ARCHIVUM MATHEMATICUM (BRNO)
Tomus 57 (2021), 83–99

GLOBAL SOLVABILITY CRITERIA FOR QUATERNIONIC RICCATI EQUATIONS

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Abstract. Some global existence criteria for quaternionic Riccati equations are established. Two of them are used to prove a completely non conjugation theorem for solutions of linear systems of ordinary differential equations.

1. Introduction

Let $a(t)$, $b(t)$, $c(t)$ and $d(t)$ be continuous quaternionic valued functions on $[t_0;+\infty)$, i.e.: $a(t) \equiv a_0(t) + ia_1(t) + ja_2(t) + ka_3(t)$, $b(t) \equiv b_0(t) + ib_1(t) + jb_2(t) + kb_3(t)$, $c(t) \equiv c_0(t) + ic_1(t) + jc_2(t) + kc_3(t)$, $d(t) \equiv d_0(t) + id_1(t) + jd_2(t) + kd_3(t)$, where $a_n(t)$, $b_n(t)$, $c_n(t)$, $d_n(t)$ $(n = \overline{0,3})$ are real valued continuous functions on $[t_0;+\infty)$, i, j, k are the imaginary unities satisfying the conditions

$$i^2 = j^2 = k^2 = ijk = -1\,, \quad ij = -ji = k\,. \tag{1.1}$$

Consider the quaternionic Riccati equation

$$q' + qa(t)q + b(t)q + qc(t) + d(t) = 0\,, \quad t \geq t_0\,. \tag{1.2}$$

Here $q = q(t)$ is the unknown continuously differentiable quaternionic valued function. Currently, there is a growing interest in quaternionic differential equations, in particular, in Eq. (1.2) in connection with their various applications (see e.g., [3]–[9]). Criteria for the existence of periodic (and, therefore, global) solutions of Eq. (1.2) with periodic coefficients were obtained in [1, 10]. Explicit global existence criteria for complex solutions of Eq. (1.2) in the case of its complex coefficients were obtained in [7].

In this paper some global existence criteria for scalar quaternionic Riccati equations are obtained. Two of them are used to prove a completely non conjugation theorem for solutions of linear systems of ordinary differential equations.

2020 *Mathematics Subject Classification*: primary 34C99.

Key words and phrases: Riccati equations, quaternions, the matrix representation of quaternions, global solvability, the solutions of linear systems satisfying of the completely non conjugation condition.

Received June 12, 2020, revised January 2021. Editor R. Šimon Hilscher.

DOI: 10.5817/AM2021-2-83

2. Auxiliary propositions

Substituting $q = q_0 - iq_1 - jq_2 - kq_3$ in (1.2), where q_0 is the real and $-q_n(\overline{1,3})$ are the imaginary parts of q, and separating the real and imaginary parts we come to the following nonlinear system

$$
\begin{cases}
q_0' + a_0(t)q_0^2 + \{b_0(t) + c_0(t) + 2[a_1(t)q_1 + a_2(t)q_2 + a_3(t)q_3]\}q_0 \\
\qquad\qquad\qquad\qquad\qquad\qquad - P(t, q_1, q_2, q_3) = 0\,; \\
q_1' + a_1(t)q_1^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_2(t)q_2 + a_3(t)q_3]\}q_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad - Q(t, q_0, q_2, q_3) = 0\,; \\
q_2' + a_2(t)q_2^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_1(t)q_1 + a_3(t)q_3]\}q_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad - R(t, q_0, q_1, q_3) = 0\,; \\
q_3' + a_3(t)q_3^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0 + a_1(t)q_1 + a_2(t)q_2]\}q_3 \\
\qquad\qquad\qquad\qquad\qquad\qquad - S(t, q_0, q_1, q_2) = 0\,;
\end{cases}
\tag{2.1}
$$

where

$$
\begin{aligned}
P(t, q_1, q_2, q_3) &\equiv a_0(t)[q_1^2 + q_2^2 + q_3^2] - (b_1(t) + c_1(t))q_1 - (b_2(t) + c_2(t))q_2 - (b_3(t) \\
&\quad + c_3(t))q_3 - d_0(t)\,; \\
Q(t, q_0, q_2, q_3) &\equiv a_1(t)[q_0^2 + q_2^2 + q_3^2] + (b_1(t) + c_1(t))q_0 + (b_3(t) - c_3(t))q_2 - (b_2(t) \\
&\quad - c_2(t))q_3 + d_1(t)\,; \\
R(t, q_0, q_1, q_3) &\equiv a_2(t)[q_0^2 + q_1^2 + q_3^2] + (b_2(t) + c_2(t))q_0 - (b_3(t) - c_3(t))q_1 + (b_1(t) \\
&\quad - c_1(t))q_3 + d_2(t)\,; \\
S(t, q_0, q_1, q_2) &\equiv a_3(t)[q_0^2 + q_1^2 + q_2^2] + (b_3(t) + c_3(t))q_0 + (b_2(t) - c_2(t))q_1 - (b_1(t) \\
&\quad - c_1(t))q_2 + d_3(t)\,;
\end{aligned}
$$

$t \geq t_0$. Consider the square matrices

$$
E \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad
I \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},
$$

$$
J \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad
K \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

It is not difficult to check that $I^2 = J^2 = K^2 = IJK = -E$, $IJ = -JI = K$. Then by (1.1) there is an one to one correspondence between the quaternions $m \equiv m_0 + im_1 + jm_2 + km_3$ and the matrices of the form $M \equiv m_0E + m_1I +$

$m_2J + m_3K$:

$$m \equiv m_0 + im_1 + jm_2 + km_3 \leftrightarrow M \equiv \begin{pmatrix} m_0 & m_1 & m_2 & -m_3 \\ -m_1 & m_0 & -m_3 & -m_2 \\ -m_2 & m_3 & m_0 & m_1 \\ m_3 & m_2 & -m_1 & m_0 \end{pmatrix} \tag{2.2}$$

The matrix M corresponding to the quaternion m by the rule (2.2) we will call the symbol of the quaternion m and will denote by $\widehat{m}$.

Let $A(t)$, $B(t)$, $C(t)$ and $D(t)$ be the symbols of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ respectively. Consider the matrix Riccati equation

$$Y' + YA(t)Y + B(t)Y + YC(t) + D(t) = 0\,, \quad t \geq t_0\,. \tag{2.3}$$

By (2.2) the solutions $q(t)$ of Eq. (1.2), existing on some interval $[t_1; t_2)$ $(t_0 \leq t_1 < t_2 \leq +\infty)$, are connected wit solutions $Y(t)$ of Eq. (2.3) by equalities

$$\widehat{q(t)} = Y(t)\,, \quad t \in [t_1; t_2),\ \widehat{q(t_1)} = Y(t_1)\,. \tag{2.4}$$

Along with Eq. (2.3) consider the system of matrix equations

$$\begin{cases} \Phi' = C(t)\Phi + A(t)\Psi\,; \\ \Psi' = -D(t)\Phi - B(t)\Psi\,, \quad t \geq t_0\,. \end{cases} \tag{2.5}$$

Here $\Phi \equiv \Phi(t)$, $\Psi \equiv \Psi(t)$ are the unknown continuously differentiable matrix functions of dimension 4×4 on $[t_0; +\infty)$. Let $Y_0(t)$ be a solution of Eq. (2.3) on $[t_1; t_2)$. The substitution

$$\Psi = Y_0(t)\Phi\,, \quad t \in [t_1; t_2)\,, \tag{2.6}$$

in (2.5) leads to the system

$$\begin{cases} \Phi' = [A(t)Y_0(t) + C(t)]\Phi\,; \\ [Y_0'(t) + Y_0(t)A(t)Y_0(t) + B(t)Y_0(t) + Y_0(t)C(t) + D(t)]\Phi = 0 \quad t \in [t_1; t_2). \end{cases}$$

Therefore $(\Phi_0(t), Y_0(t)\Phi_0(t))$ is a solution of the system (2.5) on $[t_1; t_2)$, where $\Phi_0(t)$ is a solution to the following matrix equation

$$\Phi' = [A(t)Y_0(t) + C(t)]\Phi\,, \quad t \in [t_1; t_2)\,. \tag{2.7}$$

Let $Y(t)$ $(q(t))$ be a solution to Eq. (2.3) (to Eq. (1.2)) on $[t_1; t_2)$.

Definition 2.1. The set $[t_1; t_2)$ is called the maximum existence interval for the solution $Y(t)$ of Eq. (2.3) (for the solution $q(t)$ of Eq. (1.2)), if $Y(t)$ $(q(t))$ cannot be continued to the right from t_2.

Lemma 2.1. *Let $Y(t)$ be a solution of Eq. (2.3) on $[t_1; t_2)$ $(t_0 \leq t_1 < t_2 < +\infty)$. Then $[t_1; t_2)$ is not the maximum existence interval for $Y(t)$ provided the function*

$$f_0(t) \equiv \int\limits_{t_0}^{t} \mathrm{tr}\,[A(\tau)Y(\tau)]d\tau\,, \quad t \in [t_1; t_2)\,,$$

is bounded from below on $[t_1; t_2)$.

Proof. Let $\Phi(t)$ be a solution to the matrix equation

$$\Phi' = [A(t)Y(t) + C(t)]\Phi\,, \quad t \in [t_1; t_2)\,, \quad \text{with}$$

$$\det \Phi(t_1) \neq 0\,. \tag{2.8}$$

By (2.6) and (2.7), $(\Phi(t), Y(t)\Phi(t))$ is a solution to the system (2.5) on $[t_1; t_2)$ which can be continued on $[t_0; +\infty)$ as a solution $(\Phi(t), \Psi(t))$ of the system (2.5). According to the Liouville's formula (see [8, p. 46, Theorem 1.2]) we have:

$$\det \Phi(t) = \det \Phi(t_1) \exp\Big\{ \int\limits_{t_0}^{t} \mathrm{tr}\,[A(\tau)Y(\tau) + C(\tau)]d\tau \Big\}\,, \quad t \in [t_1; t_2)\,.$$

From here from the conditions of lemma and from (2.8) it follows that $\det \Phi(t) \neq 0$, $t \in [t_1; t_3)$, for some $t_3 > t_2$. Then by (2.6) and (2.7) the matrix function $\widetilde{Y}(t) \equiv \Psi(t)\Phi^{-1}(t)$, $t \in [t_1; t_3)$, is a solution to Eq. (2.3) on $[t_1; t_3)$. Obviously $\widetilde{Y}(t)$ coincides with $Y(t)$ on $[t_1; t_2)$. Therefore $[t_1; t_2)$ is not the maximum existence interval for $Y(t)$.
The lemma is proved. □

Let $f(t)$, $g(t)$, $h(t)$, $f_1(t)$, $g_1(t)$, $h_1(t)$ be real valued continuous functions on $[t_0; +\infty)$. Consider the Riccati equations

$$y' + f(t)y^2 + g(t)y + h(t) = 0\,, \quad t \geq t_0\,; \tag{2.9}$$

$$y' + f_1(t)y^2 + g_1(t)y + h_1(t) = 0\,, \quad t \geq t_0\,. \tag{2.10}$$

and the differential inequalities

$$y' + f(t)y^2 + g(t)y + h(t) \geq 0\,, \quad t \geq t_0\,; \tag{2.11}$$

$$y' + f_1(t)y^2 + g_1(t)y + h_1(t) \geq 0\,, \quad t \geq t_0\,. \tag{2.12}$$

Remark 2.1. For $f(t) \geq 0$, $t \geq t_0$, every solution of the linear equation $y' + g(t)y + h(t) = 0$ on $[t_0; \tau_0)$ $(t_0 < \tau_0 \leq +\infty)$ is a solution of the inequality (2.11) on $[t_0; \tau_0)$.

Remark 2.2. Every solution of Eq. (2.10) on $[t_0; \tau_0)$ $(t_0 < \tau_0 \leq +\infty)$ is also a solution of the inequality (2.12) on $[t_0; \tau_0)$.

Theorem 2.1. *Let Eq.* (2.10) *has a real solution* $y_1(t)$ *on* $[t_0; \tau_0)$ $(\tau_0 \leq +\infty)$, *and let the following conditions be satisfied:* $f(t) \geq 0$,

$$\int\limits_{t_0}^{t} \exp\Big\{ \int\limits_{t_0}^{\tau} \big[f(s)(\eta_0(s) + \eta_1(s)) + g(s)\big]ds \Big\}$$
$$\times \big[(f_1(\tau) - f(\tau))y_1^2(\tau) + (g_1(\tau) - g(\tau))y_1(\tau) + h_1(\tau) - h(\tau)\big]d\tau \geq 0\,,$$
$$t \in [t_0; \tau_0)\,.$$

where $\eta_0(t)$ *and* $\eta_1(t)$ *are solutions of the inequalities* (2.11) *and* (2.12) *on* $[t_0; \tau_0)$ *such that* $\eta_j(t_0) \geq y_1(t_0)$, $j = 0, 1$. *Then for every* $\gamma_0 \geq y_1(t_0)$ *Eq.* (2.9) *has a* *solution* $y_0(t)$ *on* $[t_0;\tau_0)$, *satisfying the initial conditions* $y_0(t_0)=\gamma_0$, *and* $y_0(t)\ge y_1(t),\ \ t\in[t_0;\tau_0)$.

This theorem is proved in [4] (see [4, Theorem 3.1]).

Let $t_0<t_1<\cdots$ be a finite or infinite sequence such that $t_m\in[t_0;\tau_0]$ $(t_0<\tau_0\le+\infty)$. We assume that if $\{t_m\}$ is finite then $\max\{t_m\}=\tau_0$ otherwise $\lim\limits_{m\to+\infty} t_m=\tau_0$. Denote:

$I_{g,h}(\xi,t)\equiv\int\limits_{\xi}^{t}\exp\big\{-\int\limits_{\tau}^{t} g(s)ds\big\}h(\tau)d\tau,\ t\ge\xi\ge t_0.$

Theorem 2.2. *Let* $f(t)\ge 0,\ \ t\in[t_0;\tau_0)$, *and*

$$\int\limits_{t_k}^{t}\exp\Big\{\int\limits_{t_k}^{\tau}\big[g(s)-f(s)I_{g,h}(t_k,s)\big]ds\Big\}h(\tau)d\tau\le 0\,,\quad t\in[t_k;t_{k+1}),k=1,2,\dots\,.$$

Then for every $\gamma_0\ge 0$ *Eq.* (2.9) *has a solution* $y_0(t)$ *on* $[t_0;\tau_0)$ *satisfying the initial condition* $y_0(t_0)=\gamma_0$ *and* $y_0(t)\ge 0,\ t\in[t_0;\tau_0)$.

This theorem is proved in [5] (see [5, Theorem 4.1]).

Theorem 2.3. *Let* $\alpha(t)$ *and* $\beta(t)$ *be continuously differentiable on* $[t_0;\tau_0)$ *functions and* $\alpha(t)>0,\ \beta(t)>0,\ t\in[t_0;\tau_0)$;
A) $0\le f(t)\le\alpha(t),\ h(t)\le\beta(t),\ t\in[t_0;\tau_0)$;

B) $g(t)\ge\frac12\big[\frac{\alpha'(t)}{\alpha(t)}-\frac{\beta'(t)}{\beta(t)}\big]+2\sqrt{\alpha(t)\beta(t)},\ t\in[t_0;\tau_0)$.

Then for every $\gamma_0\ge-\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$ *Eq.* (2.9) *has a solution* $y_0(t)$ *on* $[t_0;\tau_0)$ *with* $y_0(t_0)=\gamma_0$ *and*

$$y_0(t)\ge-\sqrt{\frac{\beta(t)}{\alpha(t)}}\,,\quad t\in[t_0;\tau_0)\,.$$

This theorem is proved in [6] (see [6, Theorem 8]).

Theorem 2.4. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the same as in Theorem 2.3. If assumption* A *of Theorem 2.3 and the inequality*

D) $g(t)\le\frac12\big[\frac{\alpha'(t)}{\alpha(t)}-\frac{\beta'(t)}{\beta(t)}\big]-2\sqrt{\alpha(t)\beta(t)},\ t\in[t_0;\tau_0)$,
are valid, then for every $\gamma_0\ge\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$ *Eq.* (2.9) *has a solution* $y_0(t)$ *on* $[t_0;\tau_0)$ *with* $y_0(t_0)=\gamma_0$ *and*

$$y_0(t)\ge\sqrt{\frac{\beta(t)}{\alpha(t)}}\,,\quad t\in[t_0;\tau_0)\,.$$

This theorem is proved in [6] (see [6, Theorem 7]).

Theorem 2.5. *Let* $\alpha_m(t)$ *and* $\beta_m(t),\ \ m=1,2$, *be continuously differentiable functions on* $[t_0;\tau_0)$, *and let* $(-1)^m\alpha_m(t)>0,\ (-1)^m\beta_m(t)>0,\ t\in[t_0;\tau_0)$, $m=1,2$. *If:*

E) $\alpha_1(t) \le f(t) \le \alpha_2(t)$, $\beta_1(t) \le h(t) \le \beta_2(t)$, $t \in [t_0; \tau_0)$;

F) $g(t) \ge \frac{1}{2}\big(\frac{\alpha_m'(t)}{\alpha_m(t)} - \frac{\beta_m'(t)}{\beta_m(t)}\big) + 2(-1)^m\sqrt{\alpha_m(t)\beta_m(t)}$, $t \in [t_0; \tau_0)$, $m = 1, 2$,

then for any $y_{(0)} \in \Big[-\sqrt{\frac{\beta_2(t_0)}{\alpha_2(t_0)}}; \sqrt{\frac{\beta_1(t_0)}{\alpha_1(t_0)}}\Big]$ *Eq.* (2.9) *has a solution* $y_0(t)$ *on* $[t_0; \tau_0)$ *satisfying the initial condition* $y_0(t_0) = y_{(0)}$, *and*

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le y_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \quad t \in [t_0; \tau_0)\,.$$

This theorem is proved in [5] (see [5, Theorem 4.2]).

Let p, q, r, s, l be real numbers and let $\varepsilon > 0$.

Definition 2.2. The ordered fiver (p, q, r, s, l) is called ε-semi definite positive if:

1) $p > 0$, $l > 0$;
2) $\max\{q, r, s\} \ge \sqrt{l + \varepsilon}$ or
 $0 \le \min\{q, r, s\} \le \max\{q, r, s\} \le \sqrt{l + \varepsilon}$ and
 $q^2 + r^2 + s^2 \ge l + \varepsilon$.

Remark 2.3. From the geometrical point of view the relations 1) and 2) mean that the ball of radius $\sqrt{l + \varepsilon}$ with its center in the point (q, r, s) may be located in any such position in the space of coordinates x, y, z, that its intersection with the octant $x > 0$, $y > 0$, $z > 0$ is empty.

Consider the quadratic form

$$W(x, y, z) \equiv p\Big[\Big(x + \frac{q}{2p}\Big)^2 + \Big(y + \frac{r}{2p}\Big)^2 + \Big(z + \frac{s}{2p}\Big)^2\Big] - \frac{l}{4p}\,, \quad x, y, z \in (-\infty; +\infty)\,.$$

Lemma 2.2. *If for some* $\varepsilon > 0$ *the ordered fiver* (p, q, r, s, l) *is* ε*-semi definite positive then for every* $x \ge 0$, $y \ge 0$, $z \ge 0$ *the inequality*

$$W(x, y, z) \ge \varepsilon/4p$$

is satisfied.

Proof. For every $x \ge 0$, $y \ge 0$, $z \ge 0$ we have: if $\max\{q, r, s\} \ge \sqrt{l + \varepsilon}$, then $W(x, y, z) \ge p\frac{l+\varepsilon}{4p^2} - \frac{l}{4p} = \frac{\varepsilon}{4p}$, and if $0 \le \min\{q, r, s\} \le \max\{q, r, s\} \le \sqrt{l + \varepsilon}$, then since $q \ge 0$, $r \ge 0$, $s \ge 0$, we will get: $W(x, y, z) \ge p\big(\frac{q^2}{4p^2} + \frac{r^2}{4p^2} + \frac{s^2}{4p^2}\big) - \frac{l}{4p} \ge \frac{l+\varepsilon}{4p} - \frac{l}{4p} = \frac{\varepsilon}{4p}$.
The lemma is proved. □

3. Global solvability criteria

In this section we study the global solvability conditions of Eq. (1.2) in the case when $a_n(t) \ge 0$, $t \ge t_0$, $n = \overline{0, 3}$. The cases when $(-1)^{m_n} a_n(t) \ge 0$, $t \ge t_0$, $m_n = 0, 1$, $n = \overline{0, 3}$, $m_0 + m_1 + m_2 + m_3 > 0$ are reducible to the studying one by the simple transformations $q \to -q$, $q \to \overline{q}$, $q \to iq$, $q \to jq$, $q \to kq$ and their combinations in (1.2). Denote:

$p_{0,m}(t) \equiv b_m(t) + c_m(t)$, $m = \overline{1,3}$, $p_{1,1}(t) \equiv b_1(t) + c_1(t)$, $p_{1,2}(t) \equiv b_2(t) - c_2(t)$, $p_{1,3}(t) \equiv b_3(t) - c_3(t)$, $p_{2,1}(t) \equiv b_1(t) - c_1(t)$, $p_{2,2}(t) \equiv b_2(t) + c_2(t)$, $p_{2,3}(t) \equiv b_3(t) - c_3(t)$, $p_{3,m}(t) \equiv b_m(t) - c_m(t)$, $m = \overline{1,3}$, $t \geq t_0$.

$$D_0(t) \equiv \begin{cases} \sum\limits_{m=1}^{3} p_{0,m}^2(t) + 4a_0(t)d_0(t), & \text{if} \quad a_0(t) \neq 0\,; \\ 4d_0(t) & \text{if} \quad a_0(t) = 0\,, \end{cases}$$

$$D_n(t) \equiv \begin{cases} \sum\limits_{m=1}^{3} p_{n,m}^2(t) - 4a_n(t)d_n(t)\,, & \text{if} \quad a_n(t) \neq 0\,; \\ -4d_n(t) & \text{if} \quad a_n(t) = 0\,, \end{cases} \qquad n = \overline{1,3},\ t \geq t_0\,.$$

Let $\mathfrak{S}$ be a nonempty subset of the set $\{0,1,2,3\}$ and let $\mathfrak{O}$ be its complement, i.e., $\mathfrak{O} = \{0,1,2,3\}\backslash\mathfrak{S}$.

Theorem 3.1. *Assume $a_n(t) \geq 0$, $n \in \mathfrak{S}$ and if $a_n(t) = 0$ then $p_{n,m}(t) = 0$, $m = \overline{1,3}$, $n \in \mathfrak{S}$; $a_n(t) \equiv 0$, $n \in \mathfrak{O}$, $D_n(t) \leq 0$, $n \in \mathfrak{S}$, $t \geq t_0$.*

Then for every $\gamma_n \geq 0$, $n \in \mathfrak{S}$, $\gamma_n \in (-\infty;+\infty)G$, $n \in \mathfrak{O}$, Eq. (1.2) *has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;+\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$ and*

$$q_n(t) \geq 0\,, \qquad n \in \mathfrak{S}\,, \qquad t \geq t_0\,. \tag{3.1}$$

Moreover if for some $n \in \mathfrak{S}$, $\gamma_n > 0$, then also $q_n(t) > 0$.

Proof. Let $[t_0;T)$ be the maximum existence interval for the solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ of Eq. (1.2) satisfying the initial conditions $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$ (existence of $[t_0;T)$ follows from the theory of normal systems of ordinary differential equations and from (2.1)). Show that

$$q_n(t) \geq 0\,, \qquad t \in [t_0;T)\,,\ n \in \mathfrak{S}\,. \tag{3.2}$$

Let us prove the theorem in the case when $0 \in \mathfrak{S}$. The proof of the theorem for other nonempty $\mathfrak{S}$ can be proved by analogy. Consider the Riccati equations

$$\begin{aligned} &x' + a_0(t)x^2 + \big\{b_0(t) + c_0(t) + 2\big[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)\big]\big\}x \\ &\qquad - P(t, q_1(t), q_2(t), q_3(t)) = 0\,, \qquad t \in [t_0;T)\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &x' + a_0(t)x^2 + \big\{b_0(t) + c_0(t) + 2\big[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)\big]\big\}x \\ &\qquad = 0\,, \qquad t \in [t_0;T)\,. \end{aligned} \tag{3.4}$$

From the conditions of the theorem it follows that $P(t, q_1(t), q_2(t), q_3(t)) \geq 0$, $t \in [t_0;T)$. Then using Theorem 2.1 to the equations (3.3) and (3.4) we conclude that the solution $x(t)$ of Eq. (3.3) with $x(t_0) = \gamma_0 \geq 0$ exists on $[t_0;T)$ and is non negative (since $x_1(t) \equiv 0$ is a solution to Eq. (3.4) on $[t_0;T)$). Obviously $q_0(t)$ is a solution of Eq. (3.3). Hence $q_0(t) = x(t) \geq 0$, $t \in [t_0;T)$. By analogy can be proved the remaining inequalities (3.2). By (2.4) $Y(t) \equiv \widehat{q(t)}$, $t \in [t_0;T)$, is a solution of Eq. (2.3) on $[t_0;T)$. Then it is not difficult to verify that $\mathrm{tr}[A(t)Y(t)] = \sum\limits_{n=0}^{3} a_n(t)q_n(t) = \sum\limits_{n\in\mathfrak{S}} a_n(t)q_n(t)$, $t \in [t_0;T)$. From here and from (3.2) we have:

$$\mathrm{tr}\,[A(t)Y(t)] \geq 0\,, \qquad t \in [t_0;T)\,. \tag{3.5}$$

Show that

$$T = +\infty\,. \tag{3.6}$$

Suppose $T < +\infty$. Then by virtue of Lemma 2.1 from (3.5) it follows that $[t_0;T)$ is not the maximum existence interval for $Y(t)$. Therefore $[t_0;T)$ is not the maximum existence interval for $q(t)$. The obtained contradiction proves (3.6). From (3.6) and (3.2) it follows (3.1). Assume $\gamma_0 > 0$. By already proven the solution $\widetilde{x}(t)$ of Eq. (3.3) with $\widetilde{x}(t_0) = 0$ exists on $[t_0;+\infty)$ and is nonnegative. Then by virtue of Theorem 2.1 the solution $x(t)$ of Eq. (3.3) with $x(t_0) = \gamma_0 > 0$ exists on $[t_0;+\infty)$ and $x(t) \neq \widetilde{x}(t),\ t \geq t_0$. Therefore $x(t) > 0,\ t \geq t_0$. Obviously $x(t) \equiv q_0(t),\ t \geq t_0$. Hence $q_0(t) > 0,\ t \geq t_0$. By analogy it can be shown that if $\gamma_n > 0$ for some other $n \in \mathfrak{S}$, then also $q_n(t) > 0$, $t \geq t_0$.
The theorem is proved. □

Remark 3.1. Theorem 3.1 is a generalization of Theorem 3.1 of work [7].

Set:
$$\mathcal{L}_0(t) \equiv \big(a_0(t), -b_1(t) - c_1(t), -b_2(t) - c_2(t), -b_3(t) - c_3(t),\ D_0(t)\big)\,;$$
$$\mathcal{L}_1(t) \equiv \big(a_1(t),\ b_1(t) + c_1(t), -b_2(t) + c_2(t),\ b_3(t) - c_3(t),\ D_1(t)\big)\,;$$
$$\mathcal{L}_2(t) \equiv \big(a_2(t),\ b_1(t) - c_1(t),\ b_2(t) + c_2(t),\ b_3(t) - c_3(t),\ D_2(t)\big)\,;$$
$$\mathcal{L}_3(t) \equiv \big(a_3(t), -b_1(t) + c_1(t),\ b_2(t) - c_2(t),\ b_3(t) + c_3(t),\ D_3(t)\big)\,.$$

Theorem 3.2. *Let for some $\varepsilon > 0$ and for every $t \geq t_0$ the ordered fivers $\mathcal{L}_n(t)$, $n = \overline{0,3}$ be ε-semi definite positive. Then for every $\gamma_n > 0$, $n = \overline{0,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;+\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and*

$$q_n(t) > 0, \quad t \geq t_0\,, \qquad n = \overline{0,3}\,. \tag{3.7}$$

Proof. Let $[t_0;T)$ be the maximum existence interval for the solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ of Eq. (1.2) satisfying the initial conditions $q_n(t_0) = \gamma_n$ $n = \overline{0,3}$. Show that

$$q_n(t) \geq 0\,, \quad t \in [t_0;T) \ \ n = \overline{0,3}\,. \tag{3.8}$$

Set: $T_1 \equiv \sup\{t \in [t_0;T) : q_n(t) \geq 0,\ \ t \in [t_0;T)\ n = \overline{0,3}\}$. Suppose (3.8) is not true. Then (obviously $T_1 > t_0$)

$$T_1 < T\,. \tag{3.9}$$

On the other hand from the conditions of the theorem it follows that

$$P\big(t, q_1(t),\ q_2(t), q_3(t)\big) \geq \frac{\varepsilon}{4a_0(t)}\,, \quad Q\big(t, q_0(t), q_2(t), q_3(t)\big) \geq \frac{\varepsilon}{4a_1(t)}\,,$$

$$R\big(t, q_0(t), q_1(t), q_3(t)\big) \geq \frac{\varepsilon}{4a_2(t)}\,, \quad S\big(t, q_0(t), q_1(t), q_2(t)\big) \geq \frac{\varepsilon}{4a_3(t)},\ t \in [t_0;T_1)\,.$$

By the continuity property of the functions P, Q, R, S, q_0, q_1, q_2 and q_3 it follows that for some $T_2 > T_1$ ($T_2 < T$) the following inequalities are fulfilled:

$$\begin{cases} P\big(t, q_1(t), q_2(t), q_3(t)\big) \geq 0\,; & Q\big(t, q_0(t), q_2(t), q_3(t)\big) \geq 0\,, \\ R\big(t, q_0(t), q_1(t), q_3(t)\big) \geq 0\,; & S\big(t, q_0(t), q_1(t), q_2(t)\big) \geq 0\,, \end{cases} \tag{3.10}$$

for all $t \in [t_0; T_2)$. Consider on $[t_0; T_2)$ the Riccati equations

$$\begin{aligned} x' + a_0(t)x^2 + \{b_0(t) + c_0(t) + 2[a_1(t)q_1(t) + a_2(t)q_2(t) + a_3(t)q_3(t)]\}x \\ - P\big(t, q_1(t), q_2(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.11}$$

$$\begin{aligned} x' + a_1(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_2(t)q_2(t) + a_3(t)q_3(t)]\}x \\ - Q\big(t, q_0(t), q_2(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.12}$$

$$\begin{aligned} x' + a_2(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_1(t)q_1(t) + a_3(t)q_3(t)]\}x \\ - R\big(t, q_0(t), q_1(t), q_3(t)\big) = 0\,; \end{aligned} \tag{3.13}$$

$$\begin{aligned} x' + a_3(t)x^2 + \{b_0(t) + c_0(t) + 2[a_0(t)q_0(t) + a_1(t)q_1(t) + a_2(t)q_2(t)]\}x \\ - S\big(t, q_0(t), q_1(t), q_2(t)\big) = 0\,. \end{aligned} \tag{3.14}$$

Let $x_0(t)$, $x_1(t)$, $x_2(t)$ and $x_3(t)$ be the solutions of the equations (3.11), (3.12), (3.13) and (3.14) respectively with $x_n(t_0) = 0$, $n = \overline{0,3}$. By virtue of Theorem 2.1 from (3.10) it follows that $x_n(t)$, $n = \overline{0,3}$, exist on $[t_0; T_2)$ and are non negative. Then since $q_0(t)$, $q_1(t)$, $q_2(t)$ and $q_3(t)$ are solutions of the equations (3.11), (3.12), (3.13) and (3.14) on $[t_0; T_2)$ and $q_n(t_0) > x_n(t_0)$, $n = \overline{0,3}$, the last functions (i.e. $q_n(t)$, $n = \overline{0,3}$) are also non negative on $[t_0; T_2)$, which contradicts (3.9). The obtained contradiction proves (3.8). By virtue of Lemma 2.2 from (3.8) and from the conditions of the theorem it follows that on $[t_0; T)$ the inequalities (3.10) are fulfilled. Hence the solutions $x_n(t_0)$ ($n = \overline{0,3}$) exist on $[t_0; T)$ and are non negative. Obviously $q_0(t)$, $q_1(t)$, $q_2(t)$ and $q_3(t)$ are solutions of the equations (3.11), (3.12), (3.13) and (3.14)) respectively on $[t_0; T)$ and $q_n(t_0) > x_n(t_0)$, $n = \overline{0,3}$. Therefore $q_n(t) > 0,\ \ t \in [t_0; T),\ \ n = \overline{0,3}$. Further, the proof of the theorem is carried out similar to the proof of Theorem 3.1.
The theorem is proved. □

Theorem 3.3. *Let $a_0(t) \geq 0,\ \ a_n(t) \equiv 0,\ \ n = \overline{1,3},\ \ t \geq t_0$, and*

$$\int\limits_{t_m}^{t} \exp\Big\{\int\limits_{t_m}^{t} \big[b_0(s) + c_0(s) - I_{b_0+c_0,D_0}(t_m, s)\big]ds\Big\} D_0(\tau)d\tau \leq 0\,,$$

$$t \in [t_m; t_{m+1})\,,\ \ m = 0, 1, \ldots$$

Then for every $\gamma_0 \geq 0$, $\gamma_n \in (-\infty; +\infty)$, $n = \overline{1,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ with $q_n(t_n) = \gamma_n$, $n = \overline{0,3}$ on $[t_0; +\infty)$ and

$$q_0(t) \geq 0\,, \qquad t \geq t_0\,. \tag{3.15}$$

Proof. Let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solution of Eq. (1.2) with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and let $[t_0;T)$ be the maximum existence interval for $q(t)$. Show that

$$T = +\infty\,. \tag{3.16}$$

Consider the Riccati equation

$$y' + a_0(t)y^2 + [b_0(t) + c_0(t)]y + D_0(t) = 0\,, \qquad t \ge t_0\,. \tag{3.17}$$

By Theorem 2.2 from the conditions of the theorem it follows that for every $\gamma_0 \ge 0$ this equation has a solution $y_0(t)$ on $[t_0;+\infty)$ and $y_0(t) \ge 0$, $t \ge t_0$. Then using Theorem 2.1 to Eq. (3.11) and Eq. (3.17) and taking into account the fact that $q_0(t)$ is a solution to Eq. (3.11) we conclude that

$$q_0(t) \ge y_0(t) \ge 0\,, \qquad t \ge t_0\,. \tag{3.18}$$

Suppose $T < +\infty$. Then from (3.18) it follows that

$$\mathrm{tr}[A(t)Y(t)] = \int\limits_{t_0}^{t} a_0(s)q_0(s)ds \ge 0\,, \qquad t \in [t_0;T)\,.$$

By virtue of Lemma 2.1 from here it follows that $[t_0;T)$ is not the maximum existence interval for $q(t)$ which contradicts our supposition. The obtained contradiction proves (3.16). From (3.16) and (3.18) it follows (3.15). The theorem is proved. $\square$

Remark 3.2. Unlike of the conditions of Theorem 3.1 and Theorem 3.2 the conditions of Theorem 3.3 allow $D_0(t)$ to change sign in every $[t_m;t_{m+1})$, $m = 0, 1, \dots$.

By use of Theorem 2.3 and Theorem 2.4 analogically can be proved the following two theorems respectively.

Theorem 3.4. *Let $\alpha(t)$ and $\beta(t)$ be continuously differentiable on $[t_0;+\infty)$ functions and $\alpha(t) > 0$, $\beta(t) > 0$, $t \ge t_0$,*

(A_1) $0 \le a_0(t) \le \alpha(t)$, $D_0(t) \le \beta(t)$, $a_n(t) \equiv 0$, $n = \overline{1,3}$, $t \ge t_0$;

(B_1) $b_0(t) + c_0(t) \ge \frac{1}{2}\Big[\frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}\Big] + \sqrt{\alpha(t)\beta(t)}$, $t \ge t_0$.

Then for every $\gamma_0 \ge -\sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$, $\gamma_n \in (-\infty;+\infty)$, $n = \overline{1,3}$, Eq. (1.2) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;+\infty)$ with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and

$$q_0(t) \ge -\sqrt{\frac{\beta(t)}{\alpha(t)}}\,, \qquad t \ge t_0\,.$$

$\square$

Theorem 3.5. *Let $\alpha(t)$ and $\beta(t)$ be the same as in Theorem 3.4. If assumption (A_1) of Theorem 3.4 and the inequality*

(C_1) $b_0(t) + c_0(t) \le \frac{1}{2}\Big[\frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)}\Big] - \sqrt{\alpha(t)\beta(t)}$, $t \ge t_0$, *are valid. Then for every* $\gamma_0 \ge \sqrt{\frac{\beta(t_0)}{\alpha(t_0)}}$, $\gamma_n \in (-\infty; +\infty)$, $n = \overline{1,3}$, *Eq.* (1.2) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0; +\infty)$ *with* $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, *and*

$$q_0(t) \ge \sqrt{\frac{\beta(t)}{\alpha(t)}}, \qquad t \ge t_0 .$$

□

4. The case when $a_0(t)$ may change sign

In this section we consider the case when $a_0(t)$ my change sign. Set:

$$\left[\frac{\sqrt{\sum_{n=1}^{3}(b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0 \equiv \begin{cases} \frac{\sqrt{\sum_{n=1}^{3}(b_n(t)+c_n(t))^2}}{a_0(t)}, & \text{if} \quad a_0(t) \ne 0; \\ 0, & \text{if} \quad a_0(t) = 0, \end{cases}$$

$$\mathfrak{M}(t) \equiv \int\limits_{t_0}^{t} \|\big(d_1(\tau), d_2(\tau), d_3(\tau)\big)\| d\tau + \frac{1}{2}\sup_{\tau \in [t_0;t]} \left[\frac{\sqrt{\sum_{n=1}^{3}(b_n(\tau)+c_n(\tau))^2}}{a_0(\tau)}\right]_0,$$

$$R_\Gamma(t) \equiv |a_0(t)|(\Gamma + \mathfrak{M}(t))^2 + \sum_{n=1}^{3}|b_n(t)+c_n(t)|(\Gamma + \mathfrak{M}(t)), \qquad t \ge t_0,$$

where $\Gamma > 0$ is a parameter. For any quaternion $q \equiv q_0 + iq_1 + jq_2 + kq_3$ $(q_n \in \mathbb{R}, n = \overline{0.3})$, set $[q]_v \equiv (q_1, q_2, q_3)$.

Theorem 4.1. *Let* $\alpha_m(t)$ *and* $\beta_m(t)$, $m = 1, 2$ *be the same as in Theorem 2.5. If:*

1) $a_n(t) \equiv 0$, $n = \overline{1,3}$;
2) $\alpha_1(t) \le a_0(t) \le \alpha_2(t)$, $\beta_1(t) \le R_\Gamma(t) + d_0(t) \le \beta_2(t)$, $t \in [t_0; \tau_0)$;
3) $b_0(t) + c_0(t) \ge \frac{1}{2}\big(\frac{\alpha_m'(t)}{\alpha_m(t)} - \frac{\beta_m'(t)}{\beta_m(t)}\big) + 2(-1)^m\sqrt{\alpha_m(t)\beta_m(t)}$, $t \in [t_0; \tau_0)$, $m = 1, 2$;
4) $b_0(t) + c_0(t) \ge 2|a_0(t)|R_\Gamma(t)$, $t \in [t_0; \tau_0)$;
5) $\operatorname{supp}(b_n(t) + c_n(t)) \subset \operatorname{supp} a_0(t)$, $n = \overline{1,3}$, *the function* $\left[\frac{\sqrt{\sum_{n=1}^{3}(b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0$ *is bounded on* $[t_0; \tau_0)$ *if* $\tau_0 < +\infty$ *and is locally bounded on* $[t_0; \tau_0)$ *if* $\tau_0 = +\infty$,

then for every $\gamma_0 \in \Big[-\sqrt{\frac{\beta_2(t_0)}{\alpha_2(t_0)}}; \sqrt{\frac{\beta_1(t_0)}{\alpha_1(t_0)}}\Big]$, $\gamma_n \in \mathbb{R}$, $n = \overline{1,3}$, *with* $\|(\gamma_1, \gamma_2, \gamma_3)\| \le \Gamma$ *Eq.* (1.1) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0; \tau_0)$ *satisfying* *the initial conditions* $q_n(t_0) = \gamma_n,\ \ n = \overline{0,3}$, *and*

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; \tau_0)\,; \tag{4.1}$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v| + \mathfrak{M}(t)\,, \quad t \in [t_0; \tau_0)\,. \tag{4.2}$$

If $\tau_0 < +\infty$ *then* $q(t)$ *is continuable on* $[t_0; \tau_0]$.

Proof. Let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solution of Eq. (1.1) with $q_n(t_0) = \gamma_n$, $n = \overline{0,3}$, and let $[t_0; T)$ be the maximum existence interval for $q(t)$. We must show that

$$T \ge \tau_0\,. \tag{4.3}$$

Under the restriction 1) the system (2.1) takes the form

$$\begin{cases} q_0' + a_0(t)q_0^2 + \{b_0(t) + c_0(t)\}q_0 - P(t, q_1, q_2, q_3) = 0; \\ \widetilde{q}\,' + \mathcal{L}_{q_0}(t)\widetilde{q} - f_{q_0}(t) = 0\,, \qquad t \ge t_0\,, \end{cases} \tag{4.4}$$

where

$$f_{q_0}(t) \equiv \big(\big(b_1(t)+c_1(t)\big)q_0+d_1(t), \big(b_2(t)+c_2(t)\big)q_0+d_2(t), \big(b_3(t)+c_3(t)\big)q_0+d_3(t)\big)\,,$$

$$\mathcal{L}_{q_0}(t) \equiv$$
$$\begin{pmatrix} b_0(t) + c_0(t) + 2a_0(t)q_0 & c_3(t) - b_3(t) & b_2(t) - c_2(t) \\ b_3(t) - c_3(t) & b_0(t) + c_0(t) + 2a_0(t)q_0 & c_1(t) - b_1(t) \\ c_2(t) - b_2(t) & b_1(t) - c_1(t) & b_0(t) + c_0(t) + 2a_0(t)q_0 \end{pmatrix},$$

$t \ge t_0,\ \ \widetilde{q} \equiv (q_1, q_2, q_3)$. Since the hermitian part $\mathcal{L}^H_{q_0(t)}(t)$ of the matrix $\mathcal{L}_{q_0(t)}(t)$ is $\mathcal{L}^H_{q_0(t)}(t) = \operatorname{diag}\big\{b_0(t) + c_0(t) + 2a_0(t)q_0(t), b_0(t) + c_0(t) + 2a_0(t)q_0(t), b_0(t) + c_0(t) + 2a_0(t)q_0(t)\big\}$, by the second equation of the system (4.4) $\|[q(t)]_v\|$ we have the estimate (see [8, p. 56, Lemma 4.2]):

$$\begin{aligned} \|[q(t)]_v\| \le \|[q(t_0)]_v\| \exp\Big\{-\int\limits_{t_0}^{t} \big(b_0(\tau) + c_0(\tau) + 2a_0(\tau)q_0(\tau)\big)d\tau\Big\} \\ + \int\limits_{t_0}^{t} \exp\Big\{-\int\limits_{\tau}^{t} \big(b_0(s) + c_0(s) + 2a_0(s)q_0(s)\big)\,ds\Big\}\|f_{q_0(\tau)}(\tau)\|\,d\tau\,, \\ t \in [t_0; t_1)\,. \end{aligned} \tag{4.5}$$

From the condition 4) of the theorem it follows that

$$b_0(t) + c_0(t) + 2a_0(t)q_0(t) \ge 0\,, \qquad t \in [t_0; t_1)\,, \tag{4.6}$$

for some $t_1 > t_0$. Show that

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le q_0(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,, \qquad t \in [t_0; T_2)\,; \tag{4.7}$$

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \mathfrak{M}(t)\,, \qquad t \in [t_0 : T_2)\,. \tag{4.8}$$

From (4.5) and (4.6) it follows

$$\|[q(t)]_v\| \le \|[q(t_0)]_v\| + \frac{1}{2}\exp\Big\{-\int\limits_{t_0}^{t}\big(b_0(s)+c_0(s)+2a_0(s)q_0(s)\big)\,ds\Big\}$$
$$\times \int\limits_{t_0}^{t}\Big(\exp\Big\{\int\limits_{t_0}^{\tau}\big(b_0(s)+c_0(s)+2a_0(s)q_0(s)\big)\,ds\Big\}\Big)'$$
$$\times \left[\frac{\sqrt{\sum_{n=1}^{3}(b_n(\tau)+c_n(\tau))^2}}{a_0(\tau)}\right]_0 d\tau + \int\limits_{t_0}^{t}\|\big(d_1(\tau),d_2(\tau),d_3(\tau)\big)\|d\tau\,,$$

for $t \in [t_0; t_1)$. From here from (4.6) and 5) it follows (4.8). Since $\|[q(t_0)]_v| \le \Gamma$ from (4.8) we obtain

$$-R_\Gamma(t) + q_0(t) \le P\big(t, q_1(t), q_2(t), q_3(t)\big) \le R_\Gamma(t) + q_0(t)\,, \qquad t \in [t_0; t_1)\,.$$

From here and from 2) it follows

$$\beta_1(t) \le P\big(t, q_1(t), q_2(t), q_3(t)\big) \le \beta_2(t)\,, \qquad t \in [t_0; t_1)\,. \tag{4.9}$$

Consider the Riccati equation

$$r' + a_0(t)r^2 + \{b_0(t)+c_0(t)\}r - P\big(t, q_1(t), q_2(t), q_3(t)\big) = 0\,, \qquad t \in [t_0; t_1)\,. \tag{4.10}$$

Let $r(t)$ be a solution of this equation with $r(t_0) = q_0(t_0)$. Then by virtue of Theorem 2.1 from 1), 2) and (4.9) it follows that $r(t)$ exists on $[t_0; t_1)$ and

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}} \le r(t) \le \sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}, \qquad t \in [t_0; t_1)\,.$$

Obviously $q_0(t)$ is a solution of Eq. (4.7) on $[t_0; t_1)$. Hence by the uniqueness theorem $q_0(t)$ coincides with $r(t)$ on $[t_0; t_1)$, and therefore (4.7) is valid. Let T_1 be the upper bound of all $t_1 \in [t_0; T)$ for which (4.7)–(4.9) are satisfied. We assert that

$$T_1 = T\,. \tag{4.11}$$

Indeed otherwise from (4.7) it follows that

$$q_0(t) \ge -\sqrt{\frac{\beta_2(T_1)}{\alpha_2(T_1)}}\,.$$

From here and from 4) it follows that $b_0(t)+c_0(t)+2a_0(t)q_0(t)\ge 0,\ \ t\in[T_1;T_2)$ for some $T_2>T_1$. Hence

$$b_0(t)+c_0(t)+2a_0(t)q_0(t)\ge 0\,,\ \ t\in[t_0;T_2)\,. \tag{4.12}$$

Then repeating the arguments of the proof of (4.7) and (4.8) we conclude that

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}}\le q_0(t)\le\sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,,\qquad t\in[t_0;T_2)\,;$$
$$\|[q(t)]_v\|\le\|[q(t_0)]_v\|+\mathfrak{M}(t)\,,\qquad t\in[t_0:T_2)\,,$$

which with (4.12) contradicts the definition of T_1. The obtained contradiction proves (4.11). Thus

$$-\sqrt{\frac{\beta_2(t)}{\alpha_2(t)}}\le q_0(t)\le\sqrt{\frac{\beta_1(t)}{\alpha_1(t)}}\,,\qquad t\in[t_0;T)\,;$$
$$\|[q(t)]_v\|\le\|[q(t_0)]_v\|+\mathfrak{M}(t)\,,\qquad t\in[t_0:T)\,.$$

By virtue of Lemma 2.1 from here it follows (4.3) and fulfillment of (4.1) and (4.2). If $\tau_0<+\infty$ then by Lemma 2.1 from (4.1) and (4.2) it follows that $q(t)$ is continuable on $[t_0;\tau_0]$.
The theorem is proved. $\square$

Let $\tau_0<+\infty$. Set:

$$\mathfrak{M}^*(t)\equiv\int\limits_t^{\tau_0}\|(d_1(\tau),d_2(\tau),d_3(\tau))\|\,d\tau+\frac{1}{2}\sup_{\tau\in[t;\tau_0]}\left[\frac{\sqrt{\sum_{n=1}^3(b_n(\tau)+c_n(\tau))^2}}{a_0(\tau)}\right]_0,$$

$$R_\Gamma^*(t)\equiv|a_0(t)|(\Gamma+\mathfrak{M}^*(t))^2+\sum_{n=1}^3|b_n(t)+c_n(t)|(\Gamma+\mathfrak{M}^*(t))\,,\qquad t\in[t_0;\tau_0]\,.$$

Corollary 4.1. *Let $\alpha_m(t)$ and $\beta_m(t)$, $m=1,2$, be continuously differentiable on $[t_0;\tau_0]$ functions such that $(-1)^m\alpha_m(t)>0$, $(-1)^m\beta(t)>0$, $t\in[t_0;\tau_0]$, $m=1,2$. If:*

1) $a_n(t)\equiv 0,\ \ n=\overline{1,3}$;

1*) $\alpha_1(t)\le a_0(t)\le\alpha_2(t)$;

2*) $b_0(t)+c_0(t)\le-\frac{1}{2}\big(\frac{\alpha_m'(t)}{\alpha_m(t)}-\frac{\beta_m'(t)}{\beta_m(t)}\big)+2(-1)^m\sqrt{\alpha_m(t)\beta_m(t)},\ \ t\in[t_0;\tau_0]$, $m=1,2$;

3*) $b_0(t)+c_0(t)\le-2|a_0(t)|R_\Gamma^*(t),\ \ t\in[t_0;\tau_0]$;

4*) $\mathrm{supp}\,(b_n(t)+c_n(t))\subset\mathrm{supp}\ a_0(t),\ \ n=\overline{1,3}$, *the function* $\left[\frac{\sqrt{\sum_{n=1}^3(b_n(t)+c_n(t))^2}}{a_0(t)}\right]_0$ *is bounded on* $[t_0;\tau_0]$, *then for every* $\gamma_0 \in \Big[-\sqrt{\frac{\beta_1(\tau_0)}{\alpha_1(\tau_0)}}; \sqrt{\frac{\beta_2(\tau_0)}{\alpha_2(\tau_0)}}\Big]$, $\gamma_n \in \mathbb{R}$, $n = \overline{1,3}$, *with* $\|(\gamma_1, \gamma_2, \gamma_3)\| \le \Gamma$ *Eq.* (1.1) *has a solution* $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ *on* $[t_0; \tau_0]$ *satisfying the initial conditions* $q_n(\tau_0) = \gamma_n$, $n = \overline{0,3}$, *and*

$$-\sqrt{\frac{\beta_1(t)}{\alpha_1(t)}} \le q_0(t) \le \sqrt{\frac{\beta_2(t)}{\alpha_2(t)}}\,, \qquad t \in [t_0; \tau_0]\,; \tag{4.13}$$

$$\|[q(t)]_v\| \le \|[q(\tau_0)]_v\| + \mathfrak{M}^*(t)\,, \qquad t \in [t_0; \tau_0]\,. \tag{4.14}$$

Proof. Set: $\lambda_0 \equiv t_0 = \tau_0$, $\widetilde{a}(t) \equiv -a(\lambda_0 - t)$, $\widetilde{b}(t) \equiv -b(\lambda_0 - t)$, $\widetilde{c}(t) \equiv -c(\lambda_0 - t)$, $\widetilde{d}(t) \equiv -d(\lambda_0 - t)$, $\widetilde{a}_0(t) \equiv -a_0(\lambda_0 - t)$, $\widetilde{b}_n(t) \equiv -b_n(\lambda_0 - t)$, $\widetilde{c}_n(t) \equiv -c_n(\lambda_0 - t)$, $\widetilde{d}_n(t) \equiv -d_n(\lambda_0 - t)$,

$$\widetilde{\mathfrak{M}}(t) \equiv \int\limits_{t_0}^{t} \|(\widetilde{d}_1(\tau), \widetilde{d}_2(\tau), \widetilde{d}_3(\tau))\| d\tau + \frac{1}{2} \sup_{\tau \in [t_0; t]} \left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(\tau) + \widetilde{c}_n(\tau))^2}}{\widetilde{a}_0(\tau)}\right]_0 ,$$

$$\widetilde{R}_\Gamma(t) \equiv |\widetilde{a}_0(t)|(\Gamma + \widetilde{\mathfrak{M}}(t))^2 + \sum_{n=1}^{3} |\widetilde{b}_n(t) + \widetilde{c}_n(t)|(\Gamma + \widetilde{\mathfrak{M}}(t))\,, \qquad t \in [t_0; \tau_0]\,,$$

where

$$\left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t) + \widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}\right]_0 \equiv \begin{cases} \frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t) + \widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}\,, & \text{if } \widetilde{a}_0(t) \ne 0; \\ 0\,, & \text{if } \widetilde{a}_0(t) = 0\,. \end{cases}$$

In Eq. (1.1) make the substitution

$$q(t) = u(\lambda_0 - t)\,, \qquad t \in [t_0; \tau_0]\,.$$

we obtain

$$u' + u\widetilde{a}(t)u + \widetilde{b}(t)u + u\widetilde{c}(t) + \widetilde{d}(t) = 0\,, \qquad t \in [t_0; \tau_0]\,. \tag{4.15}$$

It is not difficult to verify that

$$\widetilde{\mathfrak{M}}(\lambda_0 - t) = \mathfrak{M}^*(t)\,, \qquad \widetilde{R}_\Gamma(\lambda_0 - t) = R^*_\Gamma(t)\,, \qquad t \in [t_0; \tau_0\,.$$

From here and from the conditions 1), 1^*)–4^*) of the corollary we get:

$$\widetilde{\alpha}_1(t) \le \widetilde{a}_0(t) \le \widetilde{\alpha}_2(t)\,, \qquad \widetilde{\beta}_1(t) \le \widetilde{R}_\Gamma(t) + \widetilde{d}_0(t) \le \widetilde{\beta}_2(t)\,,$$
$$\widetilde{b}_0(t) + \widetilde{c}_0(t) \ge 2|\widetilde{a}_0(t)|\widetilde{R}_\Gamma(t)\,,$$
$$\widetilde{b}_0(t) + \widetilde{c}_0(t) \ge \frac{1}{2}\left(\frac{\widetilde{\alpha}'_m(t)}{\widetilde{\alpha}_m(t)} - \frac{\widetilde{\beta}'_m(t)}{\widetilde{\beta}_m(t)}\right) + 2(-1)^m\sqrt{\widetilde{\alpha}_m(t)\widetilde{\beta}_m(t)}\,, \qquad t \in [t_0; \tau_0]\,,$$

where $\widetilde{\alpha}_m(t) \equiv -\alpha_{3-m}(\lambda_0 - t)$, $\widetilde{\beta}_m(t) \equiv -\beta_{3-m}(\lambda_0 - t)$, $m = 1, 2$, $t \in [t_0; \tau_0]$, $\mathrm{supp}\,(\widetilde{b}_n(t) + \widetilde{c}_n(t)) \subset \mathrm{supp}\,\widetilde{a}_0(t)$, $n = \overline{1,3}$, the function $\left[\frac{\sqrt{\sum_{n=1}^{3}(\widetilde{b}_n(t)+\widetilde{c}_n(t))^2}}{\widetilde{a}_0(t)}\right]_0$ is bounded on $[t_0; \tau_0]$. By Theorem 4.1 from here is seen that for every

$\gamma_0 \in \left[-\sqrt{\frac{\widetilde{\beta}_2(t_0)}{\widetilde{\alpha}_2(t_0)}};\sqrt{\frac{\widetilde{\beta}_1(t_0)}{\widetilde{\alpha}_1(t_0)}}\right], \gamma_n \in$ R, $n = \overline{1,3}$, with $||(\gamma_1,\gamma_2,\gamma_3)|| \le \Gamma$ Eq. (4.15) has a solution $u(t) \equiv u_0(t) - iu_1(t) - ju_2(t) - ku_3(t)$ on $[t_0;\tau_0]$ and

$$-\frac{\widetilde{\beta}_2(t)}{\widetilde{\alpha}_2(t)} \le u_0(t) \le \frac{\widetilde{\beta}_1(t)}{\widetilde{\alpha}_1(t)}\,,$$

$$\|[u(t)]_v\| \le \|[u(t_0)]_v\| + \widetilde{\mathfrak{M}}(t)\,, \qquad t \in [t_0;\tau_0]\,.$$

From here it follows that Eq. (1.1) has a solution $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ on $[t_0;\tau_0]$, satisfying the initial conditions $q_n(\tau_0) = \gamma_n, \ \ n = \overline{0,3}$ and the estimates (4.13) and (4.14) are valid.
The corollary is proved. $\square$

5. A completely non conjugation theorem

Consider the linear system

$$\begin{cases} \phi' = C(t)\phi + A(t)\psi\,; \\ \psi' = -D(t)\phi - B(t)\psi\,, \qquad t \ge t_0\,. \end{cases} \tag{5.1}$$

where $\phi = \phi(t)$ and $\psi = \psi(t)$ are the unknown continuously differentiable vector functions of dimension 4, $A(t), \ B(t), \ \ C(t)$ and $D(t)$ are the same matrix functions as in (2.5).

Definition 5.1. We will say that the solution $(\phi(t),\psi(t))$ of the system (5.1) satisfies the completely non conjugation condition if $\phi(t) \ne \theta, \ \ \psi(t) \ne \theta \ \ t \ge t_0$, where θ is the null vector of dimension 4.

Theorem 5.1. *Let the conditions of Theorem 3.1 (of Theorem 3.2) are satisfied. Then the solution* $(\phi(t),\psi(t))$ *of the system* (5.1) *with* $\psi(t_0) = (\gamma_0 E - \gamma_1 I - \gamma_2 J - \gamma_3 K)\phi(t_0) \ne \theta$, *where* $\gamma_n \ge 0, \ \ n \in \mathfrak{S}(\ne \emptyset), \ \sum\limits_{n\in\mathfrak{S}} \gamma_n \ne 0, \ \ \gamma_n \in (-\infty;+\infty), n \in \mathfrak{O}$ *(where* $\gamma_n > 0, \ n = \ \overline{0,3}$*) satisfies of the completely non conjugation condition.*

Proof. Let the conditions of Theorem 3.1 (of Theorem 3.2) be satisfied and let $q(t) \equiv q_0(t) - iq_1(t) - jq_2(t) - kq_3(t)$ be the solutions of Eq. (1.2) with $q_n(t_0) = \gamma_0$, $n = \overline{0,3}$ By virtue of Theorem 3.1 (Theorem 3.2) $q(t)$ exists on $[t_0;+\infty)$. From the condition $\sum\limits_{n\in\mathfrak{S}} \gamma_n > \ 0 \ \ (\gamma_n > 0, \ n = \overline{0,3})$ it follows that

$$q(t) \ne 0\,, \qquad t \ge t_0\,. \tag{5.2}$$

By (2.4) $Y_1(t) \equiv \widehat{q(t)}$ is a solution of Eq. (2.3) on $[t_0;+\infty)$. From (5.2) it follows that

$$\det Y_1(t) \ne 0\,, \qquad t \ge t_0\,. \tag{5.3}$$

Let $\Phi_1(t)$ be the solution of the matrix equation

$$\Phi' = [A(t)Y_1(t) + C(t)]\Phi = 0\,, \qquad t \ge t_0\,,$$

satisfying the initial condition $\Phi_1(t_0) = E$. Them by the Liouville's formula we have

$$\det \Phi(t) = \exp\left\{ \int\limits_{t_0}^{t} \mathrm{tr}\, [A(\tau)Y_1(\tau) + C(\tau)]\, d\tau \right\} > 0\,, \qquad t \geq t_0\,. \tag{5.4}$$

Let $(\phi(t), \psi(t))$ be the solution of the system (5.1) satisfying the initial condition of the theorem. Then

$$\phi(t) = \Phi(t)\phi(t_0)\,, \qquad \psi(t) = Y_1(t)\Phi(t)\phi(t_0)\,.$$

From here from (5.3) and (5.4) it follows that $\phi(t) \neq \theta,\ \ \psi(t) \neq \theta,\ t \geq t_0$.
The theorem is proved. □

Remark 5.1. Except in a special case when $A(t)$ and $D(t)$ are diagonal matrices and $C(t) = B^*(t),\ \ t \geq t_0$ (here $*$ is the transpose sign) the system (5.1) is not hamiltonian.

Acknowledgement. The author is grateful to the referee whose remarks helped to improve the paper.

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