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THE LIE GROUPOID ANALOGUE OF A SYMPLECTIC LIE GROUP

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Abstract. A symplectic Lie group is a Lie group with a left-invariant symplectic form. Its Lie algebra structure is that of a quasi-Frobenius Lie algebra. In this note, we identify the groupoid analogue of a symplectic Lie group. We call the aforementioned structure a *t-symplectic Lie groupoid*; the "t" is motivated by the fact that each target fiber of a t-symplectic Lie groupoid is a symplectic manifold. For a Lie groupoid $\mathcal{G} \rightrightarrows M$, we show that there is a one-to-one correspondence between quasi-Frobenius Lie algebroid structures on $A\mathcal{G}$ (the associated Lie algebroid) and t-symplectic Lie groupoid structures on $\mathcal{G} \rightrightarrows M$. In addition, we also introduce the notion of a *symplectic Lie group bundle* (SLGB) which is a special case of both a t-symplectic Lie groupoid and a Lie group bundle. The basic properties of SLGBs are explored.

1. Introduction

A symplectic Lie group is a Lie group G together with a left-invariant symplectic form ω [1, 5]. The associated Lie algebra structure is that of a quasi-Frobenius Lie algebra [3]; the latter is formally a Lie algebra $\mathfrak{q}$ together with a skew-symmetric, non-degenerate bilinear form β on $\mathfrak{q}$ such that

$$\beta([x,y],z) + \beta([y,z],x) + \beta([z,x],y) = 0$$

for all x, y, $z \in \mathfrak{q}$. In other words, β is a non-degenerate 2-cocycle in the Lie algebra cohomology of $\mathfrak{q}$ with values in $\mathbb{R}$ (where $\mathfrak{q}$ acts trivially on $\mathbb{R}$). For a symplectic Lie group (G,ω), the associated quasi-Frobenius Lie algebra is $(\mathfrak{g},\omega_e)$, where $\mathfrak{g} = T_eG$ is the Lie algebra defined by the left-invariant vector fields on G.

The notion of a *quasi-Frobenius Lie algebroid* (or *symplectic Lie algebroid* as it is more commonly called) was introduced independently in [6] and [14]. As one would expect, a quasi-Frobenius Lie algebroid over a point is simply a quasi-Frobenius Lie algebra. As far as the author can tell, the Lie groupoid analogue of a symplectic Lie group has not been formally identified in the literature. In other words, the following question has not yet been answered: *what is the Lie groupoid structure whose assoicated Lie algebroid is precisely a quasi-Frobenius Lie algebroid?*

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To be clear, there is a structure in the literature called a *symplectic Lie groupoid* [16]. However, it is unrelated to the notion of a symplectic Lie group. Formally, a symplectic Lie groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a symplectic form ω on $\mathcal{G}$ such that

$$\mathcal{G}_3 := \{(g,h,gh) \mid (g,h) \in \mathcal{G}_2\}$$

is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$, where $\overline{\mathcal{G}}$ is the symplectic manifold $(\mathcal{G}, -\omega)$ and

$$\mathcal{G}_2 := \{(g,h) \mid g,h \in \mathcal{G},\ s(g) = t(h)\}\,.$$

The condition that $\mathcal{G}_3$ is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$ is equivalent to the condition that

$$m^*\omega = \pi_1^*\omega + \pi_2^*\omega \tag{1.1}$$

where $m\colon \mathcal{G}_2 \longrightarrow \mathcal{G}$ denotes the multiplication map and $\pi_i\colon \mathcal{G}_2 \longrightarrow \mathcal{G}$ denotes the natural projection map for $i = 1, 2$.

Any Lie groupoid over a point is just a Lie group. Hence, one might expect that a symplectic Lie groupoid over a point is just a symplectic Lie group, but this is not the case. In fact, there are no symplectic Lie groupoids over a point. To see this, let ω be a 2-form on a Lie group G which satisfies (1.1). Let g, $h \in G$ and let

$$u := (x,y),\ v := (x',y') \in T_gG \times T_hG\,.$$

Then

$$(m^*\omega)_{(g,h)}(u,v) = \omega_{gh}\big((r_h)_*x + (l_g)_*y, (r_h)_*x' + (l_g)_*y'\big) \tag{1.2}$$

and

$$(\pi_1^*\omega)_{(g,h)}(u,v) + (\pi_2^*\omega)_{(g,h)}(u,v) = \omega_g(x,x') + \omega_h(y,y')\,. \tag{1.3}$$

Setting $h = e$, $x' = 0_g$, and $y = 0_e$ in (1.2) and (1.3) gives

$$(m^*\omega)_{(g,e)}(u,v) = \omega_g\big(x, (l_g)_*y'\big) \tag{1.4}$$

and

$$(\pi_1^*\omega)_{(g,e)}(u,v) + (\pi_2^*\omega)_{(g,e)}(u,v) = 0\,. \tag{1.5}$$

Since ω satisfies (1.1) by assumption, equations (1.4) and (1.5) imply that $\omega_g \equiv 0$ for all $g \in G$. This shows that for any Lie group G, there are no symplectic forms which satisfy (1.1). Hence, there are no symplectic Lie groupoids over a point.

In this note, we will identify the groupoid analogue of a symplectic Lie group. We call the aforementioned structure a *t-symplectic Lie groupoid*; the "t" is motivated by the fact that each target fiber of a t-symplectic Lie groupoid is a symplectic manifold. For a Lie groupoid $\mathcal{G} \rightrightarrows M$, we show that there is a one-to-one correspondence between quasi-Frobenius Lie algebroid structures on $A\mathcal{G}$ (the associated Lie algebroid) and t-symplectic Lie groupoid structures on $\mathcal{G} \rightrightarrows M$. In addition, we also introduce the notion of a *symplectic Lie group bundle* (SLGB) which is a special case of both a t-symplectic Lie groupoid and a Lie group bundle [10, 11].

The rest of this paper is organized as follows. In Section 2, we give a brief review of Lie groupoids and Lie algebroids. In Section 3, we introduce t-symplectic Lie groupoids, and establish the aforementioned one-to-one correspondence. Some basic examples of t-symplectic Lie groupoids are also presented. We conclude the paper in Section 4 by introducing SLGBs and exploring some of its basic properties. In addition, we also prove a result which is useful for the construction of nontrivial SLGBs.

2. Preliminaries

2.1. **Lie groupoids & Lie algebroids.** In this section, we give a brief review of Lie groupoids and Lie algebroids [10, 11, 12], mainly to establish the notation for the rest of the paper. We begin with the following definition:

Definition 2.1. A *Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ such that

(i) $\mathcal{G}$ and M are smooth manifolds
(ii) all structure maps are smooth
(iii) the source map $s\colon \mathcal{G} \longrightarrow M$ is a surjective submersion.

Remark 2.2. Note that condition (iii) of Definition 2.1 is equivalent to the condition that the target map $t\colon \mathcal{G} \longrightarrow M$ is a surjective submersion.

In addition, the axioms of a Lie groupoid imply that the unit map

$$u\colon M \longrightarrow \mathcal{G}$$

is a smooth embedding. As a consequence of this, we will often view M as an embedded submanifold of $\mathcal{G}$. With this viewpoint, u is simply the inclusion map.

The domain of the multiplication map m on $\mathcal{G}$ is typically denoted as

$$\mathcal{G}_2 := \{(g,h) \mid g,h \in \mathcal{G},\ s(g) = t(h)\}.$$

Give $(g,h) \in \mathcal{G}_2$, we set $gh := m(g,h)$.

Definition 2.3. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids. Let (s,t) and (s',t') denote the source and target maps of $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ respectively. Also, let u and u' denote the respective unit maps. A homomorphism from $\mathcal{G} \rightrightarrows M$ to $\mathcal{H} \rightrightarrows N$ is a pair of smooth maps $F\colon \mathcal{G} \to \mathcal{H}$ and $f\colon M \to N$ such that

(i) $F(gh) = F(g)F(h)$ for all $(g,h) \in \mathcal{G}_2$
(ii) $F \circ u = u' \circ f$
(iii) $s' \circ F = f \circ s$, $t' \circ F = f \circ t$

Example 2.4. Any Lie group is naturally a Lie groupoid over a point.

Example 2.5. Associated to any manifold M is the *pair groupoid* $M \times M \rightrightarrows M$ whose structure maps are defined as follows:

$$s(p,q) := q, \quad t(p,q) := p, \quad (p,q)(q,r) := (p,r)$$
$$u(p) := (p,p), \quad i(p,q) := (q,p)$$

for p, q, $r \in M$.

Example 2.6. Let M be a manifold with a smooth left-action by a Lie group G. Associated to (M,G) is the *action groupoid* $G \times M \rightrightarrows M$ whose structure maps are defined as follows:

$$s(g,p) := g^{-1}p\,, \quad t(g,p) := p\,, \quad (g,p)(h,g^{-1}p) := (gh,p)$$
$$u(p) := (e,p)\,, \quad i(g,p) := (g^{-1},g^{-1}p)\,.$$

for g, $h \in G$, $p \in M$.

Definition 2.7. A *Lie algebroid* is a triple (A,ρ,M) where A is a vector bundle over M and $\rho\colon A \to TM$ is a vector bundle map called the *anchor* such that
(i) $\Gamma(A)$ is a Lie algebra.
(ii) For X, $Y \in \Gamma(A)$ and $f \in C^\infty(M)$, the Lie bracket on $\Gamma(A)$ satisfies the following Leibniz-type rule:

$$[X,fY] = f[X,Y] + (\rho(X)f)Y\,.$$

Proposition 2.8. *Let (A,ρ,M) be a Lie algebroid. Then*
(i) *$\rho\colon \Gamma(A) \to \Gamma(TM)$ is a Lie algebra map, where the Lie bracket on $\Gamma(TM)$ is just the usual Lie bracket of vector fields.*
(ii) *$[fX,Y] = f[X,Y] - (\rho(Y)f)X$ for all $X,Y \in \Gamma(A)$, $f \in C^\infty(M)$.*

Proof. (i): See Lemma 8.1.4 of [7].
(ii): Direct calculation. □

Definition 2.9. Let (A,ρ,M) and (A',ρ',M) be Lie algebroids over the same base space M. A Lie algebroid homomorphism from (A,ρ,M) to (A',ρ',M) is a vector bundle map $\varphi\colon A \to A'$ such that
(i) $\varphi\colon \Gamma(A) \to \Gamma(A')$ is a Lie algebra map,
(ii) $\rho' \circ \varphi = \rho$.

Every Lie groupoid $\mathcal{G} \rightrightarrows M$ has an associated Lie algebroid $(A\mathcal{G},\rho,M)$ which arises by considering the Lie algebra of left-invariant vector fields on $\mathcal{G}$.

Definition 2.10. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A vector field $\widetilde{X}$ on $\mathcal{G}$ is *left-invariant* if

$$(l_g)_{*,h}\widetilde{X}_h = \widetilde{X}_{gh}$$

for all g, $h \in \mathcal{G}$, where $s(g) = t(h)$ and

$$l_g\colon t^{-1}\big(s(g)\big) \longrightarrow t^{-1}\big(t(g)\big)$$

is left multiplication by g.

Let

$$T^t\mathcal{G} := \ker t_* \subset T\mathcal{G}\,.$$

Since $t\colon \mathcal{G} \longrightarrow M$ is a surjective submersion, it follows that $T^t\mathcal{G}$ is a smooth sub-bundle of $T\mathcal{G}$. In addition, define

$$A\mathcal{G} := T^t\mathcal{G}|_M\,,$$

where we recall that M is identified with the embedded submanifold of $\mathcal{G}$ consisting of the unit elements. Let $\mathfrak{X}_l(\mathcal{G})$ denote the left-invariant vector fields of $\mathcal{G}$. It can be shown that $\mathfrak{X}_l(\mathcal{G})$ is closed under the ordinary Lie bracket of vector fields on $\mathcal{G}$. Consequently, $\mathfrak{X}_l(\mathcal{G})$ is a Lie algebra itself. Definition 2.10 implies that the map

$$\mathfrak{X}_l(\mathcal{G}) \longrightarrow \Gamma(A\mathcal{G})\,, \quad \widetilde{X} \mapsto \widetilde{X}|_M$$

is a vector space isomorphism. The inverse map sends a section $X \in \Gamma(A\mathcal{G})$ to the left-invariant vector field $\widetilde{X}$ on $\mathcal{G}$ defined by

$$(l_g)_{*,s(g)} X_{s(g)} = \widetilde{X}_g\,.$$

Theorem 2.11. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. For X, $Y \in \Gamma(A\mathcal{G})$, define*

$$[X, Y] := [\widetilde{X}, \widetilde{Y}]|_M\,,$$

where $\widetilde{X}$, $\widetilde{Y}$ are the left-invariant vector fields associated to X and Y respectively. Also, let

$$\rho := s_*|_{A\mathcal{G}}\,,$$

where $s\colon \mathcal{G} \longrightarrow M$ is the source map. Then $(A\mathcal{G}, \rho, M)$ is a Lie algebroid.

Proposition 2.12. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows M$ be Lie groupods and let $\varphi\colon \mathcal{G} \to \mathcal{H}$ be a Lie groupoid homomorphism, where the morphism on the base space is* id_M*. Let $\widehat{\varphi} := \varphi_*|_{A\mathcal{G}}$. Then $\widehat{\varphi}\colon A\mathcal{G} \to A\mathcal{H}$ is a Lie algebroid homomorphism.*

Example 2.13. Any Lie algebra $\mathfrak{g}$ is naturally a Lie algebroid over a point. Specifically, the Lie algebra structure on $\Gamma(\mathfrak{g})$ is induced by that of $\mathfrak{g}$ under the natural vector space isomorphism $\Gamma(\mathfrak{g}) \simeq \mathfrak{g}$, and the anchor map of $\mathfrak{g}$ is (necessarily) the zero map.

Example 2.14. The tangent bundle TM of a manifold M is naturally a Lie algebroid where the Lie bracket on $\Gamma(TM)$ is just the usual Lie bracket of vector fields on M, and the anchor map is just the identity map $\rho := \mathrm{id}_{TM}$. $(TM, \mathrm{id}_{TM}, M)$ is called the *tangent algebroid*.

A direct calculation shows that the tangent algebroid is the associated Lie algebroid of the pair groupoid $M \times M \rightrightarrows M$.

Example 2.15. Let

$$\psi\colon \mathfrak{g} \longrightarrow \Gamma(TM), \quad x \mapsto x_M := \psi(x) \in \Gamma(TM)$$

be an action of a Lie algebra $\mathfrak{g}$ on a manifold M, that is, ψ is a Lie algebra homomorphism. Consider the trivial vector bundle

$$\mathfrak{g} \times M \to M\,.$$

The sections of $\mathfrak{g} \times M$ are naturally identified with smooth $\mathfrak{g}$-valued functions on M. Given two smooth functions ϕ, $\tau\colon M \longrightarrow \mathfrak{g}$, define

$$[\phi, \tau](p) := [\phi(p), \tau(p)] + (\phi(p)_M)_p\tau - (\tau(p)_M)_p\phi \tag{2.1}$$

for all $p \in M$, where $[\phi(p), \tau(p)]$ is understood to be the Lie bracket of $\phi(p), \tau(p) \in \mathfrak{g}$ on $\mathfrak{g}$. Also, define

$$\rho\colon \mathfrak{g} \times M \longrightarrow TM\,, \quad (x, p) \mapsto (x_M)_p \in T_pM\,. \tag{2.2}$$

Then $\mathfrak{g} \times M$ is a Lie algebroid with bracket given by (2.1) and anchor map given by (2.2). $(\mathfrak{g} \times M, \rho, M)$ is called the *action algebroid*.

Now let G be a Lie group whose Lie algebra is $\mathfrak{g}$ and suppose that M has a smooth left-action by G. The G-action on M induces an action of $\mathfrak{g}$ on M which sends $x \in \mathfrak{g}$ to the vector field x_M on M given by

$$(x_M)_p := \frac{d}{dt}|_{t=0} \exp(-tx)p \in T_pM\,. \tag{2.3}$$

The action algebroid given by the $\mathfrak{g}$-action of (2.3) coincides with the associated Lie algebroid of the action groupoid $G \times M \rightrightarrows M$.

2.2. **The exterior derivative of a Lie algebroid.** Every Lie algebroid (A, ρ, M) has an *exterior derivative*

$$d_A \colon \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)\,,$$

which is analogous to the usual exterior derivative of differential forms. Formally, d_A is defined by

$$\begin{aligned}(d_A\omega)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \rho(X_i)[\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1})] \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1})\end{aligned} \tag{2.4}$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma(A)$, $i = 1, \ldots, k+1$, where $\widehat{X}_i$ denotes omission of X_i. A direct calculation shows that

$$d_A^2 = 0\,. \tag{2.5}$$

Example 2.16. The exterior derivative d_{TM} associated to the tangent algebroid $(TM, \mathrm{id}_{TM}, M)$ is just the ordinary exterior derivative of differential forms on M.

Example 2.17. For a Lie algebra $\mathfrak{g}$, the exterior derivative $d_{\mathfrak{g}}$ associated to its natural Lie algebroid structure is given explicitly by

$$(d_{\mathfrak{g}}\omega)(x_1, \ldots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_{k+1})\,,$$

for $\omega \in \wedge^k \mathfrak{g}^*$, $x_1, \ldots, x_{k+1} \in \mathfrak{g}$. From this, one sees that $d_{\mathfrak{g}}$ is just the coboundary map in the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathbb{R}$, where $\mathfrak{g}$ acts trivially on $\mathbb{R}$.

2.3. **quasi-Frobenius Lie algebroids.** As mentioned previously, a symplectic Lie algebroid over a point is a symplectic Lie algebra (or quasi-Frobenius Lie algebra as it is also called). However, the name *symplectic Lie algebra* also has a different meaning. It also refers to $\mathfrak{sp}(2n, \mathbb{R})$, the Lie algebra of the Lie group of $2n \times 2n$ symplectic matrices. For this reason, we prefer to use the name *quasi-Frobenius Lie algebroids* in place of symplectic Lie algebroids. Formally, a quasi-Frobenius Lie algebroid is defined as follows:

Definition 2.18. A *quasi-Frobenius Lie algebroid* is a Lie algebroid (A, ρ, M) together with a non-degenerate 2-cocycle ω in the Lie algebroid cohomology of (A, ρ, M), that is, $\omega \in \Gamma(\wedge^2 A^*)$ such that

(i) ω is nondegenerate

(ii) $d_A\omega = 0$.

Furthermore, if there exists $\theta \in \Gamma(A^*)$ such that $\omega = d_A\theta$, then (A, ρ, M, θ) is called a *Frobenius Lie algebroid.*

Example 2.19. Let $(\mathfrak{q}, \beta)$ be a quasi-Frobenius Lie algebra, that is, $\mathfrak{q}$ is a Lie algebra and $\beta \in \wedge^2\mathfrak{q}^*$ is a nondegenerate 2-cocycle in the Lie algebra cohomology of $\mathfrak{q}$ with values in $\mathbb{R}$ (where $\mathfrak{q}$ acts trivially on $\mathbb{R}$). Let $d_\mathfrak{q}$ denote the exterior derivative from the natural Lie algebroid structure on $\mathfrak{q}$. As noted previously, $d_\mathfrak{q}$ coincides with the coboundary map in the Lie algebra cohomology of $\mathfrak{q}$ with values in $\mathbb{R}$. Hence, $d_\mathfrak{q}\beta = 0$. Equipping $\mathfrak{q}$ with its natural Lie algebroid structure, it follows that $(\mathfrak{q}, \beta)$ is naturally a quasi-Frobenius Lie algebroid over a point.

Example 2.20. Let (M, ω) be a symplectic manifold and let

$$(TM, \mathrm{id}_{TM}, M)$$

denote the tangent algebroid. As noted previously, $d_{TM} = d$ where d is the usual exterior derivative of differential forms on M. From this, it follows that (TM, id_{TM}, M) together with ω is a quasi-Frobenius Lie algebroid over M.

3. t-symplectic Lie groupoids

In this section, we identify the Lie groupoid analogue of a symplectic Lie group. To start, recall that a symplectic Lie group is a Lie group G together with a left-invariant symplectic form ω. The condition of left-invariance simply means that

$$l_g^*\omega = \omega\,, \quad \forall\ g \in G\,, \tag{3.1}$$

where $l_g\colon G \longrightarrow G$ is left translation by $g \in G$. For a Lie groupoid $\mathcal{G} \rightrightarrows M$, where M consists of more than one point, the condition of left-invariance given by equation (3.1) is no longer applicable. In other words, while the notion of left-invariant vector fields extends from Lie groups to Lie groupoids, the notion of left-invariant differential forms does not. This is a consequence of the fact that multiplication on a groupoid $\mathcal{G} \rightrightarrows M$ is only partial whenever M consists of more than one point.

For $g \in \mathcal{G}$, the domain of l_g is not $\mathcal{G}$. Instead, one has

$$l_g\colon t^{-1}\big(s(g)\big) \xrightarrow{\sim} t^{-1}\big(t(g)\big) \hookrightarrow \mathcal{G}\,,$$

where s and t denote the source and target maps on $\mathcal{G} \rightrightarrows M$. Consequently, if one starts with a differential form ω on $\mathcal{G}$, then the pullback $(l_g)^*\omega$ is now a differential form on the embedded submanifold $t^{-1}(s(g))$, rather than on $\mathcal{G}$.

In the case of a Lie group G, every left-invariant k-form on G is uniquely determined by some element in $\wedge^k\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. Since $A\mathcal{G}$ is the analogue of $\mathfrak{g}$ for a Lie groupoid $\mathcal{G} \rightrightarrows M$ and $\Gamma(\wedge^k\mathfrak{g}^*) \simeq \wedge^k\mathfrak{g}^*$, it is natural to take the Lie groupoid analogue of left-invariant k-forms on $\mathcal{G} \rightrightarrows M$ to be in one to one correspondence with the elements of $\Gamma(\wedge^k(A\mathcal{G})^*)$. This motivates the following definition:

Definition 3.1. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A section

$$\widetilde{\omega} \in \Gamma\big(\wedge^k (T^t\mathcal{G})^*\big)$$

is *left-invariant* if

$$(l_g)^*\big(\widetilde{\omega}|_{t^{-1}(t(g))}\big) = \widetilde{\omega}|_{t^{-1}\left(s(g)\right)}$$

for all $g \in \mathcal{G}$.

Remark 3.2. Recall that for $p \in M$ and $g \in t^{-1}(p)$,

$$(T^t\mathcal{G})_g := \ker\ t_{*,g} = T_g\ t^{-1}(p)\,. \tag{3.2}$$

This implies that $\widetilde{\omega}|_{t^{-1}(p)}$ in Definition 3.1 is indeed a differential k-form on $t^{-1}(p)$.

Remark 3.3. Note that when M is a point, that is, $\mathcal{G}$ is a Lie group, we have $T^t\mathcal{G} = T\mathcal{G}$, and Definition 3.1 coincides with the usual notion of left-invariant differential forms on a Lie group.

The next result justifies Definition 3.1.

Proposition 3.4. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\Gamma_L(\wedge^k(T^t\mathcal{G})^*)$ denote the space of left-invariant sections of $\wedge^k(T^t\mathcal{G})^*$. Define*

$$\varphi\colon \Gamma_L(\wedge^k(T^t\mathcal{G})^*) \longrightarrow \Gamma(\wedge^k(A\mathcal{G})^*)$$

by $\varphi(\widetilde{\omega}) := \widetilde{\omega}|_M$, where $p \in M$ is identified with its corresponding unit element $e_p \in \mathcal{G}$ and $A\mathcal{G}$ is the Lie algebroid of $\mathcal{G} \rightrightarrows M$. Let $\widetilde{\omega}^{(p)} := \widetilde{\omega}|_{t^{-1}(p)}$ and $\omega := \varphi(\widetilde{\omega})$. Then φ is a vector space isomorphism and

$$[(d\widetilde{\omega}^{(p)})(\widetilde{X}_1, \dots, \widetilde{X}_{k+1})](g) = \big[(d_{A\mathcal{G}}\omega)(X_1, \dots, X_{k+1})\big]\big(s(g)\big)\,, \tag{3.3}$$

for all $\widetilde{\omega} \in \Gamma_L(\wedge^k(T^t\mathcal{G})^*)$, $p \in M$, $g \in t^{-1}(p)$, and $X_i \in \Gamma(A\mathcal{G})$ for $i = 1, \dots, k+1$, where $\widetilde{X}_i \in \Gamma(T^t\mathcal{G})$ is the left-invariant vector field associated to X_i.

Proof. The linearity of φ is clear. We now show that φ is injective. Let $X_i \in \Gamma(A\mathcal{G})$, $i = 1, \dots, k$ be arbitrary sections of $A\mathcal{G}$ and let $\widetilde{X}_i$, $i = 1, \dots, k$ denote the corresponding left-invariant vector fields on $\mathcal{G}$. Let $p \in M$ and $g \in t^{-1}(p)$. Note that by equation (3.2), the restriction of $\widetilde{X}_i$ to $t^{-1}(p)$ is a vector field on $t^{-1}(p)$. Let $\widetilde{\omega} \in \Gamma_L(\wedge^k(T^t\mathcal{G})^*)$ and $\omega := \widetilde{\omega}|_M$. Using the left-invariance of $\widetilde{\omega}$, we have

$$\begin{aligned}
\big[\widetilde{\omega}^{(p)}(\widetilde{X}_1, \dots, \widetilde{X}_k)\big](g) &= \widetilde{\omega}^{(p)}_g\big((\widetilde{X}_1)_g, \dots, (\widetilde{X}_k)_g\big)\\
&= \widetilde{\omega}^{(p)}_g\big((l_g)_{*,s(g)}(X_1)_{s(g)}, \dots, (l_g)_{*,s(g)}(X_k)_{s(g)}\big)\\
&= (l_g^*\widetilde{\omega}^{(p)})_{s(g)}\big((X_1)_{s(g)}, \dots, (X_k)_{s(g)}\big)\\
&= \widetilde{\omega}^{(s(g))}_{s(g)}\big((X_1)_{s(g)}, \dots, (X_k)_{s(g)}\big)\\
&= \omega_{s(g)}((X_1)_{s(g)}, \dots, (X_k)_{s(g)})\\
&= \big[\omega(X_1, \dots, X_k)\big]\big(s(g)\big)\,. \qquad (3.4)
\end{aligned}$$

Equation (3.4) can be rewritten more generally as

$$\widetilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k) = s^*[\omega(X_1, \ldots, X_k)]\,, \tag{3.5}$$

where we recall that $\widetilde{\omega}^{(p)}$ is just the restriction of $\widetilde{\omega}$ to $t^{-1}(p)$. Equation (3.5) implies that $\widetilde{\omega}$ is uniquely determined by $\omega := \widetilde{\omega}|_M \in \Gamma(\wedge^k(A\mathcal{G})^*)$. Hence, φ is injective. On the other hand, if $\beta \in \Gamma(\wedge^k(A\mathcal{G})^*)$, then one obtains an element $\widetilde{\beta} \in \Gamma_L(\wedge^k(T^t\mathcal{G})^*)$ by defining

$$\widetilde{\beta}_g(u_1, \ldots, u_k) := \beta_{s(g)}((l_{g^{-1}})_{*,g}u_1, \ldots, (l_{g^{-1}})_{*,g}u_k)\,. \tag{3.6}$$

for $g \in \mathcal{G}$, $u_1, \ldots, u_k \in (T^t\mathcal{G})_g$. From the definition, it follows that $\widetilde{\beta}|_M = \beta$. Hence, φ is also surjective which proves that φ is an isomorphism.

Next, let $X_{k+1} \in \Gamma(A\mathcal{G})$ and let $\widetilde{X}_{k+1}$ be the associated left-invariant vector field on $\mathcal{G}$. Let $g \in \mathcal{G}$. Then

$$\begin{aligned}
\big[\widetilde{X}_{k+1}(\widetilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k))\big](g) &= \big(\widetilde{X}_{k+1})_g(\widetilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k)\big) \\
&= (\widetilde{X}_{k+1})_g\big(s^*[\omega(X_1, \ldots, X_k)]\big) \\
&= \big((l_g)_{*,s(g)}(X_{k+1})_{s(g)}\big)\,([\omega(X_1, \ldots, X_k)] \circ s) \\
&= (X_{k+1})_{s(g)}\,([\omega(X_1, \ldots, X_k)] \circ s \circ l_g) \\
&= (X_{k+1})_{s(g)}\,([\omega(X_1, \ldots, X_k)] \circ s) \\
&= s_{*,s(g)}\big((X_{k+1})_{s(g)}\big)[\omega(X_1, \ldots, X_k)] \\
&= \rho\big((X_{k+1})_{s(g)}\big)[\omega(X_1, \ldots, X_k)] \qquad (3.7)
\end{aligned}$$

where the second equality follows from equation (3.5), the fifth equality follows from the fact that $s \circ l_g = s|_{t^{-1}(s(g))}$, and the last equality follows from the fact that the anchor map associated to $A\mathcal{G}$ is $\rho = s_*|_{A\mathcal{G}}$. Equation (3.7) can be written more generally as

$$\widetilde{X}_{k+1}(\widetilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k)) = s^*\left(\rho(X_{k+1})[\omega(X_1, \ldots, X_k)]\right)\,. \tag{3.8}$$

Now let $p \in M$ and $g \in t^{-1}(p)$. Then

$$\begin{aligned}
\big[(d\widetilde{\omega}^{(p)})(\widetilde{X}_1, \ldots, \widetilde{X}_{k+1})\big](g) &= \sum_{i=1}^{k+1}(-1)^{i+1}(\widetilde{X}_i)_g[\widetilde{\omega}^{(p)}(\widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widetilde{X}_{k+1})] \\
&\quad + \sum_{i<j}(-1)^{i+j}[\widetilde{\omega}^{(p)}([\widetilde{X}_i, \widetilde{X}_j], \widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widehat{\widetilde{X}}_j, \ldots, \widetilde{X}_{k+1})](g) \\
&= \sum_{i=1}^{k+1}(-1)^{i+1}(\widetilde{X}_i)_g[\widetilde{\omega}(\widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widetilde{X}_{k+1})] \\
&\quad + \sum_{i<j}(-1)^{i+j}[\widetilde{\omega}([\widetilde{X}_i, \widetilde{X}_j], \widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widehat{\widetilde{X}}_j, \ldots, \widetilde{X}_{k+1})](g)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k+1}(-1)^{i+1}\rho((X_i)_{s(g)})[\omega(X_1,\dots,\widehat{X}_i,\dots,X_{k+1})] \\
&\quad + \sum_{i<j}(-1)^{i+j}[\omega([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})](s(g)) \\
&= [(d_{A\mathcal{G}}\omega)(X_1,\dots,X_{k+1})]\big(s(g)\big)
\end{aligned}
$$

where the third equality follows from equations (3.5) and (3.8) and the fact that $[X_i,X_j] := [\widetilde{X}_i,\widetilde{X}_j]|_M$. This completes the proof. □

We now define the Lie groupoid analogue of a symplectic Lie group. The motivation for this definition will become clear shortly.

Definition 3.5. A *t-symplectic Lie groupoid* is a Lie grouopid $\mathcal{G} \rightrightarrows M$ together with a left-invariant section $\widetilde{\omega} \in \Gamma(\wedge^2(T^t\mathcal{G})^*)$ with the property that $(t^{-1}(p), \widetilde{\omega}|_{t^{-1}(p)})$ is a symplectic manifold for all $p \in M$. The section $\widetilde{\omega}$ is called a *t-symplectic form* on $\mathcal{G} \rightrightarrows M$.

Here are some immediate consequences of Definition 3.1 and Definition 3.5:

Corollary 3.6. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\widetilde{\omega} \in \Gamma(\wedge^2(T^t\mathcal{G})^*)$. Then $\widetilde{\omega}$ is a t-symplectic form iff $(t^{-1}(p), \widetilde{\omega}|_{t^{-1}(p)})$ is a symplectic manifold for all $p \in M$ and*

$$l_g\colon (t^{-1}(s(g)), \widetilde{\omega}|_{t^{-1}(s(g))}) \xrightarrow{\sim} (t^{-1}(t(g)), \widetilde{\omega}|_{t^{-1}(t(g))})$$

is a symplectomorphism for all $g \in \mathcal{G}$.

Corollary 3.7. *Let $(\mathcal{G} \rightrightarrows M, \widetilde{\omega})$ be a t-symplectic Lie groupoid. Then $\dim\mathcal{G} - \dim M$ is even.*

Proof. Let $p \in M$. By Definition 3.5, $(t^{-1}(p), \widetilde{\omega}|_{t^{-1}(p)})$ is a symplectic manifold. Hence $t^{-1}(p)$ is an even-dimensional manifold. Since t is a submersion, we have

$$\dim t^{-1}(p) = \dim\mathcal{G} - \dim M\,.$$

This completes the proof. □

Theorem 3.8. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. There is a one-to-one correspondence between quasi-Frobenius Lie algebroid structures on $(A\mathcal{G}, \rho, M)$ and t-symplectic Lie groupoid structures on $\mathcal{G} \rightrightarrows M$. This correspondence is given as follows. Let*

$$\varphi : \Gamma_L(\wedge^2(T^t\mathcal{G})^*) \xrightarrow{\sim} \Gamma(\wedge^2(A\mathcal{G})^*)\,, \quad \widetilde{\omega} \mapsto \widetilde{\omega}|_M$$

be the vector space isomorphism of Proposition 3.4. Let $\widetilde{\omega} \in \Gamma_L(\wedge^2(T^t\mathcal{G})^*)$. Then $(\mathcal{G} \rightrightarrows M, \widetilde{\omega})$ is a t-symplectic Lie groupoid iff $(A\mathcal{G}, \rho, M, \varphi(\widetilde{\omega}))$ is a quasi-Frobenius Lie algebroid.

Proof. Suppose $(A\mathcal{G}, \rho, M, \omega)$ is a quasi-Frobenius Lie algebroid. By Proposition 3.4, there exists a unique section $\widetilde{\omega} \in \Gamma_L(\wedge^2(T^t\mathcal{G})^*)$ such that $\widetilde{\omega}|_M = \omega$. From the proof of Proposition 3.4, $\widetilde{\omega}$ is given explicitly by

$$\widetilde{\omega}_g(u,v) = \omega_{s(g)}((l_{g^{-1}})_{*,g}u, (l_{g^{-1}})_{*,g}v) \tag{3.9}$$

for $g \in \mathcal{G}$ and $u, v \in (T^t\mathcal{G})_g$. Since ω is nondegenerate on $A\mathcal{G}$ and

$$(l_g)_{*,h} \colon (T^t\mathcal{G})_h \xrightarrow{\sim} (T^t\mathcal{G})_{gh}$$

is a vector space isomorphism for all $g \in \mathcal{G}$ and $h \in t^{-1}(s(g))$, it follows that $\widetilde{\omega}$ is nondegenerate on $T^t\mathcal{G}$. In particular, $\widetilde{\omega}^{(p)} := \widetilde{\omega}|_{t^{-1}(p)}$ is nondegenerate on

$$Tt^{-1}(p) = (T^t\mathcal{G})|_{t^{-1}(p)}$$

for all $p \in M$.

Now let X, $Y \in \Gamma(A\mathcal{G})$ be arbitrary and let $\widetilde{X}, \widetilde{Y} \in \Gamma(T^t\mathcal{G})$ be the associated left-invariant vector fields on $\mathcal{G}$. By Proposition 3.4,

$$\big[(d\widetilde{\omega}^{(p)})(\widetilde{X}, \widetilde{Y})\big](g) = \big[(d_{A\mathcal{G}}\omega)(X, Y)\big]\big(s(g)\big)\,, \quad \forall\, p \in M,\ g \in t^{-1}(p)\,. \tag{3.10}$$

Since $d_{A\mathcal{G}}\omega = 0$, equation (3.10) implies that $d\widetilde{\omega}^{(p)} = 0$ for all $p \in M$. Hence, $\widetilde{\omega}^{(p)}$ is a closed and nondegenerate 2-form on $t^{-1}(p)$ for all $p \in M$. Hence, $(t^{-1}(p), \widetilde{\omega}^{(p)})$ is a symplectic manifold for all $p \in M$.

On the other hand, suppose that $(\mathcal{G} \rightrightarrows M, \widetilde{\omega})$ is a t-symplectic Lie groupoid for some $\widetilde{\omega} \in \Gamma_L(\wedge^2(T^t\mathcal{G})^*)$. Let $\omega := \widetilde{\omega}|_M$. Since $(t^{-1}(p), \widetilde{\omega}|_{t^{-1}(p)})$ is a symplectic manifold and

$$(A\mathcal{G})_p = (T^t\mathcal{G})_p = T_p t^{-1}(p)$$

for all $p \in M$ (where, as usual, we identify M with the unit elements of $\mathcal{G}$), it follows immediately that $\omega := \widetilde{\omega}|_M$ is nongenerate on $A\mathcal{G}$. Since $\widetilde{\omega}$ is left-invariant, Proposition 3.4 implies equation (3.10). Since $d\widetilde{\omega}^{(p)} = 0$ for all $p \in M$ and $s \colon \mathcal{G} \longrightarrow M$ is surjective, it follows that $d_{A\mathcal{G}}\omega = 0$ as well. Hence, $(A\mathcal{G}, \rho, M, \omega)$ is a quasi-Frobenius Lie algebroid.

Since the above constructions are clearly inverse to one another, the one-to-one correspondence between t-symplectic Lie groupoid structures on $\mathcal{G} \rightrightarrows M$ and quasi-Frobenius Lie algebroid structures on $(A\mathcal{G}, \rho, M)$ is established. $\square$

We conclude this section with a few elementary examples of t-symplectic Lie groupoids.

Example 3.9. Every symplectic Lie group is naturally a t-symplectic Lie groupoid over a point (and vice versa).

Example 3.10. Let (M, ω) be a symplectic manifold and let $M \times M \rightrightarrows M$ be the pair groupoid with

$$s(p, q) := q, \qquad t(p, q) := p$$

for p, $q \in M$. Let $j \colon T^t(M \times M) \hookrightarrow T(M \times M)$ be the inclusion map and let

$$\widetilde{\omega} := j^*(s^*\omega) \in \Gamma\big(\wedge^2\,(T^t(M \times M))^*\big)$$

where j^* denotes the dual map. Then $M \times M \rightrightarrows M$ together with $\widetilde{\omega}$ is a t-symplectic Lie groupoid. The associated quasi-Frobenius Lie algebroid is just the tangent algebroid (TM, id_M, M) with ω as the nondegenerate 2-cocycle.

Example 3.11. Let M be a manifold with a smooth left-action by a Lie group G. Let $G \times M \rightrightarrows M$ be the associated action groupoid.

Now suppose that G admits a left-invariant symplectic form ω, that is, (G, ω) is a symplectic Lie group. Then ω induces a t-symplectic form $\widetilde{\omega}$ on $G \times M \rightrightarrows M$. To construct $\widetilde{\omega}$, let

$$j\colon T^t(G \times M) \hookrightarrow T(G \times M)$$

be the inclusion map and let $\pi_1\colon G \times M \longrightarrow G$ denote the natural projection map. Then $\widetilde{\omega} \in \Gamma(\wedge^2(T^t(G \times M))^*)$ is defined by $\widetilde{\omega} := j^*(\pi_1^*\omega)$.

We now verify that $\widetilde{\omega}$ satisfies the conditions of a t-symplectic form. For $p \in M$, let

$$i_p\colon t^{-1}(p) \hookrightarrow G \times M$$

be the inclusion. Then

$$\widetilde{\omega}|_{t^{-1}(p)} = i_p^*(\pi_1^*\omega) = (\pi_1 \circ i_p)^*\omega\,. \tag{3.11}$$

Equation (3.11) together with the fact that $d\omega = 0$ implies that

$$d(\widetilde{\omega}|_{t^{-1}(p)}) = 0 \tag{3.12}$$

for all $p \in M$. Now let $(g, p) \in t^{-1}(p)$ and let u, $v \in T_{(g,p)}t^{-1}(p)$. Since $t^{-1}(p) = G \times \{p\}$, it follows that

$$u = (x, 0_p)\,, \quad v = (y, 0_p)$$

for some $x, y \in T_gG$. Hence,

$$(\widetilde{\omega}|_{t^{-1}(p)})_{(g,p)}(u, v) = ((\pi_1 \circ i_p)^*\omega)_{(g,p)}(u, v) = \omega_g(x, y)\,. \tag{3.13}$$

Since ω is nondegenerate, it follows that $\widetilde{\omega}|_{t^{-1}(p)}$ is also nondegenerate. Hence, $t^{-1}(p)$ together with $\widetilde{\omega}|_{t^{-1}(p)}$ is a symplectic manifold.

All that remains to check is that for all $(g, p) \in G \times M$, the left-translation map

$$l_{(g,p)} : t^{-1}(g^{-1}p) = G \times \{g^{-1}p\} \longrightarrow t^{-1}(p) = G \times \{p\} \tag{3.14}$$

is a symplectomorphism (where we note that $s(g, p) = g^{-1}p$ and $t(g, p) = p$). This can be seen as follows:

$$\begin{aligned}
(l_{(g,p)})^*\widetilde{\omega}|_{t^{-1}(p)} &= (l_{(g,p)})^*((\pi_1 \circ i_p)^*\omega) \\
&= (\pi_1 \circ i_p \circ l_{(g,p)})^*\omega \\
&= (l_g \circ \pi_1 \circ i_{g^{-1}p})^*\omega \\
&= (\pi_1 \circ i_{g^{-1}p})^*[(l_g)^*\omega] \\
&= (\pi_1 \circ i_{g^{-1}p})^*\omega \\
&= \widetilde{\omega}|_{t^{-1}(g^{-1}p)}\,,
\end{aligned}$$

where the fifth equality follows from the fact that ω is left-invariant. This proves that $(G \times M \rightrightarrows M, \widetilde{\omega})$ is a t-symplectic Lie groupoid.

Example 3.12. Let (M, ω) be a symplectic manifold and let (G, β) be a symplectic Lie group. Let

$$M \times G \times M \rightrightarrows M$$

be the Lie groupoid defined by

(i) $s(q,g,p) := p$
(ii) $t(q,g,p) := q$
(iii) $(r,h,q)(q,g,p) := (r,hg,p)$
(iv) $u(p) := (p,e,p)$
(v) $i(q,g,p) := (p,g^{-1},q)$.
Let

$$\widehat{\omega} := \pi_2^*\beta + \pi_3^*\omega\,,$$

where $\pi_2 \colon M\times G\times M \longrightarrow G$ and $\pi_3 \colon M\times G\times M \longrightarrow M$ are the natural projection maps. Also, define the following inclusion maps

$$j \colon T^t(M\times G\times M) \hookrightarrow T(M\times G\times M)\,, \quad i_p \colon t^{-1}(p) \hookrightarrow M\times G\times M$$

for $p\in M$. Define $\widetilde{\omega} := j^*\widehat{\omega}$.

Since β and ω are symplectic forms on G and M respectively and

$$t^{-1}(p) = \{p\}\times G\times M\,,$$

it follows immediately that $(t^{-1}(p), \widetilde{\omega}|_{t^{-1}(p)})$ is a symplectic manifold for all $p\in M$. Furthermore, we have

$$\begin{aligned}
(l_{(q,g,p)})^*(\widetilde{\omega}|_{t^{-1}(q)}) &= (l_{(q,g,p)})^*(i_q^*\widehat{\omega}) \\
&= (i_q\circ l_{(q,g,p)})^*\widehat{\omega} \\
&= (i_q\circ l_{(q,g,p)})^*(\pi_2^*\beta + \pi_3^*\omega) \\
&= (\pi_2\circ i_q\circ l_{(q,g,p)})^*\beta + (\pi_3\circ i_q\circ l_{(q,g,p)})^*\omega \\
&= (l_g\circ \pi_2\circ i_p)^*\beta + (\pi_3\circ i_p)^*\omega \\
&= i_p^*\circ \pi_2^*\circ(l_g^*\beta) + i_p^*(\pi_3^*\omega) \\
&= i_p^*(\pi_2^*\beta + \pi_3^*\omega) \\
&= i_p^*\widehat{\omega} \\
&= \widetilde{\omega}|_{t^{-1}(p)},
\end{aligned}$$

for all p, $q\in M$, $g\in G$, where the seventh equality follows from the left-invariance of β. Hence, $M\times G\times M \rightrightarrows M$ together with $\widetilde{\omega}$ is a t-symplectic Lie groupoid.

4. Symplectic Lie group bundles

In this section, we introduce the notion of a *symplectic Lie group bundle* (SLGB), which combines the notion of a t-symplectic Lie groupoid with that of a Lie group bundle[1]. Formally, SLGBs are defined as follows:

Definition 4.1. A *symplectic Lie group bundle* consists of the following data: $(G,\omega,E,\pi,M,\widetilde{\omega})$, where
(i) (G,ω) is a symplectic Lie group
(ii) $\pi\colon E\to M$ is smooth fiber bundle with fiber G
(iii) $\widetilde{\omega}$ is a smooth section of $\wedge^2(\ker\pi_*)^*$
such that

[1]See [10, 11] for a review of Lie group bundles.

(a) for all $p \in M$, the fiber $E_p := \pi^{-1}(p)$ has a Lie group structure (where the smooth structure on the Lie group coincides with the smooth structure on E_p as an embedded submanifold of E)
(b) for all $p \in M$, $\gamma_p^* \widetilde{\omega}$ is a left-invariant symplectic form on E_p, where

$$\gamma_p \colon T(E_p) \hookrightarrow \ker \pi_*$$

is the inclusion
(c) there exists a system of local trivializations

$$\{\psi_i \colon \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G\}$$

such that for all i and $p \in U_i$,

$$\psi_{i,p} \colon (E_p, \gamma_p^* \widetilde{\omega}) \xrightarrow{\sim} (G, \omega)$$

is an isomorphism of symplectic Lie groups, where $\psi_{i,p}$ is the composition

$$E_p \xrightarrow{\psi_i} \{p\} \times G \xrightarrow{\sim} G$$

Proposition 4.2. *Every SLGB has a canonical t-symplectic Lie groupoid structure.*

Proof. Let $(G, \omega, E, \pi, M, \widetilde{\omega})$ be a SLGB. Since $E \to M$ is a Lie group bundle, one can also regard it as a Lie groupoid $E \rightrightarrows M$ as follows:
1. the source and target maps are defined by $s = t := \pi$
2. the unit map $u \colon M \to E$ is defined by $u(p) := 1_p$ for all $p \in M$ where 1_p is the identity element on E_p
3. the groupoid multiplication is induced by the fiber-wise group multiplication: $E_p \times E_p \to E_p$
4. the inverse map $i \colon E \to E$ is induced by the fiber-wise inverse map $E_p \to E_p$

Since $s = t = \pi$, we have

$$t^{-1}(s(x)) = t^{-1}(t(x)) = E_{\pi(x)}\,, \qquad \forall\, x \in E\,.$$

By Definition 4.1,

$$\gamma_{\pi(x)}^* \widetilde{\omega} = \widetilde{\omega}|_{E_{\pi(x)}}$$

is a left-invariant symplectic form on $E_{\pi(x)}$ for all $x \in E$. Hence,

$$l_x \colon (E_{\pi(x)}, \widetilde{\omega}|_{E_{\pi(x)}}) \xrightarrow{\sim} (E_{\pi(x)}, \widetilde{\omega}|_{E_{\pi(x)}})$$

is a symplectomorphism. By Corollary 3.6, $\widetilde{\omega}$ is a t-symplectic form on $E \rightrightarrows M$. This completes the proof. □

Proposition 4.3. *Let $\mathcal{E} = (G, \omega, E, \pi, M, \widetilde{\omega})$ be a SLGB and let (AE, ρ, M, β) be the associated quasi-Frobenius Lie algebroid (where $\mathcal{E}$ is equipped with its canonical t-symplectic Lie groupoid structure). Then*
(i) $\rho \equiv 0$
(ii) *the Lie bracket on $\Gamma(AE)$ is $C^\infty(M)$-bilinear; in particular, there is an induced Lie algebra structure on the fiber $(AE)_p$ for all $p \in M$*

(iii) *there exists a system of local trivializations*

$$\{\varphi_i\colon \pi_{AE}^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathfrak{g}\}$$

such that for all i and $p \in U_i$,

$$\varphi_{i,p}\colon ((AE)_p, \beta_p) \xrightarrow{\sim} (\mathfrak{g}, \omega_e)$$

is an isomorphism of quasi-Frobenius Lie algebras, where π_{AE} is the projection map from AE to M, $(\mathfrak{g}, \omega_e)$ is the quasi-Frobenius Lie algebra associated to (G, ω), and $\varphi_{i,p}$ is the composition

$$(AE)_p \xrightarrow{\varphi_i} \{p\} \times \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}\,.$$

Proof. (i) follows from the fact that

$$AE := (\ker t_*)|_{u(M)}$$

$\rho := s_*|_{AE}$, and $s = t = \pi$ from the proof of Proposition 4.2. (Recall that we regard AE as a vector bundle over M by identifying the unit element $1_p \in E$ with $p \in M$.)

(ii) follows from the Leibniz property of the Lie bracket on $\Gamma(AE)$ together with the fact that the anchor map ρ is identically zero.

For (iii), let

$$\{\psi_i\colon \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G\}$$

be a system of local trivialization on $\pi : E \to M$ such that for all i and $p \in U_i$,

$$\psi_{i,p}\colon (E_p, \gamma_p^*\widetilde{\omega}) \xrightarrow{\sim} (G, \omega)$$

is an isomorphism of symplectic Lie groups, where $\gamma_p\colon T(E_p) \hookrightarrow \ker \pi_*$ is the inclusion. For each i, the restriction of

$$(\psi_i)_*\colon T(\pi^{-1}(U_i)) \xrightarrow{\sim} T(U_i \times G)$$

to $AE|_{U_i}$ induces a local trivialization

$$\varphi_i\colon AE|_{U_i} \xrightarrow{\sim} U_i \times \mathfrak{g}\,.$$

Furthermore, for all i and $p \in U_i$,

$$\varphi_{i,p}\colon ((AE)_p, \beta_p) \xrightarrow{\sim} (\mathfrak{g}, \omega_e)$$

is an isomorphism of quasi-Frobenius Lie algebras. Indeed, this follows from the fact that $(AE)_p = T_{1_p}(E_p)$,

$$\beta_p = \widetilde{\omega}_{1_p} = [\gamma_p^*\widetilde{\omega}]_{1_p}$$

by Theorem 3.8, and $\psi_{i,p}$ is an isomorphism of symplectic Lie groups. This completes the proof. □

Remark 4.4. The Lie algebroid appearing in Proposition 4.3 is both a quasi-Frobenius Lie algebroid and a Lie algebra bundle[2]. For this reason, it is only natural that we call a quasi-Frobenius Lie algebroid (A, ρ, M, β) satisfying conditions (i)–(iii) of Proposition 4.3 a *quasi-Frobenius Lie algebra bundle* (QFLAB).

[2]See [10, 11] for a review of Lie algebra bundles.

We now give a characterization of general SLGBs. To this end, we will make use of the following results:

Lemma 4.5. *Let $\varphi\colon G \to H$ be a Lie group homomorphism and let $\theta \in \Omega^k(H)$ be a left-invariant k-form. Then $\varphi^*\theta \in \Omega^k(G)$ is also left-invariant.*

Proof. This is just a direct calculation:

$$\begin{aligned} l_g^*(\varphi^*\theta) &= (\varphi \circ l_g)^*\theta \\ &= (l_{\varphi(g)} \circ \varphi)^*\theta \\ &= \varphi^*(l_{\varphi(g)}^*\theta) \\ &= \varphi^*\theta\,. \end{aligned}$$

This completes the proof. □

Lemma 4.6. *Let (G, ω) be a symplectic Lie group and let M be a manifold. Let $\pi_1\colon M \times G \to M$ and $\pi_2\colon M \times G \to G$ denote the natural projections. Also, let $\tau\colon \ker(\pi_1)_* \hookrightarrow T(M \times G)$ be the inclusion. Then*

$$\big(G, \omega, M \times G, \pi_1, M, \tau^*(\pi_2^*\omega)\big)$$

is a SLGB, where for all $p \in M$, the Lie group structure on $\pi_1^{-1}(p)$ is the natural one.

Proof. For $p \in M$, let

$$\gamma_p\colon T(\{p\} \times G) \hookrightarrow \ker(\pi_1)_*$$

and

$$\iota_p\colon \{p\} \times G \hookrightarrow M \times G$$

denote the inclusion maps. Note that

$$\begin{aligned} \gamma_p^*(\tau^*(\pi_2^*\omega)) &= (\tau \circ \gamma_p)^*(\pi_2^*\omega) \\ &= (\iota_p)^*(\pi_2^*\omega) \\ &= (\pi_2 \circ \iota_p)^*\omega\,. \end{aligned} \tag{4.1}$$

Then for $g \in G$, we have

$$\begin{aligned} (l_{(p,g)})^*[\gamma_p^*(\tau^*(\pi_2^*\omega))] &= (l_{(p,g)})^*[(\pi_2 \circ \iota_p)^*\omega] \\ &= (\pi_2 \circ \iota_p)^*\omega \\ &= \gamma_p^*\big(\tau^*(\pi_2^*\omega)\big) \end{aligned}$$

where we have used (4.1) in the first and third equality and Lemma 4.5 in the second equality, where we note that

$$\pi_2 \circ \iota_p\colon \{p\} \times G \to G$$

is a Lie group isomorphism. Hence, $\gamma_p^*(\tau^*(\pi_2^*\omega))$ is left-invariant. Furthermore, (4.1) implies that ω is closed and non-degenerate, i.e., symplectic. This proves that $\big(\pi_1^{-1}(p), \gamma_p^*(\tau^*(\pi_2^*\omega))\big)$ is a symplectic Lie group. Lastly, the identity map

$$\mathrm{id}\colon M \times G \to M \times G$$

is the desired trivialization for the SLGB structure. This completes the proof. □

Proposition 4.7. *Let (G,ω) be a connected symplectic Lie group and let Aut(G,ω) be the group of automorphisms of (G,ω). Then Aut(G,ω) is a finite dimensional Lie group. Furthermore, if G is simply connected, then Aut$(G,\omega) \simeq$ Aut$(\mathfrak{g},\omega_e)$ as Lie groups, where $(\mathfrak{g},\omega_e)$ is the quasi-Frobenius Lie algebra associated to (G,ω) and Aut$(\mathfrak{g},\omega_e)$ is the group of automorphisms of $(\mathfrak{g},\omega_e)$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. In [4], Chevalley proved that the automorphism group of any finite dimensional connected Lie group is again a finite dimensional Lie group. To show that $\mathrm{Aut}(G,\omega)$ is a Lie group, it suffices to show that $\mathrm{Aut}(G,\omega)$ is a closed subset of the Lie group $\mathrm{Aut}(G)$; the closed subgroup theorem [15] then implies that $\mathrm{Aut}(G,\omega)$ is an embedded Lie subgroup of $\mathrm{Aut}(G)$.

To this end, define

$$f\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})\,, \quad \varphi \mapsto \varphi_*\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}\,.$$

Since G is connected, $\varphi \in \mathrm{Aut(G)}$ is uniquely determined by $f(\varphi) \in \mathrm{Aut}(\mathfrak{g})$. From [4], the Lie group structure on Aut(G) is obtained by using f to identify Aut(G) with im f, which is shown to be a closed subgroup of the Lie group $\mathrm{Aut}(\mathfrak{g})$ (which in turn is a closed subgroup of $GL(\mathfrak{g})$).

By definition,

$$\mathrm{Aut}(G,\omega) = \{\varphi \in \mathrm{Aut}(G) \mid \varphi^*\omega = \omega\}\,.$$

Let $\varphi \in \mathrm{Aut}(G)$ be a limit point of $\mathrm{Aut}(G,\omega)$ and let $\{\varphi_n\} \subset \mathrm{Aut}(G,\omega)$ be a sequence which converges to φ. Since f is a Lie group isomorphism from $\mathrm{Aut}(G)$ to im f (in particular - a homeomorphism), we have

$$f(\varphi_n) = (\varphi_n)_* \to f(\varphi) = \varphi_*\,.$$

Hence, for $x, y \in \mathfrak{g} = T_eG$, we have

$$\begin{aligned}
\varphi^*\omega_e(x,y) &= \omega_e(\varphi_* x, \varphi_* y)\\
&= \lim_{n\to\infty} \omega_e((\varphi_n)_* x, (\varphi_n)_* y)\\
&= \lim_{n\to\infty} (\varphi_n)^*\omega_e(x,y)\\
&= \lim_{n\to\infty} [(\varphi_n)^*\omega]_e(x,y)\\
&= \lim_{n\to\infty} \omega_e(x,y)\\
&= \omega_e(x,y)\,,
\end{aligned}$$

where we have used the fact that $\varphi_n \in \mathrm{Aut}(G,\omega)$ in the second to last equality. Hence, $\varphi^*\omega_e = \omega_e$. Since ω is left-invariant and $\varphi \in \mathrm{Aut}(G)$, it follows that $\varphi^*\omega = \omega$. This shows that $\varphi \in \mathrm{Aut}(G,\omega)$, which in turn implies that $\mathrm{Aut}(G,\omega)$ is a closed subset of $\mathrm{Aut}(G)$.

For the last part of Proposition 4.7, suppose that G is simply connected. Then

$$f\colon \mathrm{Aut}(G) \xrightarrow{\sim} \mathrm{Aut}(\mathfrak{g})$$

is a Lie group isomorphism. In addition, note that $f(\mathrm{Aut}(G,\omega)) = \mathrm{Aut}(\mathfrak{g},\omega_e)$. Hence, the restriction of f to $\mathrm{Aut}(G,\omega)$ gives a Lie group isomorphism from $\mathrm{Aut}(G,\omega)$ to $\mathrm{Aut}(\mathfrak{g},\omega_e)$. This completes the proof. ☐

The following result provides an alternate way of viewing SLGBs:

Theorem 4.8. *Let (G,ω) be a connected symplectic Lie group and let $\pi\colon E\to M$ be a smooth fiber bundle with fiber G. Then $\pi\colon E\to M$ admits the structure of a SLGB if and only if there exists a system of local trivializations*

$$\{\psi_i\colon \pi^{-1}(U_i)\to U_i\times G\}$$

for which all the transition functions take their values in the Lie group $Aut(G,\omega)$.

Proof. ($\Rightarrow$). Suppose $(G,\omega,E,\pi,M,\widetilde{\omega})$ is a SLGB. By Definition 4.1, there exists a system of local trivializations

$$\{\psi_i\colon \pi^{-1}(U_i)\xrightarrow{\sim} U_i\times G\}$$

such that

$$\psi_{i,p}\colon (E_p,\gamma_p^*\widetilde{\omega})\xrightarrow{\sim}(G,\omega)$$

is an isomorphism of symplectic Lie groups for all i and $p\in U_i$, where $\gamma_p\colon T(E_p)\hookrightarrow \ker\,(\pi_E)_*$ is the inclusion. This implies that for all i, j such that $U_i\cap U_j\neq\emptyset$ and for all $p\in U_i\cap U_j$, the map

$$\psi_{j,p}\circ\psi_{i,p}^{-1}\colon (G,\omega)\xrightarrow{\sim}(G,\omega)\quad g\mapsto\phi_{ji}(p)g$$

is an automorphism of (G,ω), where ϕ_{ji} is the transition function associated to $\psi_j\circ\psi_i^{-1}$. Hence, im $\phi_{ji}\subset \mathrm{Aut}(G,\omega)$.

($\Leftarrow$). On the other hand, suppose that $\pi\colon E\to M$ is a smooth fiber bundle with fiber G for which there exists a system of local trivializations

$$\{\psi_i\colon \pi^{-1}(U_i)\xrightarrow{\sim} U_i\times G\}$$

whose transition functions all take their values in the Lie group $\mathrm{Aut}(G,\omega)$. First, we define a Lie group structure on the fibers of E. Let $p\in M$ and let U_i be any open set such that $p\in U_i$. The (abstract) group structure on the fiber E_p is obtained by declaring

$$\psi_{i,p}\colon E_p\xrightarrow{\sim} G$$

to be a group isomorphism. Hence, for g, $h\in G$, the product, inverse, and identity on E_p are given respectively by

$$\psi_{i,p}^{-1}(g)\cdot\psi_{i,p}^{-1}(h):=\psi_{i,p}^{-1}(gh)\,,\quad (\psi_{i,p}^{-1}(g))^{-1}:=\psi_{i,p}^{-1}(g^{-1}) \tag{4.2}$$

and $1_p:=\psi_{i,p}^{-1}(e)$, where e is the identity element on G. Since $\psi_{i,p}$ is a diffeomorphism, the above product and inverse maps are smooth with respect to the manifold structure on E_p. Hence, (4.2) defines a Lie group structure on E_p.

To show that the group structure on E_p is well-defined, let U_j be another open set such that $p\in U_j$. Let

$$\phi_{ji}\colon U_i\cap U_j\to\mathrm{Aut}(G,\omega)$$

be the transition function associated to $\psi_j\circ\psi_i^{-1}$. Then

$$\psi_{i,p}^{-1}(g)=\psi_{j,p}^{-1}(\phi_{ji}(p)g)\,,\quad\forall\, g\in G\,.$$

Let $\cdot_i$ and $\cdot_j$ denote the group products defined by $\psi_{i,p}$ and $\psi_{j,p}$ respectively. Since $\phi_{ji}(p) \in \mathrm{Aut}(G,\omega)$, we have

$$\begin{aligned}\psi_{i,p}^{-1}(g) \cdot_i \psi_{i,p}^{-1}(h) &= \psi_{i,p}^{-1}(gh)\\ &= \psi_{j,p}^{-1}\big(\phi_{ji}(p)(gh)\big)\\ &= \psi_{j,p}^{-1}\big([\phi_{ji}(p)(g)][\phi_{ji}(p)(h)]\big)\\ &= \psi_{j,p}^{-1}\big(\phi_{ji}(p)(g)\big) \cdot_j \psi_{j,p}^{-1}\big(\phi_{ji}(p)(h)\big)\end{aligned}$$

for g, $h \in G$. This implies that the group product on E_p is well-defined. In a similar fashion, one can show that the identity element and the inverse map on E_p are also well-defined.

Next, for $p \in U_i$, define $\omega^{(p)} := \psi_{i,p}^{*}\omega$. Then $\omega^{(p)}$ is a symplectic form on E_p. Moreover, since $\psi_{i,p}$ is a Lie group isomorphism, Lemma 4.5 implies that $\omega^{(p)}$ is also left-invariant. From the definition of $\omega^{(p)}$, it follows that

$$\psi_{i,p} : (E_p, \omega^{(p)}) \xrightarrow{\sim} (G,\omega)$$

is an isomorphism of symplectic Lie groups. To see that the definition of $\omega^{(p)}$ is well-defined, let U_j be another open set such that $p \in U_j$. Then

$$(\psi_{j,p} \circ \psi_{i,p}^{-1})^{*}\omega = (\phi_{ji}(p))^{*}\omega = \omega$$

since $\phi_{ji}(p) \in \mathrm{Aut}(G,\omega)$. This implies that

$$\psi_{j,p}^{*}\omega = \psi_{i,p}^{*}\omega\,,$$

which proves that $\omega^{(p)}$ is well-defined.

Lastly, we construct a section $\widetilde{\omega}$ of $\wedge^2(\ker \pi_*)^*$ such that

$$\gamma_p^{*}\widetilde{\omega} = \omega^{(p)}\,, \quad \forall\, p \in M \tag{4.3}$$

where $\gamma_p\colon T(E_p) \hookrightarrow \ker \pi_*$ is the inclusion. To begin, for each i, equip the bundle

$$\pi_{1,i}\colon U_i \times G \to U_i$$

with the SLGB structure given by Lemma 4.6. The t-symplectic form on $U_i \times G$ is then $\tau_i^{*}(\pi_{2,i}^{*}\omega)$, where

$$\pi_{2,i}\colon U_i \times G \to G\,, \quad \tau_i\colon\ \ker\,(\pi_{1,i})_* \hookrightarrow T(U_i \times G)$$

are the natural maps. Let $\widetilde{\pi}_i$ denote the restriction of π to $\pi^{-1}(U_i)$ and define

$$\widetilde{\psi}_i := (\psi_i)_*|_{\ker(\widetilde{\pi}_i)_*}\,.$$

Since $\pi_{1,i} \circ \psi_i = \widetilde{\pi}_i$, it follows that

$$\widetilde{\psi}_i\colon\ \ker(\widetilde{\pi}_i)_* \xrightarrow{\sim} \ker\,(\pi_{1,i})_*$$

is a vector bundle isomorphism.

Now define $\widetilde{\omega}_i := (\widetilde{\psi}_i)^{*}[\tau_i^{*}(\pi_{2,i}^{*}\omega)]$. Let $p \in U_i$, $x \in E_p$, and

$$u, v \in \ker(\widetilde{\pi}_i)_{*,x} = \ker \pi_{*,x} = T_x(E_p)\,.$$

Then

$$
\begin{aligned}
(\widetilde{\omega}_i)_x(u,v) &= \big[(\widetilde{\psi}_i)^*[\tau_i^*(\pi_{2,i}^*\omega)]\big]_x(u,v)\\
&= [\tau_i^*(\pi_{2,i}^*\omega)]_{\psi_i(x)}\big(\widetilde{\psi}_i(u),\widetilde{\psi}_i(v)\big)\\
&= [(\pi_{2,i}^*\omega)]_{\psi_i(x)}\big(\tau_i(\widetilde{\psi}_i(u)),\tau_i(\widetilde{\psi}_i(v))\big)\\
&= [(\pi_{2,i}^*\omega)]_{\psi_i(x)}\big(\widetilde{\psi}_i(u),\widetilde{\psi}_i(v)\big)\\
&= \omega_{\pi_{2,i}\circ\psi_i(x)}\big((\pi_{2,i})_*\circ\widetilde{\psi}_i(u),(\pi_{2,i})_*\circ\widetilde{\psi}_i(v)\big)\\
&= \omega_{\psi_{i,p}(x)}\big((\psi_{i,p})_*(u),(\psi_{i,p})_*(v)\big)\\
&= ((\psi_{i,p})^*\omega)_x(u,v)\\
&= \omega_x^{(p)}(u,v)\,. \qquad (4.4)
\end{aligned}
$$

This proves that for all pairs i,j such that $U_i\cap U_j\neq\emptyset$, we have

$$\widetilde{\omega}_i=\widetilde{\omega}_j \quad \text{on} \quad \pi^{-1}(U_i)\cap\pi^{-1}(U_j)=\pi^{-1}(U_i\cap U_j)\,.$$

Hence, the $\widetilde{\omega}_i$'s glue together to form a global section $\widetilde{\omega}\in\Gamma(\wedge^2(\ker\ \pi)^*)$. Moreover, since $\gamma_p\colon T(E_p)\hookrightarrow \ker\ \pi_*$ is just the inclusion and $\widetilde{\omega}|_{\pi^{-1}(U_i)}=\widetilde{\omega}_i$ for all i, (4.4) implies $\gamma_p^*\widetilde{\omega}=\omega^{(p)}$. This completes the proof. ☐

We conclude the paper with the following corollary which provides a simple recipe for generating SLGBs:

Corollary 4.9. *Let (G,ω) be a connected symplectic Lie group and let $\pi\colon P\to M$ be any principal $Aut(G,\omega)$-bundle. Then the associated fiber bundle*

$$E:=(P\times G)/Aut(G,\omega)\to M$$

admits the structure of a SLGB, where $Aut(G,\omega)$ acts naturally on G from the left.

Proof. From the definition of the associated fiber bundle, we see that $E\to M$ is a smooth fiber bundle with fiber G which has a system of local trivializations whose transition functions all take their values in $\mathrm{Aut}(G,\omega)$. Theorem 4.8 now implies that $E\to M$ admits the structure of a SLGB. ☐

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