = \alpha\big(1 + b(t)\big)|u|^{\alpha-1}u\,. \tag{HL}$$

Here we assume that, $\alpha > 0$ is a constant, and $b \in C[0,\infty)$. A C^1-function u defined near $+\infty$ is called a solution of equation (HL) if $|u'|^{\alpha-1}u'$ is of class C^1, and (HL) is satisfied for all sufficiently large t. When $\alpha = 1$ equation (HL) reduces to the linear equation

$$u'' = \big(1 + b(t)\big)u\,. \tag{L}$$

So, equations of the type (HL) can be regarded as generalizations of linear equations. In fact for a solution u of (HL) and a constant C, Cu is also a solution of (HL); however, the sum of two solutions of (HL) is not always a solution of (HL). By these facts, equations of such types are often called half-linear equations.

Our main aim of the paper is to study the following problem:

Problem. When $b(t)$ is small, in some sense, near $+\infty$, what are the asymptotic forms of solutions of (HL)?

To get an insight into our problem, we notice the following two facts.

Fact 1.1. Let $\int^\infty |b(t)|dt < \infty$. Then linear equation (L) has two independent solutions u_1 and u_2 with the asymptotic forms

$$u_1(t) \sim e^t \quad \text{and} \quad u_2(t) \sim e^{-t} \quad \text{as} \quad t \to \infty\,,$$

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respectively, and so every nontrivial solution u of (L) has the asymptotic form

$$u(t) \sim ce^{t} \quad \text{or} \quad u(t) \sim ce^{-t} \quad \text{as} \quad t \to \infty\,,$$

for some constant $c \neq 0$. See for example [1, 2, 4]. (There are many refinements of this property.)

Fact 1.2. Let $b(t) \equiv 0$ in equation (HL), that is, let us consider the simple half-linear equation

$$(|u'|^{\alpha-1}u')' = \alpha|u|^{\alpha-1}u\,. \tag{HL_0}$$

We can solve explicitly this equation. All nontrivial solutions are given by the functions

$$ce^{t}\,,\ ce^{-t} \quad \text{with} \quad c = \text{constant} \neq 0\,,$$

and the two 2-parameter families of functions generated by generalized hyperbolic sine functions and generalized hyperbolic cosine functions. Further, every nontrivial solution u belonging to these two families has the asymptotic form $u(t) \sim ce^{t}$ as $t \to \infty$ for some constant $c \neq 0$ See in detail [3, Chapter 1].

From these observations we conjecture that, if $b(t)$ is sufficiently small near $+\infty$, then every nontrivial solution u of (HL) has the asymptotic forms

$$u(t) \sim ce^{t} \quad \text{or} \quad u(t) \sim ce^{-t} \quad \text{as} \quad t \to \infty\,,$$

for some constant $c \neq 0$. In this paper we give partial affirmative answer to this conjecture. In fact, we can show that our conjecture is true under signum conditions on $b(t)$.

To state our results we rewrite equation (HL) into the following two equations:

$$(|u'|^{\alpha-1}u')' = \alpha(1+p(t))|u|^{\alpha-1}u\,, \tag{HL_+}$$

$$(|u'|^{\alpha-1}u')' = \alpha(1-p(t))|u|^{\alpha-1}u\,. \tag{HL_-}$$

In the sequel we assume the next conditions:

(A_1) $\alpha > 0$ is a constant;

(A_2) $p \in C[0,\infty)$; $p(t)$ satisfies $p(t) \geq 0$ for (HL_+), and $p(t)$ satisfies $0 \leq p(t) \leq 1$ for (HL_-);

(A_3) $\int^{\infty} p(t)dt < \infty$.

Our main result follows:

Theorem 1.3. *Under assumptions (A_1)–(A_3), every nontrivial solution u of (HL_+) and (HL_-) has the asymptotic form*

$$u(t) \sim ce^{t} \quad \textit{or} \quad u(t) \sim ce^{-t} \quad \textit{as} \quad t \to \infty\,,$$

for some constant $c \neq 0$.

Even if the positivity of $p(t)$ is violated, we conjecture that Theorem 1.3 is still valid by assuming $\int^{\infty} |p(t)|dt < \infty$ alone. To treat equations ($HL_\pm$) under such a condition will be the theme of our future works.

The difficulty in proving Theorem 1.3 comes from mainly the following two facts: (i) The set of all solutions of a half-linear equation is not a linear space; (ii) There is not so-called variation of constants formulas for half-linear equations. So in the present paper we will give the proof of main results without employing the well-known results concerning the properties of linear equations.

This paper is organized as follows. In Section 2 we collect preparatory results which will be employed later. In Section 3 we give the proof of our main result Theorem 1.3. More precisely, in Section 3.1 we determine the asymptotic form of increasing positive solutions of $(HL_{\pm})$, while in Section 3.2 we determine that of positive decreasing solutions. The proof of Theorem 1.3 will be completed by unifying these results.

2. Preparatory results

In this section we collect preparatory results which play important roles to prove main results.

The following simple pointwise inequalities are used to estimate several integrals in the sequel.

Lemma 2.1. (i) *Let $\beta \geq 1$. Then $(1-x)^\beta \geq 1 - \beta x$ for $x \in [0,1]$.*

(ii) *Let $0 < \beta \leq 1$. Then $(1-x)^\beta \geq 1 - x$ for $x \in [0,1]$.*

(iii) *Let $0 < \beta \leq 1$. Then $(1+x)^\beta \leq 1 + x$ for $x \geq 0$.*

(iv) *Let $\beta \geq 1$ and $M > 0$ be a constant. Then there is a constant $K = K_M > 0$ such that*

$$(1+x)^\beta \leq 1 + Kx \quad \textit{for} \quad x \in [0, M]\,.$$

(In fact, we may take $K = [(1+M)^\beta - 1]/M$.)

Lemma 2.2. *Every nontrivial solution $u(t)$ of $(HL_{\pm})$ is of constant sign near $+\infty$.*

Proof. Suppose the contrary that $u(t)$ changes the sign infinitely many times near $+\infty$. Then we can find two points T_1, $T_2 > 0$ satisfying $T_1 < T_2$,

$$u(T_1) = 0\,,\ u'(T_1) > 0\,,\ u'(T_2) = 0\,, \quad \text{and} \quad u(t) > 0 \quad \text{in} \quad (T_1, T_2)\,.$$

Then an integration on $[T_1, T_2]$ of $(HL_{\pm})$ gives

$$-|u'(T_1)|^{\alpha-1}u'(T_1) = \alpha \int_{T_1}^{T_2} (1 \pm p(s))u(s)^\alpha ds > 0\,,$$

which is an obvious contradiction. The proof is complete. □

Lemma 2.3. *Every nontrivial solution $u(t)$ of $(HL_{\pm})$ satisfies exactly one of the following two properties for all sufficiently large t:*

(i) $|u'(t)| \uparrow +\infty$ *(and therefore $|u(t)| \uparrow +\infty$) as $t \to \infty$;*

(ii) $|u'(t)|$, $|u(t)| \downarrow 0$ *as $t \to +\infty$.*

Since $u(t)$ is a solution of (HL_+) (or (HL_-)) if and only if so is $-u(t)$, below we will consider mainly (eventually) positive solutions of (HL_+) (or (HL_-)).

Proof of Lemma 2.3. Let $u(t)$ be an eventually positive solution of $(HL_\pm)$. Since $\big(|u'|^{\alpha-1}u'\big)' > 0$, the function $|u'|^{\alpha-1}u'$, that is, $u'(t)$ increases.

Let $u'(t) \uparrow \infty$. Then the assertion (i) holds.

Let $u'(t) \uparrow c > 0$, a constant. Then obviously $u(t) \sim ct$ as $t \to \infty$. An integration of $(HL_\pm)$ on $[T, t]$, where T is sufficiently large, gives

$$\begin{aligned}\big(u'(t)\big)^\alpha - (u'(T))^\alpha &= \alpha \int_T^t \big(1 \pm p(s)\big)u(s)^\alpha ds \\ &\geq c_1 \int_T^t s^\alpha \big(1 - p(s)\big)\, ds\end{aligned}$$

for some constant $c_1 > 0$. By assumption (A_3) we have

$$\begin{aligned}\int_T^t s^\alpha \big(1 - p(s)\big)\, ds &\geq \int_T^t s^\alpha ds - t^\alpha \int_T^t p(s)\, ds \\ &\geq \frac{1}{\alpha+1}(t^{\alpha+1} - T^{\alpha+1}) - \Big(\int_T^\infty p(s)\, ds\Big)t^\alpha \to \infty \quad \text{as} \quad t \to \infty\,.\end{aligned}$$

This means $u'(t) \to \infty$ as $t \to \infty$, a contradiction. So the case that $u'(t) \to c \in (0, \infty)$ does not occur.

Let $u'(t) \uparrow 0$. Since $u(t) > 0$, the solution $u(t)$ decreases; and so there is a nonnegative limit $l \equiv \lim_{t\to\infty} u(t)$. To see $l = 0$ suppose the contrary that $l > 0$. Then an integration of $(HL_\pm)$ on $[T, t]$, T being sufficiently large, gives

$$\begin{aligned}[-u'(T)]^\alpha &\geq [-u'(T)]^\alpha - [-u'(t)]^\alpha \\ &= \alpha \int_T^t \big(1 \pm p(s)\big)u(s)^\alpha ds \\ &\geq \alpha l^\alpha \int_T^t \big(1 - p(s)\big)\, ds \\ &\geq \alpha l^\alpha \Big(t - T - \int_T^\infty p(s)\, ds\Big) \to \infty \text{ as } t \to \infty\,.\end{aligned}$$

This is a contradiction. So $\lim_{t\to\infty} u(t) = 0$; therefore assertion (ii) holds in this case.

Since u is a positive solution, the case that $u'(t) \uparrow c$ for some negative constant c does not occur. The proof is complete. □

The following comparison principle will be used in many places. The proof was found, for example, in [5, Lemma 4.1].

Lemma 2.4. *Suppose that* p_1, $p_2 \in C[t_0, t_1]$ *and* $0 < p_1(t) \leq p_2(t)$ *on* $[t_0, t_1]$. *Let* u_i, $i = 1, 2$, *be solutions on* $[t_0, t_1]$ *of the equations*

$$(|u_i'|^{\alpha-1}u_i)' = p_i(t)|u_i|^{\alpha-1}u_i\,, \quad i = 1, 2\,,$$

respectively, satisfying

$$u_1(t_0) \leq u_2(t_0) \quad \text{and} \quad u_1'(t_0) < u_2'(t_0)\,.$$

Then $u_1(t) < u_2(t)$ *and* $u_1'(t) < u_2'(t)$ *on* $(t_0, t_1]$.

3. Proof of the main result

In this section we give the proof of Theorem 1.3. To this end, we consider asymptotic forms of solutions of the two types indicated in Lemma 2.3 separately.

3.1. **Asymptotic form of increasing positive solutions of** $(HL_\pm)$**.** As a first step we treat eventually positive solutions u of $(HL_\pm)$ satisfying the property (i) of Lemma 2.3 : $u'(t) \uparrow \infty$ and $u(t) \uparrow \infty$ as $t \to \infty$.

Lemma 3.1. (i) *Let u be a positive solution of (HL_+) on $[T, \infty)$ satisfying the property* (i) *of Lemma 2.3 for sufficiently large $T > 0$. Then*

$$u(t) \geq ce^t\,, \quad t \geq T\,, \quad \textit{for some constant } c > 0\,. \tag{1}$$

(ii) *Let u be a positive solution of (HL_-) on $[T, \infty)$ satisfying the property (i) of Lemma 2.3 for sufficiently large $T > 0$. Then*

$$u(t) \leq ce^t\,, \quad t \geq T\,, \quad \textit{for some constant } c > 0\,. \tag{2}$$

Proof. We give only the proof of (i), because (ii) can be proved similarly.

We may assume that $u'(t) > 0$ on $[T, \infty)$. Let $c > 0$ be a sufficiently small number such that

$$u(T) > ce^T \quad \text{and} \quad u'(T) > ce^T\,.$$

Put $z(t) = ce^t$, $t \geq T$. Then z satisfies $u(T) > z(T)$, $u'(T) > z'(T)$, and

$$(|z'|^{\alpha-1}z')' = \alpha|z|^{\alpha-1}z\,, \quad t \geq T\,.$$

By Lemma 2.4 we obtain (1) as desired. □

Lemma 3.2. *Let u be a positive solution of (HL_+) or (HL_-) satisfying the property* (i) *of Lemma 2.3. Then the function $u(t)/e^t$ is eventually monotone near $+\infty$.*

Proof. Let $u(t), u'(t) > 0$ on $[T, \infty)$ and put $v(t) = u(t)/e^t$. We will show that $v'(t) \geq 0$ near ∞, or $v'(t) \leq 0$ near ∞, by contradiction.

If this is not the case, then there are three points t_1, t_2 and t_3 $(T < t_1 < t_2 < t_3)$ satisfying

$$v'(t_1)v'(t_2) < 0 \quad \text{and} \quad v'(t_1)v'(t_3) > 0\,.$$

We can assume that

$$v'(t_1) > 0,\ v'(t_2) < 0 \quad \text{and} \quad v'(t_3) > 0\,.$$

Then there are two points $\tau_1 \in (t_1, t_2)$ and $\tau_2 \in (t_2, t_3)$ such that

$$\begin{aligned} &v'(\tau_1) = 0\,, \quad v''(\tau_1) \leq 0\,, \quad \text{and} \\ &v'(\tau_2) = 0\,, \quad v''(\tau_2) \geq 0\,. \end{aligned} \tag{3}$$

On the other hand, note that $v(t)$ satisfies $v + v' > 0$, $t \geq T$, and

$$\big[\big(v + v'\big)^\alpha\big]' + \alpha\big(v + v'\big)^\alpha = \alpha\big(1 \pm p(t)\big)v^\alpha\,;$$

that is

$$v'' + 2v' + v = \big(1 \pm p(t)\big)\big(v + v'\big)^{1-\alpha}v^\alpha\,. \tag{4}$$

Let us divide the proof into two cases.

Case 1. The case where $p(t) > 0$, $t \in [T, t_3]$. By equation (4), $v''(\tau_i) = \pm p(\tau_i)v(\tau_i)$, $i = 1, 2, 3$. So $v''(\tau_1)$ and $v''(\tau_2)$ have the same signs, which is an obvious contradiction to the properties (3).

Case 2. The case where $p(t) \geq 0$, $t \in [T, t_3]$. Let $\{p_\varepsilon(t)\}_{\varepsilon > 0}$ be a family of continuous functions of $(t, \varepsilon) \in [T, t_3] \times (0, \varepsilon_0]$, $\varepsilon_0 = \text{const} > 0$, satisfying

$$p_\varepsilon(t) > p(t) \quad \text{on} \quad [T, t_3]\,, \quad \text{and} \quad \lim_{\varepsilon \to +0} \Big(\max_{[T, t_3]} \big(p_\varepsilon(t) - p(t)\big)\Big) = 0\,.$$

Further, let $z = z_\varepsilon$ be the solution of the initial value problem

$$\begin{cases} z'' + 2z' + z = \big(1 \pm p_\varepsilon(t)\big)(z + z')^{1-\alpha} z^\alpha\,, \\ z(T) = v(T)\,,\ z'(T) = v'(T)\,. \end{cases} \tag{5}$$

By the continuous dependence on the parameter [6], for sufficiently small $\varepsilon > 0, z = z_\varepsilon(t)$ exists at least for $t \in [T, t_3]$, $z(t) > 0$, $z(t) + z'(t) > 0$ for $t \in [T, t_3]$, and

$$\lim_{\varepsilon \to +0} \Big(\max_{[T, t_3]} \big| z'_\varepsilon(t) - v'(t) \big| \Big) = 0\,.$$

Let $m > 0$ be a sufficiently small number satisfying

$$v'(t_1) > m > 0\,, \quad v'(t_2) < -m < 0 \quad \text{and} \quad v'(t_3) > m > 0\,.$$

For sufficiently small $\varepsilon > 0$, we have

$$|z'_\varepsilon(t) - v'(t)| < m/2 \quad \text{for} \quad t \in [T, t_3]\,,$$

which implies that

$$\begin{aligned} &z'_\varepsilon(t_1) > v'(t_1) - (m/2) > m/2 > 0\,, \\ &z'_\varepsilon(t_2) < v'(t_2) + (m/2) < -m/2 < 0\,, \quad \text{and} \\ &z'_\varepsilon(t_3) > v'(t_3) - (m/2) > m/2 > 0\,. \end{aligned}$$

By noting that $z = z_\varepsilon$ satisfies equation (5) and $p_\varepsilon(t) > 0$ on $[T, t_3]$, we find that this is a contradiction as in Case 1.

The proof is complete. □

Proposition 3.3. *Let u be a positive solution of (HL_+) or (HL_-) satisfying the property (i) of Lemma 2.3. Then*

$$u(t) \sim ce^t \quad \textit{as} \quad t \to \infty \quad \textit{for some constant} \quad c > 0\,. \tag{6}$$

Proof of Proposition 3.3 for (HL_+). By Lemma 3.2 the function $u(t)/e^t$ is monotone near $+\infty$. If $u(t)/e^t$ decreases, then by (i) of Lemma 3.1 we find that $u(t)/e^t$ decreases to a positive constant as $t \to \infty$; and so (6) holds as desired.

Next let $u(t)/e^t$ increase near $+\infty$. We may suppose that $u' > 0$ and $u(t)/e^t$ increases on $[T, \infty)$. An integration of both sides of (HL_+) on $[T, t]$ gives

$$u'(t)^\alpha = u'(T)^\alpha + \alpha \int_T^t (1 + p(s)) u(s)^\alpha ds\,.$$

Since $u(t)/e^t$ increases, we get from the above

$$\begin{aligned} u'(t)^\alpha &\le u'(T)^\alpha + \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \int_T^t \big(e^{\alpha s} + e^{\alpha s} p(s)\big)\, ds \\ &= u'(T)^\alpha + \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \Big[\frac{1}{\alpha}(e^{\alpha t} - e^{\alpha T}) + \int_T^t e^{\alpha s} p(s)\, ds \Big] . \end{aligned}$$

Thus we obtain

$$u'(t)^\alpha \le u'(T)^\alpha + u(t)^\alpha \Big[1 + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big] . \tag{7}$$

The computation below slightly differs according to the value of α.

Firstly, let $\alpha > 1$. By the simple inequality

$$(X+Y)^{1/\alpha} \le X^{1/\alpha} + Y^{1/\alpha} \quad \text{for} \quad X, Y \ge 0 ,$$

we get from (7)

$$u'(t) \le u'(T) + u(t) \Big[1 + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big]^{1/\alpha} .$$

Further, by (iii) of Lemma 2.1 we have

$$u'(t) \le u'(T) + u(t) \Big[1 + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big] .$$

By (1) we obtain

$$\frac{u'(t)}{u(t)} \le c_1 e^{-t} + 1 + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds ,$$

for some constant $c_1 > 0$. An integration of both sides gives

$$\log \frac{u(t)}{u(T)} \le (t-T) + c_1 \int_T^t e^{-s} ds + \alpha \int_T^t \Big(e^{-\alpha s} \int_T^s e^{\alpha r} p(r)\, dr \Big)\, ds .$$

Since

$$\begin{aligned} \int_T^t \Big(e^{-\alpha s} \int_T^s e^{\alpha r} p(r)\, dr \Big)\, ds &= \frac{1}{\alpha} \int_T^t p(s) \big(1 - e^{-\alpha(t-s)}\big)\, ds \\ &\le \frac{1}{\alpha} \int_T^\infty p(s)\, ds < \infty , \end{aligned}$$

we can get

$$\log \frac{u(t)}{u(T)} \le t + O(1) , \quad \text{as} \quad t \to \infty ,$$

which implies that $u(t) = O(e^t)$ as $t \to \infty$. By recalling the assumption that $u(t)/e^t$ increases, we find that (6) holds.

Next let $0 < \alpha < 1$. From (7) we have

$$\begin{aligned} u'(t) &\le u(t) \Big[1 + \frac{u'(T)^\alpha}{u(t)^\alpha} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big]^{1/\alpha} \\ &\equiv u(t) \big(1 + B(t)\big)^{1/\alpha} . \end{aligned}$$

Here $B(t)$ is defined naturally by the last equality. Since $u(t)/e^t$ increases, we find for some constants c_2 and $c_3 > 0$

$$\begin{aligned} 0 \le B(t) &\le c_2 e^{-\alpha t} + \alpha e^{-\alpha t} \cdot e^{\alpha t} \int_T^t p(s)\, ds \\ &\le c_3 + \alpha \int_T^\infty p(s)\, ds < \infty \,. \end{aligned}$$

Therefore by (iv) of Lemma 2.1 we obtain for some constant $K > 0$

$$u'(t) \le u(t) \Big[1 + \frac{K u'(T)^\alpha}{u(t)^\alpha} + K\alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big] \,.$$

Dividing the both sides by $u(t)$, and integrating on $[T, t]$, we have

$$\begin{aligned} \log \frac{u(t)}{u(T)} &\le t - T + c_2 \int_T^t e^{-\alpha s}\, ds + K\alpha \int_T^t \Big(e^{-\alpha s} \int_T^s e^{\alpha r} p(r)\, dr \Big)\, ds \\ &\le t + O(1) + K \int_T^\infty p(s)\, ds \,. \end{aligned}$$

as $t \to \infty$. So $u(t) = O(e^t)$ as $t \to \infty$, which implies that (6) holds as before.

This completes the proof. □

Proof of Proposition 3.3 for (HL_-)**.** The argument here is parallel to that in the proof of Proposition 3.3 for (HL_+). By Lemma 3.2 the function $u(t)/e^t$ is monotone near $+\infty$. If $u(t)/e^t$ increases, then by (ii) of Lemma 3.1 we find that $u(t)/e^t$ increases to a positive constant as $t \to \infty$; and so (6) holds as desired.

Next let $u(t)/e^t$ decrease near $+\infty$. We may suppose that $u' > 0$ and $u(t)/e^t$ decreases on $[T, \infty)$. An integration of both sides of (HL_-) on $[T, t]$ gives

$$u'(t)^\alpha = u'(T)^\alpha + \alpha \int_T^t (1 - p(s)) u(s)^\alpha\, ds \,.$$

Employing the decreasing property of $u(t)/e^t$, we get

$$\begin{aligned} u'(t)^\alpha &\ge \alpha \int_T^t \big(1 - p(s)\big) e^{\alpha s} \Big[\frac{u(s)}{e^s} \Big]^\alpha ds \\ &\ge \alpha \frac{u(t)^\alpha}{e^{\alpha t}} \int_T^t \big(e^{\alpha s} - e^{\alpha s} p(s) \big)\, ds \\ &= u(t)^\alpha \Big[1 - \Big(e^{-\alpha(t-T)} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s} p(s)\, ds \Big) \Big] \\ &\equiv u(t)^\alpha \big(1 - B(t) \big) \,. \end{aligned} \tag{8}$$

Here of course, $B(t)$ is defined naturally by the last equality. Since $0 \le p(s) \le 1$ by the assumption (A_2), we observe that

$$0 \le B(t) \le e^{-\alpha(t-T)} + \alpha e^{-\alpha t} \int_T^t e^{\alpha s}\, ds = 1 \,, \quad \text{for} \quad t \ge T \,.$$

So by (i) and (ii) of Lemma 2.1 we obtain from (8)

$$u'(t) \geq u(t)\Big[1 - c\Big(e^{-\alpha(t-T)} + \alpha e^{-\alpha t}\int_T^t e^{\alpha s}p(s)\,ds\Big)\Big]\,, \tag{9}$$

where $c > 0$ is a constant given by

$$c = \begin{cases} 1/\alpha & \text{if} \quad 0 < \alpha < 1\,; \\ 1 & \text{if} \quad \alpha > 1\,. \end{cases}$$

As before, we get from (9)

$$\begin{aligned} \log\frac{u(t)}{u(T)} &\geq t - T - c\int_T^t e^{-\alpha(s-T)}\,ds - c\alpha\int_T^t e^{-\alpha s}\int_T^s e^{\alpha r}p(r)\,dr\,ds \\ &= t + O(1) \quad \text{as} \quad t \to \infty\,. \end{aligned}$$

So, $u(t)/e^t \geq c_4 > 0$ for some constant c_4, and we find that (6) holds.

This completes the proof. □

3.2. **Asymptotic form of decreasing positive solutions of** $(HL_\pm)$**.** In this subsection we treat eventually positive solutions u of $(HL_\pm)$ satisfying the property (ii) of Lemma 2.3: $u'(t) \uparrow 0$ and $u(t) \downarrow 0$ as $t \to \infty$.

To state auxiliary results, we consider two equations of the form of $(HL_\pm)$ for a moment:

$$\big(|W'|^{\beta-1}W'\big)' = Q(t)|W|^{\beta-1}W\,, \qquad t \geq 0\,, \tag{A_Q}$$

$$\big(|w'|^{\beta-1}w'\big)' = q(t)|w|^{\beta-1}w\,, \qquad t \geq 0\,. \tag{A_q}$$

Here we assume that $\beta > 0$ is a constant, Q, $q \in C[0,\infty)$, and they satisfy

$$Q(t) \geq q(t) > 0\,, \quad t \geq 0\,,$$

and

$$\int^\infty q(t)\,dt = \infty\,.$$

Let $T \geq 0$ and $h > 0$ be arbitrary numbers. Then by [5, Theorem 5.1], equation (A_Q) has only one positive solution W on $[T,\infty)$ satisfying $W(T) = h, W(t) \downarrow 0$ and $W'(t) \uparrow 0$ as $t \to \infty$. Similarly equation (A_q) has only one positive solution w on $[T,\infty)$ satisfying $w(T) = h$, $w(t) \downarrow 0$ and $w'(t) \uparrow 0$ as $t \to \infty$. Such solutions are often called *positive decaying solutions*. Note that positive solutions of $(HL_\pm)$ satisfying the property (ii) of Lemma 2.3 are positive decaying solutions of $(HL_\pm)$.

For example, the positive decaying solution u of the equation

$$\big(|u'|^{\beta-1}u'\big)' = \beta|u|^{\beta-1}u\,,$$

passing through the point (T,h) is given by $u(t) = he^{-(t-T)}$. The following comparison lemma is important to prove our main results.

Lemma 3.4. *Let W and w be positive decaying solutions of equation (A_Q) and (A_q) on $[T,\infty)$, respectively, passing through the point (T,h), $T \geq 0$, $h > 0$. Then, $W(t) \leq w(t)$ for $t > T$.*

Proof. The proof is done by contradiction. Suppose the contrary that $W(t) > w(t)$ for some $t > T$. Then we can find an interval $[t_0, t_1] \subset [T, \infty)$ such that

$$W(t_0) = w(t_0)\,, \quad \text{and} \quad W(t) > w(t)\,, \quad \text{in} \quad (t_0, t_1]\,. \tag{10}$$

We claim that $W'(\tau) > w'(\tau)$ for some $\tau \in [t_0, t_1]$. For, if there are no such points, that is, if $W'(t) \le w'(t)$ on $[t_0, t_1]$, then the function $W(t) - w(t)$ is nonincreasing on $[t_0, t_1]$. So $W(t) - w(t) \le W(t_0) - w(t_0) = 0$. However this contradicts to (10). Hence $W'(\tau) > w'(\tau)$ for some $\tau \in [t_0, t_1]$.

Since $W(\tau) > w(\tau)$, Lemma 2.4 implies that $W(t) > w(t)$ for $t \ge \tau$. From (A_Q) and (A_q) we obtain

$$\begin{aligned}
&|W'(t)|^{\beta-1}W'(t) - |w'(t)|^{\beta}w'(t) \\
&\qquad = |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau) \\
&\qquad\quad + \int_\tau^t \Big[Q(s)W(s)^\beta - q(s)w(s)^\beta\Big]\,ds \\
&\qquad > |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau)\,, \quad \text{for} \quad t \ge \tau\,.
\end{aligned}$$

Since $\lim_{t\to\infty} W'(t) = \lim_{t\to\infty} w'(t) = 0$, by letting $t \to \infty$ we obtain

$$0 \ge |W'(\tau)|^{\beta-1}W'(\tau) - |w'(\tau)|^{\beta-1}w'(\tau) > 0\,.$$

This is a contradiction to the definition of τ. This completes the proof. □

Lemma 3.5. (i) *Let u be a positive solution of equation (HL_+) on $[T, \infty)$ satisfying the property* (ii) *of Lemma 2.3 for sufficiently large $T > 0$. Then*

$$u(t) \le ce^{-t}\,, \quad t \ge T\,, \quad \textit{for some constant} \;\; c > 0\,. \tag{11}$$

(ii) *Let u be a positive solution of equation (HL_-) on $[T, \infty)$ satisfying the property* (ii) *of Lemma 2.3 for sufficiently large $T > 0$. Then*

$$u(t) \ge ce^{-t}\,, \quad t \ge T\,, \quad \textit{for some constant} \;\; c > 0\,. \tag{12}$$

Proof. We give only the proof of (i), because (ii) can be proved similarly.

Let $z(t)$ be the positive decaying solution of equation

$$\big(|z'|^{\alpha-1}z'\big)' = \alpha|z|^{\alpha-1}z\,,$$

passing through the point $(T, u(T))$; that is, $z(t) = u(T)e^{-(t-T)}$. Since $\alpha \le \alpha\big(1 + p(t)\big)$, Lemma 3.4 implies that

$$u(t) \le z(t) \equiv u(T)e^{-(t-T)}\,, \quad t \ge T\,,$$

which show that (11) holds. This completes the proof. □

Lemma 3.6. *Let u be a positive solution of (HL_-) or (HL_+) satisfying the property* (ii) *of Lemma 2.3. Then the function $u(t)/e^{-t}$ is eventually monotone.*

Proof. Put $v = u(t)/e^{-t}$, Then $v - v' > 0$ and v satisfies

$$v'' - 2v' + v = \big(1 \pm p(t)\big)(v - v')^{1-\alpha}v^\alpha\,,$$

for large t. If $v'(\tilde{t}) = 0$ for some sufficiently large $\tilde{t}$, then $v''(\tilde{t}) = \pm p(\tilde{t})v(\tilde{t})$. So arguing as in the proof of Lemma 3.2, we find that $u(t)/e^{-t}(\equiv v(t))$ is eventually monotone. This completes the proof. $\square$

Proposition 3.7. *Let u be a positive solution of (HL_+) or (HL_-) satisfying the property* (ii) *of Lemma 2.3. Then*

$$u(t) \sim ce^{-t} \quad \textit{as} \quad t \to \infty \quad \textit{for some constant} \;\; c > 0\,. \tag{13}$$

Proof of Proposition 3.7 for (HL_+). By Lemma 3.6 the function $u(t)/e^{-t}$ is eventually monotone. If $u(t)/e^{-t}$ increases, then by (i) of Lemma 3.5 we find that $u(t)/e^{-t}$ converges to a positive constant as $t \to \infty$; so (13) holds.

Next let $u(t)/e^{-t}$ decrease near $+\infty$. We may suppose that $u' < 0$ and $u(t)/e^{-t}$ decreases on $[T, \infty)$. Since $u'(\infty) = 0$, from (HL_+) we have

$$\big[-u'(t)\big]^\alpha = \alpha \int_t^\infty \big(1 + p(s)\big)u(s)^\alpha \, ds\,.$$

The monotonicity of $e^t u(t)$ implies that

$$\begin{aligned}
[-u(t)]^\alpha &= \alpha \int_t^\infty e^{-\alpha s}\big(1 + p(s)\big)\big[e^s u(s)\big]^\alpha \, ds \\
&\le \alpha e^{\alpha t} u(t)^\alpha \int_t^\infty e^{-\alpha s}\big(1 + p(s)\big)\, ds\,.
\end{aligned}$$

Thus

$$-u'(t) \le u(t)\Big(1 + \alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds\Big)^{1/\alpha}.$$

Firstly let $\alpha > 1$. Then by (iii) of Lemma 2.1 we obtain

$$-u'(t) \le u(t)\Big(1 + \alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds\Big), \tag{14}$$

that is

$$-\frac{u'(t)}{u(t)} \le 1 + \alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds\,.$$

An integration on $[T, t]$ gives

$$\begin{aligned}
\log \frac{u(T)}{u(t)} &\le t - T + \alpha \int_T^t e^{\alpha s} \int_s^\infty e^{-\alpha r} p(r)\, dr\, ds \\
&\le t - T + \alpha \int_T^\infty e^{\alpha s} \int_s^\infty e^{-\alpha r} p(r)\, dr\, ds \\
&= t - T + \int_T^\infty \big(1 - e^{-\alpha(s-T)}\big)p(s)\, ds \\
&\le t + O(1) \quad \text{as} \quad t \to \infty\,.
\end{aligned}$$

Therefore, $u(t) \ge c_1 e^{-t}$ for some constant $c_1 > 0$. Since $u(t)/e^{-t}$ decreases, we find that (13) holds.

Secondly, let $0 < \alpha < 1$. As before we get (14). Note that,

$$0 \le \alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds \le \alpha e^{\alpha t} \cdot e^{-\alpha t} \int_t^\infty p(s)\, ds \le \int_T^\infty p(s)\, ds\,.$$

Then, (iv) of Lemma 2.1 implies that for some constant $K > 0$ we obtain

$$-u'(t) \le u(t)\Big[1 + K\alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds\Big]\,.$$

So arguing as in the case that $\alpha > 1$, we can get $u(t) \ge c_2 e^{-t}$ for some constant $c_2 > 0$; and hence (13) holds. This completes the proof. $\square$

Proof of Proposition 3.7 for (HL_-)**.** By Lemma 3.6 the function $u(t)/e^{-t}$ is eventually monotone. If $u(t)/e^{-t}$ decreases, then (ii) of Lemma 3.5 implies that $u(t)/e^{-t}$ converges to a positive constant as $t \to \infty$; and so (13) holds.

Let us consider the case where $u(t)/e^{-t}$ increases. We may suppose that $u' < 0$ and $u(t)/e^{-t}$ decreases on $[T, \infty)$. From (HL_-) we have

$$\big[-u'(t)\big]^\alpha = \alpha \int_t^\infty \big(1 - p(s)\big) u(s)^\alpha\, ds.$$

The monotonicity of $u(t)/e^{-t}$ implies that

$$\big[-u'(t)\big]^\alpha \ge \alpha e^{\alpha t} u(t)^\alpha \int_t^\infty \Big(e^{-\alpha s} - p(s) e^{-\alpha s}\Big)\, ds\,,$$

that is

$$\big[-u'(t)\big]^\alpha \ge u(t)^\alpha \Big[1 - \alpha e^{\alpha t} \int_t^\infty p(s) e^{-\alpha s}\, ds\Big]\,.$$

Notice that

$$\begin{aligned}\alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds &\le \alpha e^{\alpha t} \cdot e^{-\alpha t} \int_t^\infty p(s)\, ds \\ &\le \int_T^\infty p(s)\, ds \le 1\,, \quad t \ge T\,,\end{aligned}$$

for sufficiently large T. Therefore (i) and (ii) of Lemma 2.1 implies that,

$$-u'(t) \ge u(t)\Big[1 - c\alpha e^{\alpha t} \int_t^\infty e^{-\alpha s} p(s)\, ds\Big], \quad t \ge T\,, \tag{15}$$

where c is a constant given by

$$c = \begin{cases} 1/\alpha & if\ \ 0 < \alpha < 1\,; \\ 1 & if\ \ \alpha > 1\,. \end{cases}$$

Dividing the both sides of (15) by $u(t)$, and integrating the resulting inequality on $[T,t]$, we obtain

$$\begin{aligned}\log\frac{u(T)}{u(t)} &\geq t-T-c\alpha\int_T^t\Big(e^{\alpha s}\int_s^\infty e^{-\alpha r}p(r)\,dr\Big)\,ds\\ &\geq t-T-c\alpha\int_T^\infty\Big(e^{\alpha s}\int_s^\infty e^{-\alpha r}p(r)\,dr\Big)\,ds\\ &= t-T-c\int_T^\infty\big(1-e^{-\alpha(s-T)}\big)p(s)\,ds\\ &= t+O(1)\quad\text{as}\quad t\to\infty\,.\end{aligned}$$

Therefore, $u(t)\leq c_2e^{-t}$ for some constant $c_2>0$. Since $u(t)/e^{-t}$ increases, we find that (13) holds. This completes the proof. □

As stated in the introduction, our main result Theorem 1.3 is a direct consequence of Propositions 3.3 and 3.7.

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