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REMARKS ON THE EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS, II

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Abstract. We consider the half-linear differential equation of the form

$$(p(t)|x'|^{\alpha}\operatorname{sgn} x')' + q(t)|x|^{\alpha}\operatorname{sgn} x = 0\,, \quad t \ge t_0\,,$$

under the assumption that $p(t)^{-1/\alpha}$ is integrable on $[t_0,\infty)$. It is shown that if a certain condition is satisfied, then the above equation has a pair of nonoscillatory solutions with specific asymptotic behavior as $t \to \infty$.

1. Introduction

In this paper we consider the half-linear ordinary differential equation

$$(p(t)|x'|^{\alpha}\operatorname{sgn} x')' + q(t)|x|^{\alpha}\operatorname{sgn} x = 0\,, \quad t \ge t_0\,, \tag{1.1}$$

where α is a positive constant, and $p(t)$ and $q(t)$ are real-valued continuous functions on $[t_0,\infty)$ and $p(t) > 0$ for $t \ge t_0$.

If $\alpha = 1$, then (1.1) becomes the linear equation

$$(p(t)x')' + q(t)x = 0\,, \quad t \ge t_0\,. \tag{1.2}$$

It is known that basic results and qualitative results for the linear equation (1.2) can be generalized to the half-linear equation (1.1). The important works for (1.1) are summarized in the book of Došlý and Řehák [2].

Recently the problem of asymptotic behavior of nonoscillatory solutions of (1.2) or (1.1) is investigated in the framework of regular variation. We refer the reader, for instance, to [5, 6, 7, 8, 9, 10, 11, 14, 15] and the references therein. The classical theory on regularly varying functions is suited for the case $p(t) \equiv 1$ in (1.1). However the classical theory is not sufficient to properly describe the possible asymptotic behavior of nonoscillatory solutions of (1.1) for the case $p(t) \not\equiv 1$ in (1.1). In fact, the asymptotic behavior of a nonoscillatory solution of (1.1) is strongly affected by

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the condition on $p(t)$, more precisely, by the condition

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}}\,ds = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}}\,ds < \infty\,.$$

Then the notion of regular variation in the classical sense has been generalized by Jaroš and Kusano [5].

Let $R(t)$ be a continuously differentiable function on some neighborhood $[T_0,\infty)$ of infinity and satisfy

$$R'(t) > 0 \quad \text{for} \;\; t \ge T_0 \quad \text{and} \quad \lim_{t\to\infty} R(t) = \infty\,.$$

For simplicity, we suppose that a function $f(t)$ is positive and continuously differentiable on $[T_0,\infty)$. Then, $f(t)$ is said to be a regularly varying function with respect to $R(t)$ if and only if $f(t)$ can be written in the form

$$f(t) = c(t)\exp\left\{\int_T^t \frac{R'(s)}{R(s)}\mu(s)\,ds\right\}, \quad t \ge T\,, \tag{1.3}$$

for some $T > T_0$ and some continuous functions $c(t)$ and $\mu(t)$ such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \mu(t) = \mu \in \boldsymbol{R}\,.$$

The real number μ is called the index of $f(t)$. If $c(t) \equiv c$ (positive constant) in (1.3), then $f(t)$ is said to be a normalized regularly varying function (of index μ) with respect to $R(t)$. We omit the description of the original definition of regular variation with respect to $R(t)$. For details, see the paper [5]. It is worth noting that $f(t)$ is a normalized regularly varying function of index μ with respect to $R(t)$ if and only if

$$\lim_{t\to\infty} \frac{R(t)}{R'(t)}\frac{f'(t)}{f(t)} = \mu\,.$$

The set of normalized regularly varying functions of index μ with respect to $R(t)$ is denoted by $\text{n–RV}_R(\mu)$.

The half-linear equation (1.1) of the case

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}}\,ds = \infty$$

has been discussed in [13]. In the present paper we consider the case

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}}\,ds < \infty\,. \tag{1.4}$$

Then the function $\pi(t)$ can be defined by

$$\pi(t) = \int_t^{\infty} \frac{1}{p(s)^{1/\alpha}}\,ds\,, \quad t \ge t_0\,. \tag{1.5}$$

Now, put

$$E(\alpha) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{1.6}$$

If c is a constant such that $c < E(\alpha)$, then the equation

$$|\sigma|^{(\alpha+1)/\alpha} + \sigma + c = 0 \tag{1.7}$$

has two real roots σ_1, σ_2 ($\sigma_1 < \sigma_2$). They satisfy $\sigma_1 < -(\alpha+1)E(\alpha) < \sigma_2$.

Under the assumption that

$$\int_{t_0}^{\infty} \pi(s)^{\alpha+1} q(s)\,ds = \lim_{t\to\infty} \int_{t_0}^{t} \pi(s)^{\alpha+1} q(s)\,ds \quad \text{exists and is finite,} \tag{1.8}$$

we define the function $Q_{\pi,\alpha}(t)$ by

$$Q_{\pi,\alpha}(t) = \int_t^{\infty} \pi(s)^{\alpha+1} q(s)\,ds\,, \quad t \ge t_0\,. \tag{1.9}$$

Then the condition

$$\lim_{t\to\infty} \frac{Q_{\pi,\alpha}(t)}{\pi(t)} = \alpha c \quad (c < E(\alpha)) \tag{1.10}$$

plays an important role. The following theorem has been proved by Jaroš, Kusano and Tanigawa [7, Theorem 3.1]. As usual, the asterisk notation

$$\xi^{\gamma *} = |\xi|^{\gamma} \operatorname{sgn} \xi, \quad \xi \in \boldsymbol{R}\,, \quad \gamma > 0\,,$$

is used.

Theorem A. *Consider the equation* (1.1) *under the condition* (1.4)*. Define the function* $\pi(t)$ *by* (1.5)*. Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* σ_1, σ_2 ($\sigma_1 < \sigma_2$) *be the real roots of* (1.7)*.*

If (1.1) *has a solution* $x_i(t) \in \text{n–RV}_{1/\pi}(\sigma_i^{(1/\alpha)*})$ ($i = 1$ *or* 2)*, then* (1.10) *is satisfied. Conversely, if* (1.10) *is satisfied, then* (1.1) *has a pair of solutions* $x_i(t) \in \text{n–RV}_{1/\pi}(\sigma_i^{(1/\alpha)*})$ ($i = 1$ *and* 2)*.*

Observe that $x_i(t)$ belongs to $\text{n–RV}_{1/\pi}(\sigma_i^{(1/\alpha)*})$ if and only if

$$\lim_{t\to\infty} p(t)^{1/\alpha} \pi(t) \frac{x_i'(t)}{x_i(t)} = \sigma_i^{(1/\alpha)*}.$$

Throughout the paper the following fact plays an essential part. Let $x(t)$ be a nonoscillatory solution of (1.1). We suppose that $x(t) > 0$ for $t \ge T$ ($> t_0$). Put

$$y(t) = p(t)\Big(\frac{x'(t)}{x(t)}\Big)^{\alpha *}, \quad t \ge T\,. \tag{1.11}$$

Then, $y(t)$ satisfies the generalized Riccati differential equation

$$y'(t) = -q(t) - \alpha p(t)^{-1/\alpha} |y(t)|^{(\alpha+1)/\alpha}\,, \quad t \ge T\,. \tag{1.12}$$

Conversely, if $y(t)$ is a solution of (1.12) on $[T, \infty)$, then

$$x(t) = \exp\Big(\int_T^t p(s)^{-1/\alpha} y(s)^{(1/\alpha)*}\,ds\Big)\,, \quad t \ge T\,, \tag{1.13}$$

is a positive solution of (1.1) on $[T, \infty)$. The proof is immediate.

In this paper we first note that the condition (1.8) is necessary for the existence of a solution $x(t)$ which belongs to the class n–$\mathrm{RV}_{1/\pi}(\mu)$ for some $\mu \in \boldsymbol{R}$. Then, a natural integral form of (1.12) is

$$\begin{aligned}\pi(t)^{\alpha+1}y(t) = Q_{\pi,\alpha}(t) + (\alpha+1)\int_t^\infty p(s)^{-1/\alpha}\pi(s)^\alpha y(s)\,ds \\ + \alpha\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\alpha+1}|y(s)|^{(\alpha+1)/\alpha}\,ds\,, \quad t \geq T\,,\end{aligned} \tag{1.14}$$

where $Q_{\pi,\alpha}(t)$ is defined by (1.9). Actually, we have the following proposition.

Proposition 1.1. *Consider the equation* (1.1) *under the condition* (1.4)*. Define the function* $\pi(t)$ *by* (1.5)*. If the equation* (1.1) *has a nonoscillatory solution* $x(t)$*,* $x(t) > 0$ $(t \geq T)$*, such that*

$$\lim_{t\to\infty} p(t)^{1/\alpha}\pi(t)\frac{x'(t)}{x(t)} = \mu \tag{1.15}$$

for some $\mu \in \boldsymbol{R}$*, then* (1.8) *holds, and moreover, the function* $y(t)$ *defined by* (1.11) *satisfies* (1.14)*.*

Proposition 1.1 is essentially proved in [7]. See the proof of the "only if" part of Theorem 3.1 in [7].

The purpose of this paper is to show that the last statement of Theorem A can be refined as follows:

Theorem 1.1. *Consider the equation* (1.1) *under the condition* (1.4)*. Define the function* $\pi(t)$ *by* (1.5)*. Assume that* (1.8) *holds and define* $Q_{\pi,\alpha}(t)$ *by* (1.9)*. Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* σ_1*,* σ_2 *be the real roots of* (1.7) *such that* $\sigma_1 < \sigma_2$ *and* $\sigma_2 \neq 0$*. Suppose that* (1.10) *holds and put*

$$\varepsilon(t) = \frac{Q_{\pi,\alpha}(t)}{\pi(t)} - \alpha c\,, \quad t \geq t_0\,. \tag{1.16}$$

If

$$\int_{t_0}^\infty \frac{|\varepsilon(t)|}{p(t)^{1/\alpha}\pi(t)}\,dt < \infty\,, \tag{1.17}$$

then (1.1) *has a pair of solutions* $x_i(t)$ $(i = 1$ *and* $2)$ *such that*

$$\begin{cases} x_i(t) \sim \pi(t)^{-\mu_i} & \text{as } t \to \infty\,, \\ x_i'(t) \sim \mu_i p(t)^{-1/\alpha}\pi(t)^{-\mu_i-1} & \text{as } t \to \infty\,, \end{cases} \tag{1.18}$$

where $\mu_i = \sigma_i^{(1/\alpha)*}$ $(i = 1, 2)$*.*

Here, the notation $f(t) \sim g(t)$ as $t \to \infty$ means that

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = 1\,.$$

Note that the solution $x_i(t)$ satisfying (1.18) belongs to the class n–$\mathrm{RV}_{1/\pi}(\mu_i)$, $i = 1, 2$.

Recent results on regularly varying solutions of (1.1) are found, e.g., in [8, 10, 14, 15]. The paper [10] is a systematic survey of the theory of existence and asymptotic behavior of nonoscillatory solutions of (1.1) and (1.2) by means of regularly varying functions (in the sense of Karamata). In [8] the detailed information about the asymptotic behavior of generalized regularly varying solutions of (1.1) is obtained. Řehák [14] and Řehák and Taddei [15] considered the equation (1.1) of the case $q(t) < 0$ and obtained the precise asymptotic behavior of nonoscillatory (monotone) solutions of (1.1) with the aid of the Karamata theory of regular variation and the de Haan theory, and, in particular, the conditions which guarantee that all (nonoscillatory) solutions of (1.1) are regularly varying were given.

The result in this paper is new even for the linear equation (1.2). However, an analogous result to Theorem 1.1 is derived from Theorem 9.1 in [4]. As an auxiliary equation, consider the Euler equation

$$(p(t)y')' + \frac{c}{p(t)\pi(t)^2} y = 0,$$

which has a principal solution $y_1(t) = \pi(t)^{-\mu_1}$ and a nonprincipal solution $y_2(t) = \pi(t)^{-\mu_2}$. Applying Theorem 9.1 in [4], we have the following result. If

$$\int^{\infty} \frac{|p(t)\pi(t)^2 q(t) - c|}{p(t)\pi(t)}\, dt < \infty\,, \tag{1.19}$$

then (1.2) has a pair of solutions $x_i(t)$ ($i = 1$ and 2) such that (1.18) with $\alpha = 1$ holds. Moreover (see Exercise 9.4 in [4]), if $p(t)\pi(t)^2 q(t) - c$ does not change signs, and (1.2) has a solution $x_i(t)$ satisfying (1.18) with $\alpha = 1$ for either $i = 1$ or $i = 2$, then (1.19) holds.

Theorem 1.1 for the case of $\alpha = 1$ yields the result of [13, Theorem 1.3 ($\alpha = 1$)]. To see this, we introduce the transformation of variables

$$\tau = 1 + \int_{t_0}^{t} \frac{1}{p(s)}\, ds\,, \quad y(\tau) = \frac{x(t)}{\tau}\,.$$

Then the equation of the form (1.1) with $\displaystyle\int_{t_0}^{\infty} p(s)^{-1} ds = \infty$ reduces to the equation

$$\frac{d}{d\tau}\Big(\tau^2 \frac{d}{d\tau} y\Big) + \tau^2 p(t) q(t) y = 0\,, \quad \tau \geq 1\,.$$

Since this is an equation of the type in the present paper, we are able to apply Theorem 1.1 ($\alpha = 1$). Then it is easy to check that the result of [13, Theorem 1.3 ($\alpha = 1$)] is derived from Theorem 1.1 ($\alpha = 1$).

Conversely, Theorem 1.1 for the case of $\alpha = 1$ may be derived from the result of [13, Theorem 1.3 ($\alpha = 1$)]. In fact, the change of variables

$$\tau = \Big(\int_{t}^{\infty} \frac{1}{p(s)}\, ds\Big)^{-1}, \quad z(\tau) = \tau x(t)$$

transforms the equation (1.1) into the equation

$$\frac{d^2}{d\tau^2} z + \frac{1}{\tau^4} p(t) q(t) z = 0\,, \quad \tau \geq \Big(\int_{t_0}^{\infty} \frac{1}{p(t)}\, dt\Big)^{-1},$$

to which the result of [13, Theorem 1.3 ($\alpha = 1$)] can be applicable.

The proof of Theorem 1.1 is similar to that used in [7]. The key idea of the proof of Theorem 1.1 is to use Hardy-type integral inequalities. For the classical Hardy inequality, see Hardy et al. [3, Theorem 330]. In [1], Beesack gave a systematic treatment of some analogues and extensions of the classical Hardy inequality. Then Beesack employed half-linear differential equations of the special form and corresponding generalized Riccati differential equations. We adapt the method of Beesack [1], and give two kinds of Hardy-type integral inequalities which are necessary for the proof of Theorem 1.1. These are stated and proved in Section 2. The proof of Theorem 1.1 is given in Section 3.

2. Hardy-type inequalities

Suppose that $g(t)$ is a continuous and positive function on an interval $[a, \infty)$ such that

$$\int_a^\infty g(s)\, ds < \infty\,. \tag{2.1}$$

Then we define the function $\gamma(t)$ by

$$\gamma(t) = \int_t^\infty g(s)\, ds\,, \quad t \ge a\,. \tag{2.2}$$

As a general inequality we have, for $\rho > 1$,

$$\xi^\rho + (\rho - 1)\eta^\rho - \rho\xi\eta^{\rho-1} \ge 0 \qquad (\xi \ge 0,\ \eta \ge 0)\,. \tag{2.3}$$

The inequality (2.3) plays an important part in this section.

Theorem 2.1. *Let ρ and r be constants such that $\rho > 1$ and $r > 1$, respectively. Suppose that $g(t)$ is continuous on $[a, \infty)$, and $g(t) > 0$ ($t \ge a$), and (2.1) holds. Define the function $\gamma(t)$ by (2.2). Suppose further that $f(t)$ is continuous on $[a, \infty)$ and satisfies*

$$\int_a^\infty g(s)|f(s)|\, ds < \infty\,, \quad \text{and} \tag{2.4}$$

$$\int_a^\infty g(s)\gamma(s)^{\rho-r}|f(s)|^\rho\, ds < \infty\,. \tag{2.5}$$

Then we have

$$\begin{aligned} &\int_a^\infty g(s)\gamma(s)^{-r}\Big[\int_s^\infty g(\sigma)|f(\sigma)|\, d\sigma\Big]^\rho ds \\ &\qquad \le \Big(\frac{\rho}{r-1}\Big)^\rho \int_a^\infty g(s)\gamma(s)^{\rho-r}|f(s)|^\rho\, ds\,. \end{aligned} \tag{2.6}$$

Equality holds if and only if $f(t) \equiv 0$ on $[a, \infty)$.

Proof. We adapt the method of Beesack [1]. Define the functions $\varphi(t)$ and $\psi(t)$ by

$$\varphi(t) = \Big(\frac{\rho}{r-1}\Big)^\rho \gamma(t)^{\rho-r} \quad \text{and} \quad \psi(t) = \gamma(t)^{-r}$$

for $t \geq a$. Clearly, $\varphi(t) > 0$ and $\psi(t) > 0$ $(t \geq a)$. It is easily seen that the function

$$x(t) = \gamma(t)^{(r-1)/\rho}, \quad t \geq a,$$

satisfies $x'(t) < 0$ $(t \geq a)$, and that $x = x(t)$ is a positive solution of the half-linear differential equation

$$(g(t)^{-\rho+1}\varphi(t)|x'|^{\rho-1}\operatorname{sgn} x')' + g(t)\psi(t)|x|^{\rho-1}\operatorname{sgn} x = 0, \quad t \geq a.$$

Therefore the function

$$y(t) = -\Big(\frac{x'(t)}{x(t)}\Big)^{(\rho-1)*} (> 0)$$

satisfies the generalized Riccati differential equation

$$\big(g(t)^{-\rho+1}\varphi(t)y(t)\big)' = g(t)\psi(t) + (\rho-1)g(t)^{-\rho+1}\varphi(t)y(t)^{\rho/(\rho-1)} \tag{2.7}$$

for $t \geq a$.

Applying the inequality (2.3) to the case

$$\xi = g(t)|f(t)|, \quad \eta = y(t)^{1/(\rho-1)}\int_t^\infty g(\sigma)|f(\sigma)|\,d\sigma,$$

we obtain

$$\begin{aligned} g(t)^\rho|f(t)|^\rho &+ (\rho-1)y(t)^{\rho/(\rho-1)}\Big[\int_t^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^\rho \\ &- \rho g(t)|f(t)|y(t)\Big[\int_t^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^{\rho-1} \geq 0, \quad t \geq a. \end{aligned}$$

Multiply the above inequality by $g(t)^{-\rho+1}\varphi(t)$ and integrate from a to t $(t \geq a)$. Then it is seen that the function

$$\begin{aligned} I(t) \equiv &\int_a^t g(s)\varphi(s)|f(s)|^\rho ds \\ &+ (\rho-1)\int_a^t g(s)^{-\rho+1}\varphi(s)y(s)^{\rho/(\rho-1)}\Big[\int_s^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \\ &- \rho\int_a^t g(s)^{-\rho+1}\varphi(s)y(s)g(s)|f(s)|\Big[\int_s^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^{\rho-1} ds \end{aligned}$$

is nonnegative for $t \geq a$. Denote the last term of $I(t)$ by $I_3(t)$. An integration by parts and use of (2.7) imply

$$\begin{aligned} I_3(t) = &\, g(t)^{-\rho+1}\varphi(t)y(t)\Big[\int_t^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho \\ &- g(a)^{-\rho+1}\varphi(a)y(a)\Big[\int_a^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho \\ &- \int_a^t \big\{g(s)\psi(s) + (\rho-1)g(s)^{-\rho+1}\varphi(s)y(s)^{\rho/(\rho-1)}\big\} \\ &\times \Big[\int_s^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^\rho ds, \quad t \geq a. \end{aligned}$$

Therefore, since $I(t) \geq 0$ $(t \geq a)$, we obtain

$$
\begin{aligned}
&\int_a^t g(s)\psi(s)\Big[\int_s^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \\
&\quad\le \int_a^t g(s)\varphi(s)|f(s)|^\rho ds + g(t)^{-\rho+1}\varphi(t)y(t)\Big[\int_t^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^\rho \\
&\qquad - g(a)^{-\rho+1}\varphi(a)y(a)\Big[\int_a^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^\rho .
\end{aligned}
\tag{2.8}
$$

We claim that

$$
\lim_{t\to\infty} g(t)^{-\rho+1}\varphi(t)y(t)\Big[\int_t^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho = 0\,. \tag{2.9}
$$

To prove (2.9), we note that $y(t)$ is explicitly given by

$$
y(t) = \Big(\frac{\rho}{r-1}\Big)^{-\rho+1} g(t)^{\rho-1}\gamma(t)^{-\rho+1}\,, \quad t\ge a\,,
$$

and so

$$
g(t)^{-\rho+1}\varphi(t)y(t) = \frac{\rho}{r-1}\gamma(t)^{1-r}\,.
$$

Using the Hölder inequality, we have

$$
\begin{aligned}
\int_t^\infty g(\sigma)|f(\sigma)|\,d\sigma &\le \Big[\int_t^\infty g(\sigma)\gamma(\sigma)^{-(\rho-r)/(\rho-1)}\,d\sigma\Big]^{(\rho-1)/\rho} \\
&\quad\times \Big[\int_t^\infty g(\sigma)\gamma(\sigma)^{\rho-r}|f(\sigma)|^\rho\,d\sigma\Big]^{1/\rho} \\
&= A\gamma(t)^{(r-1)/\rho}\Big[\int_t^\infty g(\sigma)\gamma(\sigma)^{\rho-r}|f(\sigma)|^\rho\,d\sigma\Big]^{1/\rho},
\end{aligned}
$$

where

$$
A = \Big(\frac{\rho-1}{r-1}\Big)^{(\rho-1)/\rho} > 0\,.
$$

Therefore we find that

$$
\begin{aligned}
0 &\le g(t)^{-\rho+1}\varphi(t)y(t)\Big[\int_t^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho \\
&\le \frac{\rho}{r-1}A^\rho\int_t^\infty g(\sigma)\gamma(\sigma)^{\rho-r}|f(\sigma)|^\rho d\sigma\,.
\end{aligned}
$$

Then, by (2.5), we see that (2.9) holds as claimed.

Let $t\to\infty$ in (2.8). Then, noting that (2.9) holds, we find

$$
\begin{aligned}
&\int_a^\infty g(s)\psi(s)\Big[\int_s^\infty g(\sigma)|f(\sigma)|\,\sigma\Big]^\rho ds \\
&\quad\le \int_a^\infty g(s)\varphi(s)|f(s)|^\rho ds - g(a)^{-\rho+1}\varphi(a)y(a)\Big[\int_a^\infty g(\sigma)|f(\sigma)|d\sigma\Big]^\rho,
\end{aligned}
$$

and so

$$
\int_a^\infty g(s)\psi(s)\Big[\int_s^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \le \int_a^\infty g(s)\varphi(s)|f(s)|^\rho\,ds\,. \tag{2.10}
$$

The inequality (2.10) is identical with the inequality (2.6). The equality holds in (2.10) if and only if

$$g(a)^{-\rho+1}\varphi(a)y(a)\Big[\int_a^\infty g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho = 0\,, \quad \text{i.e.} \quad f(t) \equiv 0 \text{ on } [a,\infty)\,.$$

This finishes the proof of Theorem 2.1. □

Theorem 2.2. *Let ρ and r be constants such that $\rho > 1$ and $r < 1$, respectively. Suppose that $g(t)$ is continuous on $[a,\infty)$, and $g(t) > 0$ $(t \ge a)$, and (2.1) holds. Define the function $\gamma(t)$ by (2.2). Let $b > a$. Suppose that $f(t)$ is continuous on $[a,b]$. Then we have*

$$\begin{aligned}(2.11)\qquad &\int_a^b g(s)\gamma(s)^{-r}\Big[\int_a^s g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \\ &\le \Big(\frac{\rho}{1-r}\Big)^\rho \int_a^b g(s)\gamma(s)^{\rho-r}|f(s)|^\rho\,ds\,.\end{aligned}$$

Equality holds if and only if $f(t) \equiv 0$ on $[a,b]$.

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. We put

$$\varphi(t) = \Big(\frac{\rho}{1-r}\Big)^\rho \gamma(t)^{\rho-r} \quad \text{and} \quad \psi(t) = \gamma(t)^{-r}$$

for $t \in [a,b]$. It is easy to see that the function

$$x(t) = \gamma(t)^{-(1-r)/\rho}\,, \quad a \le t \le b\,,$$

satisfies $x'(t) > 0$ $(a \le t \le b)$, and that $x = x(t)$ is a positive solution of the half-linear differential equation

$$(g(t)^{-\rho+1}\varphi(t)|x'|^{\rho-1}\operatorname{sgn} x')' + g(t)\psi(t)|x|^{\rho-1}\operatorname{sgn} x = 0\,, \quad a \le t \le b\,.$$

Therefore the function

$$y(t) = \Big(\frac{x'(t)}{x(t)}\Big)^{(\rho-1)*}(>0)$$

satisfies the generalized Riccati differential equation

$$(2.12)\qquad \big(g(t)^{-\rho+1}\varphi(t)y(t)\big)' + g(t)\psi(t) + (\rho-1)g(t)^{-\rho+1}\varphi(t)y(t)^{\rho/(\rho-1)} = 0$$

for $t \in [a,b]$.

Apply the inequality (2.3) to the case

$$\xi = g(t)|f(t)|\,, \quad \eta = y(t)^{1/(\rho-1)}\int_a^t g(\sigma)|f(\sigma)|\,d\sigma\,.$$

Then, it is found that the number

$$\begin{aligned}J \equiv &\int_a^b g(s)\varphi(s)|f(s)|^\rho\,ds \\ &+ (\rho-1)\int_a^b g(s)^{-\rho+1}\varphi(s)y(s)^{\rho/(\rho-1)}\Big[\int_a^s g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \\ &- \rho\int_a^b g(s)^{-\rho+1}\varphi(s)y(s)g(s)|f(s)|\Big[\int_a^s g(\sigma)|f(\sigma)|\,d\sigma\Big]^{\rho-1} ds\end{aligned}$$

is nonnegative. Integrating the last term of J by parts and using (2.12), we find that

$$\int_a^b g(s)\psi(s)\Big[\int_a^s g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds$$
$$\le \int_a^b g(s)\varphi(s)|f(s)|^\rho\,ds - g(b)^{-\rho+1}\varphi(b)y(b)\Big[\int_a^b g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho,$$

and so

$$\int_a^b g(s)\psi(s)\Big[\int_a^s g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho ds \le \int_a^b g(s)\varphi(s)|f(s)|^\rho\,ds\,. \tag{2.13}$$

The inequality (2.13) is identical with (2.11). The equality holds in (2.13) if and only if

$$g(b)^{-\rho+1}\varphi(b)y(b)\Big[\int_a^b g(\sigma)|f(\sigma)|\,d\sigma\Big]^\rho = 0\,, \quad \text{i.e.} \quad f(t)\equiv 0 \text{ on } [a,b]\,.$$

This completes the proof of Theorem 2.2. $\square$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we use the following lemmas.

Lemma 3.1. *Let $\mu \neq 0$ be fixed. Let w and ε be real numbers satisfying $|w| < \infty$ and $|\varepsilon| \le |\mu|^\alpha/4$, respectively. The function*

$$\begin{aligned} F(w,\varepsilon) = {} & |w+\mu^{\alpha*}+\varepsilon|^{(\alpha+1)/\alpha} - |\mu^{\alpha*}+\varepsilon|^{(\alpha+1)/\alpha} \\ & - \frac{\alpha+1}{\alpha}(\mu^{\alpha*}+\varepsilon)^{(1/\alpha)*}w \end{aligned} \tag{3.1}$$

satisfies

$$0 \le F(w,\varepsilon) \le K(\alpha)|\mu|^{-\alpha+1}w^2 \quad \Big(|w| \le \frac{|\mu|^\alpha}{4},\ |\varepsilon| \le \frac{|\mu|^\alpha}{4}\Big),$$

where

$$K(\alpha) = \begin{cases} \dfrac{\alpha+1}{2\alpha^2}\Big(\dfrac{3}{2}\Big)^{(-\alpha+1)/\alpha} & (0<\alpha\le 1),\\[2mm] \dfrac{\alpha+1}{2\alpha^2}\Big(\dfrac{1}{2}\Big)^{(-\alpha+1)/\alpha} & (\alpha>1). \end{cases}$$

Note that the function $F(w,\varepsilon)$ defined by (3.1) arises naturally in [6, 7]. For a brief proof of Lemma 3.1, see Naito [12, Lemma 2.4].

Lemma 3.2. *Let $\mu \neq 0$. Then*

$$\big||\mu^{\alpha*}+\varepsilon|^{(\alpha+1)/\alpha} - |\mu|^{\alpha+1}\big| \le 2\frac{\alpha+1}{\alpha}|\mu||\varepsilon| \quad \textit{and} \tag{3.2}$$

$$|(\mu^{\alpha*}+\varepsilon)^{(1/\alpha)*} - \mu| \le \frac{2}{\alpha}|\mu|^{-\alpha+1}|\varepsilon| \tag{3.3}$$

for all sufficiently small $|\varepsilon|$.

Since

$$\lim_{\varepsilon\to 0}\frac{|\mu^{\alpha*}+\varepsilon|^{(\alpha+1)/\alpha}-|\mu|^{\alpha+1}}{\varepsilon}=\frac{\alpha+1}{\alpha}\mu \qquad \text{and}$$

$$\lim_{\varepsilon\to 0}\frac{(\mu^{\alpha*}+\varepsilon)^{(1/\alpha)*}-\mu}{\varepsilon}=\frac{1}{\alpha}|\mu|^{-\alpha+1} \qquad (\mu\neq 0)\,,$$

Lemma 3.2 is obvious.

Now, let c be a real constant such that $c<E(\alpha)$, where $E(\alpha)$ is defined by (1.6). We suppose that (1.8) and (1.10) hold. Here, $\pi(t)$ and $Q_{\pi,\alpha}(t)$ are defined by (1.5) and (1.9), respectively. For the roots σ_1 and σ_2 ($\sigma_1<\sigma_2$) of (1.7), set $\mu_i=\sigma_i^{(1/\alpha)*}$ ($i=1,2$). Then, μ_1 and μ_2 are solutions of the equation

$$|\mu|^{\alpha+1}+\mu^{\alpha*}+c=0, \tag{3.4}$$

and satisfy $\mu_1<-\alpha/(\alpha+1)<\mu_2$. Let $x(t)$ be a nonoscillatory solution of (1.1) and satisfy the asymptotic condition of the form (1.18). We suppose that $x(t)>0$ for $t\geq T$, and define the function $y(t)$ by (1.11). According to Proposition 1.1, the function $y(t)$ satisfies (1.14).

Let $\mu=\mu_1$ or $\mu=\mu_2\neq 0$. We define the function $w(t)$ by

$$w(t)=\pi(t)^\alpha y(t)-\mu^{\alpha*}-\varepsilon(t)\,,\quad t\geq T\,,$$

where $\varepsilon(t)$ is given by (1.16). Since $\varepsilon(t)\to 0$ ($t\to\infty$), we may suppose that $|\varepsilon(t)|\leq|\mu|^\alpha/4$ for $t\geq T$. Using (3.4) and (1.14), we have

$$\begin{aligned} w(t)=&-\alpha|\mu|^{\alpha+1}-(\alpha+1)\mu^{\alpha*}\\ &+\frac{\alpha+1}{\pi(t)}\int_t^\infty p(s)^{-1/\alpha}\left(w(s)+\mu^{\alpha*}+\varepsilon(s)\right)ds\\ &+\frac{\alpha}{\pi(t)}\int_t^\infty p(s)^{-1/\alpha}\left|w(s)+\mu^{\alpha*}+\varepsilon(s)\right|^{(\alpha+1)/\alpha}ds \end{aligned} \tag{3.5}$$

for $t\geq T$. Then it is easy to see that

$$\begin{aligned} p(t)^{1/\alpha}\pi(t)w'(t)=&\,w(t)+\alpha|\mu|^{\alpha+1}+(\alpha+1)\mu^{\alpha*}\\ &-(\alpha+1)\left(w(t)+\mu^{\alpha*}+\varepsilon(t)\right)\\ &-\alpha\left|w(t)+\mu^{\alpha*}+\varepsilon(t)\right|^{(\alpha+1)/\alpha}\,,\quad t\geq T. \end{aligned}$$

This equality can be rewritten as

$$\begin{aligned} p(t)^{1/\alpha}\pi(t)w'(t)=&-\{(\alpha+1)\mu+\alpha\}\,w(t)-(\alpha+1)\varepsilon(t)\\ &-\alpha\left\{|\mu^{\alpha*}+\varepsilon(t)|^{(\alpha+1)/\alpha}-|\mu|^{\alpha+1}\right\}\\ &-(\alpha+1)\left\{(\mu^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu\right\}w(t)\\ &-\alpha F(w(t),\varepsilon(t))\,,\quad t\geq T\,, \end{aligned}$$

where $F(w,\varepsilon)$ is defined by (3.1).

For simplicity of notation, we put

$$f_1(t)=|\mu^{\alpha*}+\varepsilon(t)|^{(\alpha+1)/\alpha}-|\mu|^{\alpha+1},\quad f_2(t)=(\mu^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu, \tag{3.6}$$

and so

$$p(t)^{1/\alpha}\pi(t)w'(t) = -\left\{(\alpha+1)\mu+\alpha\right\}w(t) - (\alpha+1)\varepsilon(t) \tag{3.7}$$
$$- \alpha f_1(t) - (\alpha+1)f_2(t)w(t)$$
$$- \alpha F\big(w(t),\varepsilon(t)\big)$$

for $t \ge T$. Since $\varepsilon(t) \to 0$ $(t\to\infty)$, it follows from Lemma 3.2 that

$$|f_1(t)| \le 2\frac{\alpha+1}{\alpha}|\mu||\varepsilon(t)|\,, \quad |f_2(t)| \le \frac{2}{\alpha}|\mu|^{-\alpha+1}|\varepsilon(t)| \tag{3.8}$$

for all large t. Without loss of generality we assume that (3.8) holds for $t \ge T$.

In the following, we distinguish the cases

$$(\alpha+1)\mu+\alpha<0 \qquad \text{and} \qquad (\alpha+1)\mu+\alpha>0\,.$$

Let us consider the first case $(\alpha+1)\mu+\alpha<0$, i.e., $\mu=\mu_1$. In this case we set

$$\beta = -(\alpha+1)\mu_1 - \alpha \ (>0)\,.$$

Then, (3.7) yields

$$(\pi(t)^\beta w(t))' = -(\alpha+1)p(t)^{-1/\alpha}\pi(t)^{\beta-1}\varepsilon(t) \tag{3.9}$$
$$- \alpha p(t)^{-1/\alpha}\pi(t)^{\beta-1}f_1(t)$$
$$- (\alpha+1)p(t)^{-1/\alpha}\pi(t)^{\beta-1}f_2(t)w(t)$$
$$- \alpha p(t)^{-1/\alpha}\pi(t)^{\beta-1}F\big(w(t),\varepsilon(t)\big)$$

for $t \ge T$. Note that $x(t)$ is assumed to satisfy the asymptotic condition of the form (1.18) with $i=1$. Therefore we have

$$\pi(t)^\alpha y(t) = p(t)\pi(t)^\alpha\big(x'(t)/x(t)\big)^{\alpha*} \to \mu_1^{\alpha*} \quad \text{as} \quad t\to\infty\,,$$

and so $w(t)$ with $\mu=\mu_1$ tends to 0 as $t\to\infty$. In particular, $\pi(t)^\beta w(t)$ tends to 0 as $t\to\infty$. Therefore, it follows from (3.9) that

$$w(t) = (\alpha+1)\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}\varepsilon(s)\,ds \tag{3.10}$$
$$+ \alpha\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}f_1(s)\,ds$$
$$+ (\alpha+1)\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}f_2(s)w(s)\,ds$$
$$+ \alpha\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}F\big(w(s),\varepsilon(s)\big)\,ds, \quad t\ge T\,.$$

We are now ready for the proof of Theorem 1.1 of the case $i=1$.

Proof of Theorem 1.1 **of the case** $i=1$**.** Since $\varepsilon(t)\to 0$ $(t\to\infty)$, we can take $T>t_0$ sufficiently large so that $|\varepsilon(t)| \le |\mu_1|^\alpha/4$ for $t\ge T$. Define the functions $f_1(t)$ and $f_2(t)$ by (3.6) with $\mu=\mu_1$. We may suppose that (3.8) $(\mu=\mu_1)$ holds for $t \ge T$. Put

$$\eta(t) = \pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}|\varepsilon(s)|\,ds\,, \quad t\ge T\,. \tag{3.11}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, the function $\eta(t)$ is well defined and $\eta(t) \to 0$ $(t \to \infty)$. We put

$$M(\alpha, \mu_1) = 3(\alpha + 1)(1 + 2|\mu_1|).$$

Since $\eta(t) \to 0$ $(t \to \infty)$, we may suppose that

$$M(\alpha, \mu_1)\eta(t) \le |\mu_1|^\alpha/4\,, \quad t \ge T\,.$$

Denote by W the set of all functions $w \in C[T, \infty)$ such that

$$|w(t)| \le M(\alpha, \mu_1)\eta(t)\,, \quad t \ge T\,. \tag{3.12}$$

Moreover, keeping (3.10) in mind, we define the operator $\mathcal{F} : W \to C[T, \infty)$ by

$$\begin{aligned}(\mathcal{F}w)(t) = (\alpha + 1)\pi(t)^{-\beta} &\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}\varepsilon(s)\, ds \\ &+ \alpha\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}f_1(s)\, ds \\ &+ (\alpha + 1)\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}f_2(s)w(s)ds \\ &+ \alpha\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}F\big(w(s), \varepsilon(s)\big)\, ds\end{aligned}$$

for $t \ge T$. Here, $F(w, \varepsilon)$ is given by (3.1) with $\mu = \mu_1$. As is easily verified, the set W is a nonempty closed convex subset of the Fréchet space $C[T, \infty)$ of all continuous functions on $[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. Note that if $w \in W$, then $|w(t)| \le |\mu_1|^\alpha/4$ for $t \ge T$, and so, by Lemma 3.1,

$$0 \le F\big(w(t), \varepsilon(t)\big) \le K(\alpha)|\mu_1|^{-\alpha+1}w(t)^2\,, \quad t \ge T\,. \tag{3.13}$$

Then it is easily observed that $\mathcal{F}w$ is well defined for $w \in W$.

Let $w \in W$. Then, by (3.8) with $\mu = \mu_1$ and (3.12) and (3.13), we have

$$\begin{aligned}|(\mathcal{F}w)(t)| \le (\alpha + 1)\pi(t)^{-\beta} &\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}|\varepsilon(s)|\, ds \\ &+ 2(\alpha + 1)|\mu_1|\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}|\varepsilon(s)|\, ds \\ &+ \frac{2(\alpha + 1)}{\alpha}|\mu_1|^{-\alpha+1}M(\alpha, \mu_1)\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}|\varepsilon(s)|\eta(s)\, ds \\ &+ \alpha K(\alpha)|\mu_1|^{-\alpha+1}M(\alpha, \mu_1)^2\pi(t)^{-\beta}\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}\eta(s)^2\, ds\end{aligned}$$

for $t \ge T$. For the integral in the last term of the right-hand side of the above, apply the Hardy-type inequality (2.6) in Theorem 2.1 to the case $a = t$, $\rho = 2$, $r = \beta + 1$, and

$$g(t) = p(t)^{-1/\alpha} \quad \text{and} \quad f(t) = \pi(t)^{\beta-1}|\varepsilon(t)|\,.$$

Then it can be concluded that

$$\begin{aligned}
&\int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}\eta(s)^2 ds \\
&\quad = \int_t^\infty p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\left[\int_s^\infty p(\sigma)^{-1/\alpha}\pi(\sigma)^{\beta-1}|\varepsilon(\sigma)|d\sigma\right]^2 ds \\
&\quad \le \left(\frac{2}{\beta}\right)^2 \int_t^\infty p(s)^{-1/\alpha}\pi(s)^{\beta-1}|\varepsilon(s)|^2\, ds, \quad t \ge T\,.
\end{aligned}$$

Therefore, $|(\mathcal{F}w)(t)|$ is estimated as follows:

$$\begin{aligned}
|(\mathcal{F}w)(t)| &\le (\alpha+1)\eta(t) + 2(\alpha+1)|\mu_1|\eta(t) \\
&\quad + \frac{2(\alpha+1)}{\alpha}|\mu_1|^{-\alpha+1}M(\alpha,\mu_1)\Big[\sup_{s\ge t}\eta(s)\Big]\eta(t) \\
&\quad + \alpha K(\alpha)|\mu_1|^{-\alpha+1}M(\alpha,\mu_1)^2\Big(\frac{2}{\beta}\Big)^2\Big[\sup_{s\ge t}|\varepsilon(s)|\Big]\eta(t)
\end{aligned}$$

for $t \ge T$. Since $\sup_{s\ge t}\eta(s) \to 0$ and $\sup_{s\ge t}|\varepsilon(s)| \to 0$ as $t \to \infty$, we can suppose that

$$\frac{2(\alpha+1)}{\alpha}|\mu_1|^{-\alpha+1}\Big[\sup_{s\ge t}\eta(s)\Big] \le \frac{1}{3}\,, \quad t \ge T\,,$$

and

$$\alpha K(\alpha)|\mu_1|^{-\alpha+1}M(\alpha,\mu_1)\Big(\frac{2}{\beta}\Big)^2\Big[\sup_{s\ge t}|\varepsilon(s)|\Big] \le \frac{1}{3}\,, \quad t \ge T\,.$$

Then we obtain

$$|(\mathcal{F}w)(t)| \le M(\alpha,\mu_1)\eta(t)\,, \quad t \ge T\,.$$

This means that

(i) $\mathcal{F}$ maps W into W.

Moreover it can be checked that

(ii) $\mathcal{F}$ is continuous on W;

(iii) $\mathcal{F}W$ is uniformly bounded and equicontinuous at every point of $[T,\infty)$.

The Schauder-Tychonoff fixed point theorem implies that $\mathcal{F}$ has a fixed element $w \in W$: $w(t) = (\mathcal{F}w)(t)$, $t \ge T$. It is clear that this fixed element $w(t)$ satisfies (3.10) and (3.12). Since $\eta(t) \to 0$ $(t \to \infty)$, it follows from (3.12) that $w(t) \to 0$ as $t \to \infty$. In addition, it can be shown without difficulty that $w(t)$ satisfies (3.5) $(\mu = \mu_1)$ for $t \ge T$. Put

$$y(t) = \frac{w(t) + \mu_1^{\alpha *} + \varepsilon(t)}{\pi(t)^\alpha}\,, \qquad t \ge T\,. \tag{3.14}$$

Then we find that $y(t)$ satisfies (1.14). Moreover, it can be shown that $y(t)$ satisfies (1.12). Therefore, the function $x(t)$ which is defined by (1.13) is a positive solution of (1.1) on $[T,\infty)$. Furthermore, we have

$$p(t)\pi(t)^\alpha\Big(\frac{x'(t)}{x(t)}\Big)^{\alpha *} = \pi(t)^\alpha y(t) = w(t) + \mu_1^{\alpha *} + \varepsilon(t) \to \mu_1^{\alpha *}$$

as $t \to \infty$, and so

$$\lim_{t\to\infty} p(t)^{1/\alpha}\pi(t)\frac{x'(t)}{x(t)} = \mu_1 \,. \tag{3.15}$$

Since $\mu_1 = \sigma_1^{(1/\alpha)*}$, this implies that $x(t)$ belongs to the class n–$\mathrm{RV}_{1/\pi}(\sigma_1^{(1/\alpha)*})$.

Note that the arguments up to now give a proof of the last statement ($i = 1$) of Theorem A. The advantage of the arguments here is the bound (3.12). In fact, it follows from (1.11) and (3.14) that

$$\begin{aligned}\frac{x'(t)}{x(t)} &= \frac{(w(t)+\mu_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}}{p(t)^{1/\alpha}\pi(t)}\\ &= \frac{\mu_1}{p(t)^{1/\alpha}\pi(t)} + \frac{(w(t)+\mu_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu_1}{p(t)^{1/\alpha}\pi(t)}\,, \qquad t \ge T.\end{aligned} \tag{3.16}$$

Noting that $\varepsilon(t) \to 0$ and $w(t) \to 0$ ($t \to \infty$) and using (3.3) in Lemma 3.2, we see that

$$|(w(t)+\mu_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu_1| \le \frac{2}{\alpha}|\mu_1|^{-\alpha+1}\{|w(t)|+|\varepsilon(t)|\} \tag{3.17}$$

for all large t. By (3.16), we have

$$x(t) = \frac{x(T_1)}{\pi(T_1)^{-\mu_1}}\exp\Big(\int_{T_1}^t \frac{(w(s)+\mu_1^{\alpha*}+\varepsilon(s))^{(1/\alpha)*}-\mu_1}{p(s)^{1/\alpha}\pi(s)}\,ds\Big)\pi(t)^{-\mu_1} \tag{3.18}$$

for $t \ge T_1$, where T_1 is a constant and is taken sufficiently large. It should be noticed that the assumption (1.17), together with the condition $\varepsilon(t) \to 0$ ($t \to \infty$), implies

$$\int_{T_1}^\infty \frac{\eta(t)}{p(t)^{1/\alpha}\pi(t)}\,dt < \infty\,, \tag{3.19}$$

where $\eta(t)$ is given by (3.11). The verification of this fact is left to the reader. Then, by (3.12) and (3.19), we have

$$\int_{T_1}^\infty \frac{|w(t)|}{p(t)^{1/\alpha}\pi(t)}\,dt < \infty\,. \tag{3.20}$$

By (1.17), (3.17), (3.18) and (3.20), it is found that $x(t)$ is written in the form

$$x(t) = c_0(t)\pi(t)^{-\mu_1} \quad \text{with} \quad c_0(t) \to c_0 \in (0,\infty) \;\text{ as }\; t \to \infty\,. \tag{3.21}$$

Then we have

$$x'(t) = c_0(t)p(t)^{-1/\alpha}\pi(t)^{-\mu_1-1}p(t)^{1/\alpha}\pi(t)\frac{x'(t)}{x(t)},$$

and so (3.15) implies that $x'(t)$ is written in the form

$$x'(t) = c_1(t)p(t)^{-1/\alpha}\pi(t)^{-\mu_1-1} \quad \text{with} \quad c_1(t) \to c_0\mu_1 \;\text{ as }\; t \to \infty. \tag{3.22}$$

In general, if $x(t)$ is a solution of the half-linear equation (1.1) and if c is a constant, then $cx(t)$ is also a solution of (1.1). Therefore, without loss of generality, we may suppose that $c_0 = 1$ in (3.21) and (3.22). This shows (1.18) with $i = 1$. The proof of Theorem 1.1 of the case $i = 1$ is complete. □

In order to discuss the second case $(\alpha+1)\mu+\alpha>0$, let us return to (3.7). Note that $\mu=\mu_2$ for this case. We set

$$\beta=(\alpha+1)\mu_2+\alpha\ (>0)\,.$$

Then, (3.7) yields

$$\begin{aligned}\big(\pi(t)^{-\beta}w(t)\big)' &= -(\alpha+1)p(t)^{-1/\alpha}\pi(t)^{-\beta-1}\varepsilon(t)\\ &\quad-\alpha p(t)^{-1/\alpha}\pi(t)^{-\beta-1}f_1(t)\\ &\quad-(\alpha+1)p(t)^{-1/\alpha}\pi(t)^{-\beta-1}f_2(t)w(t)\\ &\quad-\alpha p(t)^{-1/\alpha}\pi(t)^{-\beta-1}F\big(w(t),\varepsilon(t)\big)\,,\quad t\ge T\,.\end{aligned}$$

Therefore we have

$$\begin{aligned}w(t)&=\pi(T)^{-\beta}w(T)\pi(t)^{\beta}\\ &\quad-(\alpha+1)\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\varepsilon(s)\,ds\\ &\quad-\alpha\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_1(s)\,ds\\ &\quad-(\alpha+1)\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_2(s)w(s)\,ds\\ &\quad-\alpha\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}F\big(w(s),\varepsilon(s)\big)\,ds\,,\quad t\ge T\,.\end{aligned}\tag{3.23}$$

We can now give the proof of Theorem 1.1 of the case $i=2$.

Proof of Theorem 1.1 **of the case** $i=2$**.** Since $\varepsilon(t)\to 0$ $(t\to\infty)$, we can choose $T>t_0$ sufficiently large so that $|\varepsilon(t)|\le|\mu_2|^{\alpha}/4$ for $t\ge T$. Define the functions $f_1(t)$ and $f_2(t)$ by (3.6) with $\mu=\mu_2$. We may suppose that (3.8) $(\mu=\mu_2)$ holds for $t\ge T$. Put

$$\eta(t;T)=\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}|\varepsilon(s)|\,ds\,,\quad t\ge T\,.$$

Since $\varepsilon(t)\to 0$ $(t\to\infty)$, we have $\eta(t;T)\to 0$ $(t\to\infty)$. It is clear that

$$\eta(t;T)\le\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\,ds\Big[\sup_{s\ge T}|\varepsilon(s)|\Big]\le\frac{1}{\beta}\Big[\sup_{s\ge T}|\varepsilon(s)|\Big]\tag{3.24}$$

for $t\ge T$. We put

$$M(\alpha,\mu_2)=3(\alpha+1)(1+2|\mu_2|)\,.$$

Since $\sup_{s\ge T}|\varepsilon(s)|\to 0$ as $T\to\infty$, we may suppose that

$$M(\alpha,\mu_2)\frac{1}{\beta}\Big[\sup_{s\ge T}|\varepsilon(s)|\Big]\le|\mu_2|^{\alpha}/4\,.$$

Denote by W the set of all functions $w\in C[T,\infty)$ such that

$$|w(t)|\le M(\alpha,\mu_2)\eta(t;T)\,,\quad t\ge T\,.\tag{3.25}$$

For $w \in W$, we have $w(T) = 0$. Then, keeping (3.23) in mind, we define the operator $\mathcal{F}\colon W \to C[T,\infty)$ by

$$\begin{aligned}(\mathcal{F}w)(t) = &-(\alpha+1)\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\varepsilon(s)\,ds \\ &-\alpha\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_1(s)\,ds \\ &-(\alpha+1)\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_2(s)w(s)\,ds \\ &-\alpha\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}F\big(w(s),\varepsilon(s)\big)\,ds\end{aligned}$$

for $t \ge T$. Here, $F(w,\varepsilon)$ is given by (3.1) with $\mu = \mu_2$.

Let $w \in W$. We have $|w(t)| \le |\mu_2|^{\alpha}/4$ for $t \ge T$, and so, by Lemma 3.1,

$$0 \le F\big(w(t),\varepsilon(t)\big) \le K(\alpha)|\mu_2|^{-\alpha+1}w(t)^2\,, \quad t \ge T\,. \tag{3.26}$$

Therefore, by (3.8) with $\mu = \mu_2$ and (3.25) and (3.26), we have

$$\begin{aligned}|(\mathcal{F}w)(t)| \le\ &(\alpha+1)\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}|\varepsilon(s)|\,ds \\ &+2(\alpha+1)|\mu_2|\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}|\varepsilon(s)|\,ds \\ &+\frac{2(\alpha+1)}{\alpha}|\mu_2|^{-\alpha+1}M(\alpha,\mu_2)\pi(t)^{\beta} \\ &\times\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}|\varepsilon(s)|\eta(s;T)\,ds \\ &+\alpha K(\alpha)|\mu_2|^{-\alpha+1}M(\alpha,\mu_2)^2\pi(t)^{\beta}\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\eta(s;T)^2\,ds\end{aligned}$$

for $t \ge T$. For the third term of the right-hand side of the above, we use (3.24). For the last term of the right-hand, we apply the Hardy-type inequality (2.11) in Theorem 2.2 to the case $a = T$, $b = t$, $\rho = 2$, $r = -\beta + 1$ and

$$g(t) = p(t)^{-1/\alpha} \quad \text{and} \quad f(t) = \pi(t)^{-\beta-1}|\varepsilon(t)|\,.$$

Then we find that

$$\begin{aligned}&\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\eta(s;T)^2\,ds \\ &\quad= \int_T^t p(s)^{-1/\alpha}\pi(s)^{\beta-1}\Big[\int_T^s p(\sigma)^{-1/\alpha}\pi(\sigma)^{-\beta-1}|\varepsilon(\sigma)|d\sigma\Big]^2\,ds \\ &\quad\le \Big(\frac{2}{\beta}\Big)^2\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}|\varepsilon(s)|^2\,ds\,, \quad t \ge T\,.\end{aligned}$$

Therefore, $|(\mathcal{F}w)(t)|$ is estimated in the following way:

$$\begin{aligned}|(\mathcal{F}w)(t)| \le (\alpha+1)\eta(t;T) &+ 2(\alpha+1)|\mu_2|\eta(t;T)\\ &+ \frac{2(\alpha+1)}{\alpha}|\mu_2|^{-\alpha+1}M(\alpha,\mu_2)\frac{1}{\beta}\Big[\sup_{s\ge T}|\varepsilon(s)|\Big]\eta(t;T)\\ &+ \alpha K(\alpha)|\mu_2|^{-\alpha+1}M(\alpha,\mu_2)^2\Big(\frac{2}{\beta}\Big)^2\Big[\sup_{s\ge T}|\varepsilon(s)|\Big]\eta(t;T)\end{aligned}$$

for $t \ge T$. Since $\sup_{s\ge T}|\varepsilon(s)| \to 0$ as $T \to \infty$, we can suppose that

$$\frac{2(\alpha+1)}{\alpha}|\mu_2|^{-\alpha+1}\frac{1}{\beta}\Big[\sup_{s\ge T}|\varepsilon(s)|\Big] \le \frac{1}{3}$$

and

$$\alpha K(\alpha)|\mu_2|^{-\alpha+1}M(\alpha,\mu_2)\Big(\frac{2}{\beta}\Big)^2\Big[\sup_{s\ge T}|\varepsilon(s)|\Big] \le \frac{1}{3}.$$

Then we deduce that

$$|(\mathcal{F}w)(t)| \le M(\alpha,\mu_2)\eta(t;T)\,, \quad t \ge T\,.$$

This means that

(i) $\mathcal{F}$ maps W into W.

Moreover it can be checked without difficulty that

(ii) $\mathcal{F}$ is continuous on W;

(iii) $\mathcal{F}W$ is uniformly bounded and equicontinuous at every point of $[T,\infty)$.

The Schauder-Tychonoff fixed point theorem implies that $\mathcal{F}$ has a fixed element w in W. This fixed element $w(t)$ satisfies

$$\begin{aligned}w(t) = -(\alpha+1)\pi(t)^\beta &\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}\varepsilon(s)\,ds\\ &- \alpha\pi(t)^\beta\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_1(s)\,ds\\ &- (\alpha+1)\pi(t)^\beta\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}f_2(s)w(s)\,ds\\ &- \alpha\pi(t)^\beta\int_T^t p(s)^{-1/\alpha}\pi(s)^{-\beta-1}F\big(w(s),\varepsilon(s)\big)\,ds\,, \quad t \ge T\,,\end{aligned}$$

and has the bound (3.25). It is obvious that $w(t) \to 0$ as $t \to \infty$. Further, $w(t)$ satisfies (3.5) ($\mu = \mu_2$) for $t \ge T$. Put

$$y(t) = \frac{w(t) + \mu_2^{\alpha*} + \varepsilon(t)}{\pi(t)^\alpha}\,, \quad t \ge T\,, \tag{3.27}$$

and define $x(t)$ by (1.13). Then, as in the first case, we see that $x(t)$ is a positive solution of (1.1) on $[T,\infty)$, and

$$p(t)\pi(t)^\alpha\Big(\frac{x'(t)}{x(t)}\Big)^{\alpha*} = \pi(t)^\alpha y(t) = w(t) + \mu_2^{\alpha*} + \varepsilon(t) \to \mu_2^{\alpha*}$$

as $t \to \infty$. Consequently,

$$\lim_{t\to\infty} p(t)^{1/\alpha}\pi(t)\frac{x'(t)}{x(t)} = \mu_2\,. \tag{3.28}$$

Since $\mu_2 = \sigma_2^{(1/\alpha)*}$, this implies that $x(t)$ belongs to the class n–$\mathrm{RV}_{1/\pi}(\sigma_2^{(1/\alpha)*})$.

As in the first case, it follows from (1.11) and (3.27) that

$$\begin{aligned}\frac{x'(t)}{x(t)} &= \frac{(w(t)+\mu_2^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}}{p(t)^{1/\alpha}\pi(t)}\\ &= \frac{\mu_2}{p(t)^{1/\alpha}\pi(t)} + \frac{(w(t)+\mu_2^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu_2}{p(t)^{1/\alpha}\pi(t)}\,,\quad t\ge T\,.\end{aligned} \tag{3.29}$$

Using (3.3) in Lemma 3.2, we see that

$$|(w(t)+\mu_2^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\mu_2| \le \frac{2}{\alpha}|\mu_2|^{-\alpha+1}\big\{|w(t)|+|\varepsilon(t)|\big\}$$

for all large t. By (3.29), the solution $x(t)$ is expressed as

$$x(t) = \frac{x(T_1)}{\pi(T_1)^{-\mu_2}}\exp\Big(\int_{T_1}^{t}\frac{(w(s)+\mu_2^{\alpha*}+\varepsilon(s))^{(1/\alpha)*}-\mu_2}{p(s)^{1/\alpha}\pi(s)}\,ds\Big)\pi(t)^{-\mu_2}$$

for $t \ge T_1$, where T_1 is taken sufficiently large. Note that the assumption (1.17) implies

$$\int_{T_1}^{\infty}\frac{\eta(t;T)}{p(t)^{1/\alpha}\pi(t)}\,dt < \infty\,.$$

Then, by (3.25), we get

$$\int_{T_1}^{\infty}\frac{|w(t)|}{p(t)^{1/\alpha}\pi(t)}\,dt < \infty\,.$$

Therefore, as in the first case, it is seen that $x(t)$ is written in the form

$$x(t) = c_0(t)\pi(t)^{-\mu_2}\quad\text{with}\quad c_0(t)\to c_0\in(0,\infty)\ \text{ as }\ t\to\infty\,. \tag{3.30}$$

Then, by (3.28), we have

$$x'(t) = c_1(t)p(t)^{-1/\alpha}\pi(t)^{-\mu_2-1}\quad\text{with}\quad c_1(t)\to c_0\mu_2\ \text{ as }\ t\to\infty\,. \tag{3.31}$$

Without loss of generality, we may suppose that $c_0 = 1$ in (3.30) and (3.31). This implies (1.18) with $i = 2$. The proof of Theorem 1.1 of the case $i = 2$ is complete. □

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