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ON SOME DIOPHANTINE EQUATIONS INVOLVING BALANCING NUMBERS

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Abstract. In this paper, we find all the solutions of the Diophantine equation $B_1^p + 2B_2^p + \cdots + kB_k^p = B_n^q$ in positive integer variables (k, n), where B_i is the i^{th} balancing number if the exponents p, q are included in the set $\{1, 2\}$.

1. Introduction

In 1999, A. Behera, and G.K. Panda [2] studied balancing numbers $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)\,, \tag{1.1}$$

for some positive integer r, in which case the number r is called a *balancer* or a *cobalancing number*. If n is a balancing number with balancer r, then $\frac{n(n-1)}{2} = nr + \frac{r(r+1)}{2}$. This means that

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad \text{and} \quad n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}\,. \tag{1.2}$$

Let B_n denote the n^{th} balancing number and b_n the n^{th} cobalancing number. Then,

$$\begin{aligned} B_1 = 1,\ B_2 = 6 \quad &\text{and} \quad B_{n+1} = 6B_n - B_{n-1}\,, \quad \text{for } n \geq 2\,,\\ b_1 = 0,\ b_2 = 2 \quad &\text{and} \quad b_{n+1} = 6b_n - b_{n-1} + 2\,, \quad \text{for } n \geq 2\,. \end{aligned}$$

From (1.2), we see that B_n is a balancing number if and only if $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2 + 8b_n + 1$ is a perfect square. The numbers

$$C_n = \sqrt{8B_n^2 + 1} \quad \text{and} \quad c_n = \sqrt{8b_n^2 + 8b_n + 1}$$

are then called the n^{th} Lucas-balancing number and the n^{th} Lucas-cobalancing number, respectively. P.K. Ray [10] derived some nice results on balancing numbers and Pell numbers which are given by

$$P_0 = 0,\ P_1 = 1 \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2}\,,$$

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for $n \geq 2$. More generally, for $n \geq 0$, $P_{-n} = (-1)^{n+1} P_n$ (extension of the sequence for negative subscripts). Since an integer x is a balancing number if and only if $8x^2 + 1$ is a square, we set $8x^2 + 1 = y^2$, so that $y^2 - 8x^2 = 1$, for some integer $y \neq 0$, which is a Pell's equation. The fundamental solution is $(x_1, y_1) = (1, 3)$. So $y_n + x_n\sqrt{8} = (3 + \sqrt{8})^n$ for $n \geq 1$ and hence $y_n - x_n\sqrt{8} = (3 - \sqrt{8})^n$. Put $\gamma = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$. The Binet's formula for balancing numbers is

$$B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}\,, \quad \text{for } n \geq 1\,. \tag{1.3}$$

Putting $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, the roots of the characteristic quadratic equation $x^2 - 2x - 1 = 0$ of the Pell's sequence $(P_n)_{n \geq 0}$, the Binet's formula for P_n is

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}\,, \quad \text{for } n \in \mathbb{Z}\,.$$

This easily implies that the inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{1.4}$$

hold, for $n \geq 1$. Since $\alpha^2 = \gamma$ and $\beta^2 = \delta$, we easily get that $B_n = \dfrac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}$ for $n \geq 1$. Thus, there is a correspondence between balancing numbers and Pell numbers. More precisely, we have that $B_n = \frac{P_{2n}}{2}$. See [7], [8] and [9] for further details.

The Diophantine equation

$$\sum_{j=1}^{k} jF_j^p = F_n^q$$

has been studied in 2018 by G. Soydan, L. Németh, and L. Szalay [11], where F_i is the i^{th} Fibonacci number. They solved this equation for $(p, q) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Further, they conjectured that the only non-trivial solutions are given by:

$$F_4^2 = 9 = F_1 + 2F_2 + 3F_3,\ F_8 = 21 = F_1 + 2F_2 + 3F_3 + 4F_4,\ F_4^3 = 27 = F_1^3 + 2F_2^3 + 3F_3^3\,.$$

Later in 2019, K. Gueth, F. Luca and L. Szalay [4] confirmed the conjecture, for $\max\{p, q\} \leq 10$. This result is again improved by Altassan and Luca [1] who proved that if such equation is satisfied, then $\max\{k, n, p, q\} \leq 10^{2500}$. The authors of this paper studied a similar equation where the Fibonacci sequence is replaced by the Pell sequence. See [12].

A question is what will happen if Fibonacci numbers are replaced by balancing numbers. Therefore, in this paper, we investigate the Diophantine equation

$$B_1^p + 2B_2^p + \cdots + kB_k^p = B_n^q, \tag{1.5}$$

in positive integers k and n, where p and q are fixed in $\{1, 2\}$. We consider $B_1^p = 1 = B_1^q$ as a trivial solution to (1.5). The main results proved in this paper are described as follows.

Theorem 1. *The Diophantine equation*

$$B_1 + 2B_2 + \cdots + kB_k = B_n \tag{1.6}$$

has only the trivial solution $(k, n) = (1, 1)$.

Theorem 2. *The Diophantine equation*

$$B_1^2 + 2B_2^2 + \cdots + kB_k^2 = B_n^2 \tag{1.7}$$

possesses only the trivial solution $(k, n) = (1, 1)$.

Theorem 3. *The Diophantine equation*

$$B_1 + 2B_2 + \cdots + kB_k = B_n^2 \tag{1.8}$$

possesses only the trivial solution $(k, n) = (1, 1)$.

Theorem 4. *The Diophantine equation*

$$B_1^2 + 2B_2^2 + \cdots + kB_k^2 = B_n \tag{1.9}$$

possesses only the trivial solution $(k, n) = (1, 1)$.

We will prove our main results using modular arithmetic. We organize this paper as follows. In Section 2, we will recall some known properties and prove key lemmas. Our main results will be proved in Sections 3–6.

2. Some useful lemmas

In this section, we present some useful lemmas. Some of them are a few well-known results and we also prove some preliminary results. We start by recalling Euler's totient function, denoted φ, which is defined for each positive integer n by the number of integers k in the range $1 \le k \le n$ such that $\gcd(n, k) = 1$. The following lemma is a well-known result. One can see Theorem 2.8 of [6].

Lemma 1 (Euler's Totient Theorem)**.** *Let n be a positive integer. For each non-zero integer a relatively prime to n,*

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

The next lemma is a collection of well-known results. One can see for instance Proposition 2.1, Proposition 2.2, Proposition 2.3, and Proposition 2.6 in [3] and [5].

Lemma 2. *Let k and n be arbitrary positive integers.*

(i) $B_{k+n}B_{k-n} - B_k^2 = -B_n^2$ *if* $k > n$ *(Catalan's identity). In particular,* $B_{n-1}B_{n+1} - B_n^2 = -1$ *(Cassini's identity) and* $B_{2n+1} = B_{n+1}^2 - B_n^2$.

(ii) $\sum_{j=1}^{k} B_j = \frac{B_{k+1} - B_k - 1}{4}$.

(iii) $gcd(B_k, B_n) = B_{gcd(k,n)}$. *In particular,* B_k *and* B_n *are coprime if and only if* k *and* n *are coprime.*

(iv) $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$.

(v) $P_{k+n} = P_kP_{n+1} + P_{k-1}P_n$. *In particular,* $P_{2n+1} = P_n^2 + P_{n+1}^2$.

Remark 5. Properties (iv) and (v) hold for any integers k and n, using the formula of the extension to negative subscripts.

We will prove the following results.

Lemma 3. *Let k be a positive integer.*

(i) $6B_kB_{k-1} = B_k^2 + B_{k-1}^2 - 1$ *and* $33B_k^2 - B_{k-1}^2 = B_{2k+1} - 2$ *if* $k \geq 2$.

(ii) $\sum_{j=1}^{k} B_j^2 = \frac{B_{2k+1}-2k-1}{32}$.

(iii) $\sum_{j=1}^{k} jB_j = \frac{k(B_{k+1}-B_k)-B_k}{4}$.

(iv) $\sum_{j=1}^{k} jB_j^2 = \frac{kB_{2k+1}-B_k^2-k(k+1)}{32}$.

(v) $B_{k+n} - B_{k+1-n} = (B_{k+1} - B_k)(P_{2n-1} + P_{2n-2})$, *for* $n \in \{1, 2, \ldots, k\}$. *In particular,* $B_{k+n} \equiv B_{k+1-n} \pmod{(B_{k+1} - B_k)}$.

(vi) $4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} = P_{2k-4n+3}\,(B_{k+1} - B_k)$, *for* $n \in \{1, 2, \ldots, k\}$. *In particular,* $4B_{k+1-n}^2 \equiv 4B_n^2 - P_{4n-1} \pmod{(B_{k+1} - B_k)}$.

Proof.
• The property (i) can be obtained easily, using the recurrence formula of $(B_n)_{n\geq 1}$ and Lemma 2 (i).

• We prove property (ii). It is obvious to see that the property is true for $k = 1$. Assume that $k \geq 2$. We have

$$\sum_{j=1}^{k} B_j^2 = \sum_{j=1}^{k} \left(\frac{P_{2j}}{2}\right)^2 = \frac{1}{4}\sum_{j=1}^{k} P_{2j}^2\,. \tag{2.1}$$

Observe that

$$\begin{aligned}
\sum_{j=1}^{k} P_{2j}^2 &= 4 + \sum_{j=2}^{k}(2P_{2j-1} + P_{2j-2})^2 = 4 + \sum_{j=2}^{k}(4P_{2j-1}^2 + P_{2j-2}^2 + 4P_{2j-1}P_{2j-2}) \\
&= 4 + \sum_{j=2}^{k}\left(4P_{2j-1}^2 - P_{2j-2}^2 + 2P_{2j-2}P_{2j}\right) = 4 + \sum_{j=2}^{k}\left(6P_{2j-1}^2 - P_{2j-2}^2 - 2\right) \\
&= 4 + \sum_{j=2}^{k}\left(6(6P_{2j-2}^2 - P_{2j-3}^2 + 2) - P_{2j-2}^2 - 2\right) \\
&= 4 + \sum_{j=2}^{k}\left(35P_{2j-2}^2 - 6P_{2j-3}^2 + 10\right) \\
&= 4 + \sum_{j=2}^{k}\left(34P_{2j-2}^2 + (P_{2j-2}^2 - 6P_{2j-3}^2 + 2) + 8\right) \\
&= 4 + \sum_{j=2}^{k}\left(34P_{2j-2}^2 - P_{2j-4}^2 + 8\right) \\
&= 33\sum_{j=1}^{k} P_{2j}^2 - 33P_{2k}^2 + P_{2k-2}^2 + 8k - 4\,.
\end{aligned}$$

In the above chain, we used Lemma 2 (iv) to get $P_{2j-2}P_{2j} = P_{2j-1}^2 - 1$. So

$$\sum_{j=1}^{k} P_{2j}^2 = \frac{33P_{2k}^2 - P_{2k-2}^2 - 8k + 4}{32}. \tag{2.2}$$

Now (2.1) and (2.2) imply that

$$\sum_{j=1}^{k} B_j^2 = \frac{33P_{2k}^2 - P_{2k-2}^2 - 8k+4}{128} = \frac{33B_k^2 - B_{k-1}^2 - 2k+1}{32} = \frac{B_{2k+1} - 2k-1}{32},$$

where we used (i) to get $33B_k^2 - B_{k-1}^2 = B_{2k+1} - 2$. Then, property (ii) is proved.

• The next step is to prove (iii). We have

$$\begin{aligned}
\sum_{j=1}^{k} jB_j &= 13+\sum_{j=3}^{k} jB_j = 13+\sum_{j=3}^{k}(6jB_{j-1} - jB_{j-2}) = 13+6\sum_{j=3}^{k} jB_{j-1} - \sum_{j=3}^{k} jB_{j-2} \\
&= 13 + 6\sum_{j=3}^{k}[(j-1)B_{j-1} + B_{j-1}] - \sum_{j=3}^{k}[(j-2)B_{j-2} + 2B_{j-2}] \\
&= 13 + 6\sum_{j=2}^{k-1} jB_j + 6\sum_{j=2}^{k-1} B_j - \sum_{j=1}^{k-2} jB_j - 2\sum_{j=1}^{k-2} B_j \\
&= 1 + 5\sum_{j=1}^{k} jB_j + 4\sum_{j=1}^{k} B_j - (5k+4)B_k + (k+1)B_{k-1}.
\end{aligned}$$

So, we get

$$\begin{aligned}
\sum_{j=1}^{k} jB_j &= \frac{(5k+4)B_k - (k+1)B_{k-1} - 4\sum_{j=1}^{k} B_j - 1}{4} \\
&= \frac{(5k+5)B_k - (k+1)B_{k-1} - B_{k+1}}{4} \\
&= \frac{(k+1)(B_{k+1} - B_k) - B_{k+1}}{4} = \frac{k(B_{k+1} - B_k) - B_k}{4},
\end{aligned}$$

where we used Lemma 2 (ii). Then property (iii) is proved.

• Now, we will take care of (iv). One can easily see that the property is true for $k = 1$. Assume that $k \geq 2$.

$$\begin{aligned}
\sum_{j=1}^{k} jB_j^2 &= B_1^2 + 2B_2^2 + \sum_{j=3}^{k} j(6B_{j-1} - B_{j-2})^2 \\
&= 73+\sum_{j=3}^{k} j(36B_{j-1}^2 - 12B_{j-1}B_{j-2} + B_{j-2}^2) \\
&= 73+\sum_{j=3}^{k} j(34B_{j-1}^2 - B_{j-2}^2 + 2) = 73 + 34\sum_{j=3}^{k} jB_{j-1}^2 - \sum_{j=3}^{k} jB_{j-2}^2 + 2\sum_{j=3}^{k} j
\end{aligned}$$

$$
\begin{aligned}
&= 73 + 34\sum_{j=3}^{k}\left[(j-1)B_{j-1}^2 + B_{j-1}^2\right] - \sum_{j=3}^{k}\left[(j-2)B_{j-2}^2 + 2B_{j-2}^2\right] + 2\sum_{j=3}^{k} j\\
&= 67 + 34\sum_{j=2}^{k-1} jB_j^2 + 34\sum_{j=2}^{k-1} B_j^2 - \sum_{j=1}^{k-2} jB_j^2 - 2\sum_{j=1}^{k-2} B_j^2 + 2\sum_{j=1}^{k} j\\
&= 33\sum_{j=1}^{k} jB_j^2 + 32\sum_{j=1}^{k} B_j^2 - 33kB_k^2 + (k+1)B_{k-1}^2 - 32B_k^2 + k(k+1) - 1\,,
\end{aligned}
$$

where we used Lemma 3 (i) to get that $12B_{j-1}B_{j-2} = 2B_{j-2}^2 + 2B_{j-1}^2 - 2$. Then,

$$
\begin{aligned}
32\sum_{j=1}^{k} jB_j^2 &= 33kB_k^2 - (k+1)B_{k-1}^2 + 32B_k^2 - k(k+1) + 1 - 32\sum_{j=1}^{k} B_j^2\\
&= k(33B_k^2 - B_{k-1}^2) + 33B_k^2 - B_{k-1}^2 - B_k^2 - k(k+1) + 1 - B_{2k+1} + 2k + 1\\
&= kB_{2k+1} - B_k^2 - k(k+1)\,, \qquad (2.3)
\end{aligned}
$$

where we used (ii) and (i). Finally, (2.3) implies that

$$
\sum_{j=1}^{k} jB_j^2 = \frac{kB_{2k+1} - B_k^2 - k(k+1)}{32}\,,
$$

as expected.

• Next, we will prove (v). Let n be an element of $\{1, 2, \ldots, k\}$. We have

$$
\begin{aligned}
(B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) &= \Big(\frac{P_{2k+2} - P_{2k}}{2}\Big)(P_{2n-1} + P_{2n-2})\\
&= P_{2k+1}(P_{2n-1} + P_{2n-2})\,. \qquad (2.4)
\end{aligned}
$$

On the other hand, using Lemma 2 (v), we obtain

$$
\begin{aligned}
P_{2n+2k} &= P_{2n}P_{2k+1} + P_{2n-1}P_{2k}\\
&= 2P_{2k+1}(P_{2n-1} + P_{2n-2}) - P_{2n-2}P_{2k+1} + P_{2n-1}P_{2k}\,. \qquad (2.5)
\end{aligned}
$$

Then, we get

$$
P_{2k+1}(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} + P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k}). \qquad (2.6)
$$

Equations (2.4) and (2.6) imply that

$$
(B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} + P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k}). \qquad (2.7)
$$

Using again Lemma 2 (v), we have that

$$
\begin{aligned}
P_{2k+2-2n} = P_{2-2n+2k} &= P_{2-2n}P_{2k+1} + P_{1-2n}P_{2k}\\
&= -P_{2n-2}P_{2k+1} + P_{2n-1}P_{2k}\,, \qquad (2.8)
\end{aligned}
$$

where we used formulas of the extension of the sequence of Pell numbers for negative subscripts. Equation (2.8) implies that

$$P_{2n-2}P_{2k+1} - P_{2n-1}P_{2k} = -P_{2k+2-2n}\,. \tag{2.9}$$

Now, equations (2.7) and (2.9) imply that

$$(B_{k+1} - B_k)(P_{2n-1} + P_{2n-2}) = \frac{1}{2}(P_{2n+2k} - P_{2k+2-2n}) = B_{k+n} - B_{k+1-n}\,,$$

and property (v) is proved.

• Finally, we will deal with property (vi). Let n be an element of $\{1, 2, \ldots, k\}$. Using the Binet's formula for $(P_n)_{n\geq 0}$, we obtain

$$\begin{aligned} P_{2k-4n+3}(B_{k+1} - B_k) &= P_{2k-4n+3}\Big(\frac{P_{2k+2} - P_{2k}}{2}\Big) = P_{2k-4n+3}P_{2k+1} \\ &= \frac{1}{8}\left(\alpha^{4k-4n+4} + \beta^{4k-4n+4} - \alpha^{2k-4n+3}\beta^{2k+1} - \alpha^{2k+1}\beta^{2k-4n+3}\right)\,. \end{aligned} \tag{2.10}$$

Observe that

$$\begin{aligned} -\alpha^{2k-4n+3}&\beta^{2k+1} - \alpha^{2k+1}\beta^{2k-4n+3} \\ &= -\,\alpha^{2k+1}\beta^{2k+1}\alpha^2\alpha^{-4n} - \alpha^{2k+1}\beta^{2k+1}\beta^2\beta^{-4n} = \alpha^2\alpha^{-4n} + \beta^2\beta^{-4n} \\ &= \alpha^{4n-2} + \beta^{4n-2}\,, \end{aligned} \tag{2.11}$$

where we used the fact that $\alpha^{2k+1}\beta^{2k+1} = (-1)^{2k+1} = -1$. Then, equations (2.10) and (2.11) imply that

$$P_{2k-4n+3}\,(B_{k+1} - B_k) = \frac{1}{8}\left(\alpha^{4k-4n+4} + \beta^{4k-4n+4} + \alpha^{4n-2} + \beta^{4n-2}\right)\,. \tag{2.12}$$

On the other hand, again using the Binet's formula for $(P_n)_{n\geq 0}$, we get that

$$\begin{aligned} 4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} = P_{2k+2-2n}^2 - P_{2n}^2 + P_{2n-1}^2 + P_{2n}^2 &= P_{2k+2-2n}^2 + P_{2n-1}^2 \\ &= \frac{1}{8}\left(\alpha^{4k-4n+4} + \beta^{4k-4n+4} + \alpha^{4n-2} + \beta^{4n-2}\right)\,, \end{aligned} \tag{2.13}$$

where we used Lemma 2 (v) to get that $P_{4n-1} = P_{2n-1}^2 + P_{2n}^2$. From equations (2.12) and (2.13), we conclude that

$$4B_{k+1-n}^2 - 4B_n^2 + P_{4n-1} = P_{2k-4n+3}\,(B_{k+1} - B_k)\,,$$

as expected. □

Lemma 4. *Let $(U_n)_{n\geq 1}$ be the sequence defined by $U_n = B_{2n+1} - B_{2n-1}$ and k be a positive integer.*

(i) *The sequence $(U_n)_{n\geq 1}$ satisfies $U_1 = 34$, $U_2 = 1154$ and $U_n = 34U_{n-1} - U_{n-2}$, for $n \geq 3$.*

(ii) *$U_{k+n} - U_{k+1-n} = 32B_{2n-1}B_{2k+1}$, for $n \in \{1, 2, \ldots, k\}$. In particular, $U_{k+n} \equiv U_{k+1-n} \pmod{B_{2k+1}}$, for $n \in \{1, 2, \ldots, k\}$.*

Proof. One can check easily (i). We prove (ii). Let $n \in \{1, 2, \ldots, k\}$. We have

$$U_{k+n} - U_{k+1-n} = B_{2k+2n+1} - B_{2k+2n-1} - (B_{2k-2n+3} - B_{2k-2n+1}) . \tag{2.14}$$

By, Lemma 3 (i), $B_{2k+2n+1} = 33B_{k+n}^2 - B_{k+n-1}^2 + 2$ and by Lemma 2 (i), $B_{2k+2n-1} = B_{k+n}^2 - B_{k+n-1}^2$, so that $B_{2k+2n+1} - B_{2k+2n-1} = 32B_{k+n}^2 + 2$. Similarly, $B_{2k-2n+3} - B_{2k-2n+1} = 32B_{k-n+1}^2 + 2$. Then, equation (2.14) becomes

$$U_{k+n} - U_{k+1-n} = 32(B_{k+n}^2 - B_{k-n+1}^2) = 32B_{2n-1}B_{2k+1} ,$$

where we used Lemma 2 (i) to get that $B_{k+n}^2 - B_{k-n+1}^2 = B_{2n-1}B_{2k+1}$. Then, $U_{k+n} - U_{k+1-n} = 32B_{2n-1}B_{2k+1}$, as expected, completing the proof of (ii). □

Lemma 5. *Let k be a positive integer. Then, we have*

$$B_k^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}} ,$$

where φ is Euler's totient function.

Proof. We have

$$B_k(B_k - B_{k+2}) = B_k^2 - B_kB_{k+2} = B_k^2 - B_{k+1}^2 + 1 \equiv 1 \pmod{B_{2k+1}} , \tag{2.15}$$

since by Lemma 2 (i) to get that $B_kB_{k+2} = B_{k+1}^2 - 1$ and $B_k^2 - B_{k+1}^2 = -B_{2k+1}$. Multiplying both sides of (2.15) by $B_k^{\varphi(B_{2k+1})-1}$, we get that

$$(B_k - B_{k+2})B_k^{\varphi(B_{2k+1})} \equiv B_k^{\varphi(B_{2k+1})-1} \pmod{B_{2k+1}} . \tag{2.16}$$

Since k and $2k+1$ are coprime, B_k and B_{2k+1} are coprime (see Lemma 2 (iii)). Then, by Lemma 1, $B_k^{\varphi(B_{2k+1})} \equiv 1 \pmod{B_{2k+1}}$, so that (2.16) leads to

$$B_k^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}} .$$

□

3. Proof of Theorem 1

For $k = 1, 2, \ldots, 5$, one can easily find the solutions mentioned in the statement of Theorem 1. So, we assume from now that $k \geq 6$. Using Lemma 3 (iii), equation (1.6) leads to

$$k = \frac{4B_n + B_k}{B_{k+1} - B_k} \in \mathbb{N} .$$

This last equation implies that $4B_n + B_k \equiv 0 \pmod{(B_{k+1} - B_k)}$. So, we study the sequence $(B_m)_{m\geq 1}$ modulo $B_{k+1} - B_k$ if k is fixed. Note that we just indicate a suitable value congruent to B_m modulo $B_{k+1} - B_k$, not always the smallest non-negative remainders. Since $B_{k+i} \equiv B_{k+1-i} \pmod{(B_{k+1} - B_k)}$, for $i \in \{1, 2, \ldots, k\}$ (see Lemma 3 (v)), the period having length $4k+2$ can be given by

$$\overbrace{1, 6, 35, \ldots, B_k}^{k}, \overbrace{B_k, B_{k-1}, B_{k-2}, \ldots, 35, 6, 1}^{k}, \overbrace{0, -1, -6, -35, \ldots, -B_k}^{k+1},$$

$$\overbrace{-B_k, -B_{k-1}, -B_{k-2}, \ldots, -35, -6, -1, 0}^{k+1} .$$

So either $B_n \equiv 0$ or $\pm B_i \pmod{(B_{k+1} - B_k)}$, for some $i \in \{1, 2, \ldots, k\}$. Hence,

$$4B_n + B_k \equiv B_k \text{ or } \pm 4B_i + B_k \pmod{(B_{k+1} - B_k)}.$$

Assume that $4B_n + B_k \equiv B_k \pmod{(B_{k+1} - B_k)}$. We have

$$0 < B_k < B_{k+1} - B_k = 6B_k - B_{k-1} - B_k = 5B_k - B_{k-1},$$

so that $B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$. Thus $4B_n + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$.

Assume now that $4B_n + B_k \equiv \pm 4B_i + B_k \pmod{(B_{k+1} - B_k)}$, $i \in \{1, 2, \ldots, k\}$.

- If $1 \leq i \leq k - 1$, then we get that

$$0 < -4B_{k-1} + B_k \leq \pm 4B_i + B_k \leq 4B_{k-1} + B_k < 5B_k - B_{k-1} = B_{k+1} - B_k.$$

So $\pm 4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$.

- If $i = k$, then

$$4B_i + B_k = 5B_k = B_{k+1} - B_k + B_{k-1} \equiv B_{k-1} \pmod{(B_{k+1} - B_k)}.$$

But, since $k \geq 6$, we have $0 < B_{k-1} < 5B_k - B_{k-1} = B_{k+1} - B_k$, so that

$$B_{k-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}.$$

Then, $-4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$. Similarly, $-4B_i + B_k = -3B_k$. Since

$$-5B_k + B_{k-1} = -B_{k+1} + B_k \equiv 0 \pmod{(B_{k+1} - B_k)},$$

we get that $-3B_k \equiv 2B_k - B_{k-1} \pmod{(B_{k+1} - B_k)}$. But $2B_k - B_{k-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}$ since $0 < 2B_k - B_{k-1} < B_{k+1} - B_k$. Then, $-4B_i + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$.

In conclusion, we get that

$$4B_n + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)},$$

for $k \geq 6$. So equation (1.6) has no more solutions when $k \geq 6$. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

When $k = 1, 2, \ldots, 5$, one can easily find the solution given in Theorem 2. So, we assume from now that $k \geq 6$. Using Lemma 3 (*iv*), equation (1.7) implies that

$$k = \frac{32B_n^2 + B_k^2 + k(k+1)}{B_{2k+1}}. \tag{4.1}$$

Observe that $32B_n^2 = 33B_n^2 - B_{n-1}^2 - (B_n^2 - B_{n-1}^2) = B_{2n+1} - 2 - B_{2n-1}$ (see Lemma 3 (i) and Lemma 2 (i)), so that equation (4.1) becomes

$$k = \frac{U_n + B_k^2 + k(k+1) - 2}{B_{2k+1}} \in \mathbb{N}, \tag{4.2}$$

where $(U_m)_{m\geq 1}$ is the sequence defined by $U_m = B_{2m+1} - B_{2m-1}$. By Lemma 4 (i), $U_1 = 34$, $U_2 = 1154$ and $U_m = 34U_{m-1} - U_{m-2}$, for $m \geq 3$. Equation (4.2) implies that $U_n + B_k^2 + k(k+1) - 2 \equiv 0 \pmod{B_{2k+1}}$. So, we study the sequence $(U_m)_{m\geq 1}$ modulo B_{2k+1}, if k is fixed. Since $U_{k+i} \equiv U_{k+1-i} \pmod{B_{2k+1}}$, for $i \in \{1, 2, \ldots, k\}$ (see Lemma 4 (ii)), the period having length $2k+1$ can be given by

$$\overbrace{34, 1154, \ldots, U_{k-1}, U_k}^{k}\,\overbrace{U_k, U_{k-1}, \ldots, 1154, 34, 2}^{k+1}\,.$$

So either $U_n \equiv 2$ or $U_j \pmod{B_{2k+1}}$, for some $j \in \{1, 2, \ldots, k\}$. Hence, we have

$$U_n + B_k^2 + k(k+1) - 2 \equiv B_k^2 + k(k+1) \text{ or } U_j + B_k^2 + k(k+1) - 2 \pmod{B_{2k+1}}.$$

Assume that $U_n + B_k^2 + k(k+1) - 2 \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$. Using Lemma 3 (i), we have

$$\begin{aligned} B_{2k+1} &= 33B_k^2 - B_{k-1}^2 + 2 > 32B_k^2 = B_k^2 + 31\frac{P_{2k}^2}{4} \\ &> B_k^2 + 7P_{2k}^2 > B_k^2 + k(k+1)\,, \end{aligned}$$

where we used the fact that $7P_{2k}^2 \geq 7\alpha^{4(k-1)} > k(k+1)$, for $k \geq 1$. Hence, $0 < B_k^2 + k(k+1) < B_{2k+1}$, so that $B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$. Then, we get

$$U_n + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}.$$

Assume now that

$$U_n + B_k^2 + k(k+1) - 2 \equiv U_j + B_k^2 + k(k+1) - 2 \pmod{B_{2k+1}},$$

for some $j \in \{1, 2, \ldots, k\}$.

- If $j = k$, then we get that

$$\begin{aligned} U_k + B_k^2 + k(k+1) - 2 &= B_{2k+1} - B_{2k-1} + B_k^2 + k(k+1) - 2 \\ &= B_{2k+1} + B_{k-1}^2 + k(k+1) - 2 \\ &\equiv B_{k-1}^2 + k(k+1) - 2 \pmod{B_{2k+1}}, \end{aligned} \tag{4.3}$$

where, we used Lemma 2 (i) to get that $B_{2k-1} = B_k^2 - B_{k-1}^2$. Using the previous case, we have that

$$0 < B_{k-1}^2 + k(k+1) - 2 < B_k^2 + k(k+1) < B_{2k+1}\,,$$

so that

$$B_{k-1}^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}. \tag{4.4}$$

Congruences (4.3) and (4.4) imply that $U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}$,

- If $1 \leq j \leq k-1$, then we get that

$$\begin{aligned} U_j + B_k^2 + k(k+1) - 2 &\leq U_{k-1} + B_k^2 + k(k+1) - 2 \\ &= B_{2k-1} - B_{2k-3} + B_k^2 + k(k+1) - 2 \\ &= B_{2k-1} + B_{k-2}^2 + B_{2k-1} + k(k+1) - 2 \\ &= 2B_{2k-1} + k(k+1) + B_{k-2}^2 - 2\,, \end{aligned} \tag{4.5}$$

where, we used Lemma 2 (i) to get that $B_{2k-3} = B_{k-1}^2 - B_{k-2}^2$ and $B_k^2 - B_{k-1}^2 = B_{2k-1}$. On the other hand, we have

$$\begin{aligned}
&B_{2k+1} - 2B_{2k-1} - k(k+1) - B_{k-2}^2 + 2 \\
&\quad = B_{k+1}^2 - B_k^2 - 2(B_k^2 - B_{k-1}^2) - k(k+1) - B_{k-2}^2 + 2 \\
&\quad = 36B_k^2 - 12B_kB_{k-1} + B_{k-1}^2 - 3B_k^2 + 2B_{k-1}^2 - B_{k-2}^2 - k(k+1) + 2 \\
&\quad = 31B_k^2 + B_{k-1}^2 - B_{k-2}^2 - k(k+1) + 4 \\
&\quad > 31B_k^2 - k(k+1) + 4 = \frac{31}{4}P_{2k}^2 - k(k+1) + 4 > 0\,.
\end{aligned} \tag{4.6}$$

In the above chain, we used Lemma 3 (i) to get that $12B_kB_{k-1} = 2B_k^2 + 2B_{k-1}^2 - 2$, as well as the fact that $\frac{31}{4}P_{2k}^2 > 7\alpha^{4(k-1)} > k(k+1) - 4$, for $k \geq 1$. Inequality (4.6) implies that

$$2B_{2k-1} + k(k+1) + B_{k-2}^2 - 2 < B_{2k+1}\,. \tag{4.7}$$

Now, inequalities (4.5) and (4.7) imply that $U_j + B_k^2 + k(k+1) - 2 < B_{2k+1}$. Finally, we have $0 < U_j + B_k^2 + k(k+1) - 2 < B_{2k+1}$, so that

$$U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}\,.$$

In conclusion, we have

$$U_j + B_k^2 + k(k+1) - 2 \not\equiv 0 \pmod{B_{2k+1}}\,,$$

for $j \in \{1, 2, \ldots, k\}$. Therefore, equation (1.7) has no more solutions for $k \geq 6$. This completes the proof of Theorem 2.

5. Proof of Theorem 3

The proof of this theorem is similar to the proof of Theorem 1. For $k = 1, 2, \ldots, 5$, one can easily find the solution mentioned in the statement of Theorem 3. So, we assume from now that $k \geq 6$. By Lemma 3 (iii), equation (1.8) implies that

$$k = \frac{4B_n^2 + B_k}{B_{k+1} - B_k} \in \mathbb{N}\,.$$

This last equation implies that $4B_n^2 + B_k \equiv 0 \pmod{(B_{k+1} - B_k)}$. So, we study here the sequence $\left(B_m^2\right)_{m\geq 1}$ modulo $B_{k+1} - B_k$, if k is fixed. The period having length $2k+1$ can be deduced from the range

$$\overbrace{1^2, 6^2, 35^2, \ldots, B_k^2}^{k}, \overbrace{B_k^2, B_{k-1}^2, B_{k-2}^2, \ldots, 35^2, 6^2, 1^2, 0}^{k+1}\,.$$

So we have $B_n^2 \equiv 0$ or $B_i^2 \pmod{(B_{k+1} - B_k)}$, for some $i \in \{1, 2, \ldots, k\}$. Thus

$$4B_n^2 + B_k \equiv B_k \ \text{ or } \ 4B_i^2 + B_k \pmod{(B_{k+1} - B_k)}\,.$$

• Assume that $4B_n^2 + B_k \equiv B_k \pmod{(B_{k+1} - B_k)}$. We have $0 < B_k < B_{k+1} - B_k$, so that $B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$. So $4B_n^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$.

• Assume that $4B_n^2 + B_k \equiv 4B_i^2 + B_k \pmod{(B_{k+1} - B_k)}$, for some $i \in \{1, 2, \ldots, k\}$.

We put $m = \lfloor \frac{k}{2} \rfloor$. Suppose that $1 \le i \le m$. Then, we have

$$4B_i^2 + B_k = P_{2i}^2 + B_k \le P_{2m}^2 + B_k \le P_k^2 + B_k \,. \tag{5.1}$$

On the other hand, we get

$$\begin{aligned} B_{k+1} - B_k &= 4B_k - B_{k-1} + B_k = \frac{4P_{2k} - P_{2k-2}}{2} + B_k \\ &= \frac{8P_{2k-1} + 3P_{2k-2}}{2} + B_k = \frac{8P_{k-1}^2 + 8P_k^2 + 3P_{2k-2}}{2} + B_k \\ &= 4P_{k-1}^2 + 4P_k^2 + 3\frac{P_{2k-2}}{2} + B_k > P_k^2 + B_k \,. \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), one can see that $4B_i^2 + B_k < B_{k+1} - B_k$. Thus we obtain

$$0 < 4B_i^2 + B_k < B_{k+1} - B_k \,,$$

so that $4B_i^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)}$, for $i \in \{1, 2, \ldots, m\}$.

It remains to prove that for $i \in \{1, 2, \ldots, m+1\}$,

$$4B_{k+1-i}^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)} \,.$$

By Lemma 3 (vi), we have $4B_{k+1-i}^2 + B_k \equiv 4B_i^2 + B_k - P_{4i-1} \pmod{(B_{k+1} - B_k)}$, for $i \in \{1, 2, \ldots, m+1\}$. So, it suffices to prove that for $i \in \{1, 2, \ldots, m+1\}$,

$$4B_i^2 + B_k - P_{4i-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)} \,.$$

Let i be an element of the set $\{1, 2, \ldots, m+1\}$. From the previous argument, we get $4B_i^2 + B_k < B_{k+1} - B_k$. Then, we deduce that

$$4B_i^2 + B_k - P_{4i-1} < B_{k+1} - B_k \,. \tag{5.3}$$

We will prove that $4B_i^2 + B_k - P_{4i-1} > 0$. We have

$$\begin{aligned} 4B_i^2 + B_k - P_{4i-1} &= P_{2i}^2 + \frac{P_{2k}}{2} - P_{2i-1}^2 - P_{2i}^2 = \frac{2P_{2k-1} + P_{2k-2}}{2} - P_{2i-1}^2 \\ &\ge \frac{2P_{k-1}^2 + 2P_k^2 + P_{2k-2}}{2} - P_{2m+1}^2 \\ &\ge P_{k-1}^2 + P_k^2 - P_{k+1}^2 + \frac{P_{2k-2}}{2} > 0 \,. \end{aligned} \tag{5.4}$$

Now, from (5.3) and (5.4), we have $0 < 4B_i^2 + B_k - P_{4i-1} < B_{k+1} - B_k$, so that $4B_i^2 + B_k - P_{4i-1} \not\equiv 0 \pmod{(B_{k+1} - B_k)}$. Hence, we obtain

$$4B_{k+1-i}^2 + B_k \not\equiv 0 \pmod{(B_{k+1} - B_k)} \,,$$

for $i \in \{1, 2, \ldots, m+1\}$. This completes the proof of Theorem 3.

6. Proof of Theorem 4

For $k = 1, 2, \ldots, 5$, one can easily find the solution mentioned in the statement of Theorem 4. So, we assume that $k \ge 6$. By Lemma 2 (i), one can see that

$B_{2k}^2 = B_{2k-1}B_{2k+1} + 1 \equiv 1 \pmod{B_{2k+1}}$. Using Lemma 3 (iv), equation (1.9) implies that

$$k = \frac{32B_n + B_k^2 + k(k+1)}{B_{2k+1}}\,. \tag{6.1}$$

Equation (6.1) implies particularly that $32B_n + B_k^2 + k(k+1) \equiv 0 \pmod{B_{2k+1}}$. So, here we study the sequence $(B_m)_{m\geq 1}$ modulo B_{2k+1} if k is fixed. The period can be deduced from the range

$$\overbrace{1, 6, 35, \ldots, B_{2k}, 0}^{2k+1} \overbrace{-B_{2k}, -6B_{2k}, -35B_{2k}, \ldots, -B_{2k-1}B_{2k}, -B_{2k}^2, 0}^{2k+1}$$

of length $4k+2$ since $B_{2k}^2 \equiv 1 \pmod{B_{2k+1}}$. So either $B_n \equiv 0 \pmod{B_{2k+1}}$ or $B_n \equiv B_j$ or $-B_jB_{2k} \pmod{B_{2k+1}}$, for some $j \in \{1, 2, \ldots, 2k\}$. Hence, we have

$$32B_n + B_k^2 + k(k+1) \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$$

or

$$32B_i + B_k^2 + k(k+1) \equiv 32B_j + B_k^2 + k(k+1) \pmod{B_{2k+1}}$$

or

$$32B_i + B_k^2 + k(k+1) \equiv -32B_jB_{2k} + B_k^2 + k(k+1) \pmod{B_{2k+1}},$$

for some $j \in \{1, 2, \ldots, 2k\}$. Therefore, we will distinguish three cases.

Case 1: $32B_n + B_k^2 + k(k+1) \equiv B_k^2 + k(k+1) \pmod{B_{2k+1}}$. Using Lemma 3 (i), we have

$$B_{2k+1} = 33B_k^2 - B_{k-1}^2 + 2 > 32B_k^2 = B_k^2 + 31\frac{P_{2k}^2}{4} > B_k^2 + k(k+1)\,,$$

since $31\frac{P_{2k}^2}{4} > 7P_{2k}^2 > 7\alpha^{4(k-1)} > k(k+1)$. Hence, we obtain, $0 < B_k^2 + k(k+1) < B_{2k+1}$, so that $B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$. Then, we deduce that

$$32B_n + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}\,.$$

Case 2: $32B_n + B_k^2 + k(k+1) \equiv 32B_j + B_k^2 + k(k+1) \pmod{B_{2k+1}}$, for some $j \in \{1, 2, \ldots, 2k\}$.

- If $j = 2k$, then we get that

$$\begin{aligned} 32B_{2k} + B_k^2 + k(k+1) &= 5B_{2k+1} + 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) \\ &\equiv 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) \pmod{B_{2k+1}}\,. \end{aligned} \tag{6.2}$$

We will check that

$$2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}\,. \tag{6.3}$$

Indeed, one can see that

$$5B_{2k-1} = B_{2k} + B_{2k-2} - B_{2k-1} < B_{2k}\,,$$

so that

$$2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < 3B_{2k} + B_k^2 + k(k+1)\,. \tag{6.4}$$

On the other hand, using $B_{2k} = \frac{B_{2k+1}+B_{2k-1}}{6}$ and Lemma 2 (i), we get that

$$\begin{aligned} & B_{2k+1} - 3B_{2k} - B_k^2 - k(k+1) \\ & \quad = 16B_k^2 + B_{k-1}^2 - 6B_kB_{k-1} - k(k+1) \\ & \quad = 15B_k^2 + 1 - k(k+1) = 15\frac{P_{2k}^2}{4} + 1 - k(k+1) > 0\,, \end{aligned} \tag{6.5}$$

where we used Lemma 3 (i) to get that $6B_kB_{k-1} = B_k^2 + B_{k-1}^2 - 1$ and the fact that $15\frac{P_{2k}^2}{4} > 3\alpha^{4(k-1)} > k(k+1)$, for $k \geq 1$. Inequality (6.5) implies that

$$3B_{2k} + B_k^2 + k(k+1) < B_{2k+1}\,. \tag{6.6}$$

From (6.4) and (6.6), we get $2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}$. So inequality (6.3) is proved and finally

$$0 < 2B_{2k} + 5B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}\,. \tag{6.7}$$

Now, (6.2) and (6.7) imply that

$$32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}\,.$$

- If $1 \leq j \leq 2k-1$, then we have

$$32B_j + B_k^2 + k(k+1) \leq 32B_{2k-1} + B_k^2 + k(k+1)\,. \tag{6.8}$$

Furthermore, using Lemma 2 (i) and the recurrence formula, we get

$$\begin{aligned} & B_{2k+1} - 32B_{2k-1} - B_k^2 - k(k+1) \\ & \quad = 2B_k^2 + 33B_{k-1}^2 - 12B_kB_{k-1} - k(k+1) \\ & \quad = 31B_{k-1}^2 + 2 - k(k+1) > 0, \end{aligned} \tag{6.9}$$

where we used again Lemma 3 (i), as well as the fact that $31B_{k-1}^2 > 7P_{2k-2}^2 > 7\alpha^{4(k-2)} > k(k+1)$, for $k \geq 1$. Inequality (6.9) implies that

$$32B_{2k-1} + B_k^2 + k(k+1) < B_{2k+1}\,. \tag{6.10}$$

Now, (6.8) and (6.10) imply that $32B_j + B_k^2 + k(k+1) < B_{2k+1}$. Finally, we have $0 < 32B_j + B_k^2 + k(k+1) < B_{2k+1}$, so that $32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$.

In all subcases, we have $32B_j + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$. So, we obtain

$$32B_n + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}\,.$$

Case 3: $32B_n + B_k^2 + k(k+1) \equiv -32B_jB_{2k} + B_k^2 + k(k+1) \pmod{B_{2k+1}}$, for some $j \in \{1, 2, \ldots, 2k\}$. We will prove that $-32B_jB_{2k} + B_k^2 + k(k+1) \not\equiv 0 \pmod{B_{2k+1}}$. Assume that $-32B_jB_{2k} + B_k^2 + k(k+1) \equiv 0 \pmod{B_{2k+1}}$ in order to get a contradiction. Then, one can see that

$$B_k^2 + k(k+1) \equiv 32B_jB_{2k} \pmod{B_{2k+1}}\,.$$

Since $B_{2k}^{\varphi(B_{2k+1})} \equiv 1 \pmod{B_{2k+1}}$, multiplying both sides by $B_{2k}^{\varphi(B_{2k+1})-1}$, we get

$$\left[B_k^2 + k(k+1)\right] B_{2k}^{\varphi(B_{2k+1})-1} \equiv 32B_j \pmod{B_{2k+1}}\,. \tag{6.11}$$

By Lemma 5, $B_{2k}^{\varphi(B_{2k+1})-1} \equiv B_k - B_{k+2} \pmod{B_{2k+1}}$. Then, (6.11) implies that

$$[B_k^2 \ + k(k+1)]\,(B_k - B_{k+2}) \equiv 32B_j \pmod{B_{2k+1}},$$

i.e.

$$32B_j \ + \left[B_k^2 + k(k+1)\right](B_{k+2} - B_k) \equiv 0 \pmod{B_{2k+1}}.$$

This leads to

$$32B_j + \left[B_k^2 + k(k+1)\right](B_{k+2} - B_k) - B_kB_{2k+1} \equiv 0 \pmod{B_{2k+1}}, \tag{6.12}$$

since $B_kB_{2k+1} \equiv 0 \pmod{B_{2k+1}}$. Observe that

$$\begin{aligned}
&B_k^2 + k(k+1)]\,(B_{k+2} - B_k) - B_kB_{2k+1} \\
&= \left[B_k^2 + k(k+1)\right](6B_{k+1} - 2B_k) - B_k\left(B_{k+1}^2 - B_k^2\right) \\
&= B_kB_{k+1}(6B_k - B_{k+1}) - B_k^3 + 6k(k+1)B_{k+1} - 2k(k+1)B_k \qquad (6.13)\\
&= B_kB_{k+1}B_{k-1} - B_k^3 + 6k(k+1)B_{k+1} - 2k(k+1)B_k \\
&= 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k\,,
\end{aligned}$$

where we used Lemma 2 (i) to get that $B_{2k+1} = B_{k+1}^2 - B_k^2$, $B_{k+1}B_{k-1} = B_k^2 - 1$. Then, (6.12) and (6.13) imply that

$$32B_j + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k \equiv 0 \pmod{B_{2k+1}}, \tag{6.14}$$

- Suppose that $1 \le j \le 2k-2$. We will prove that

$$0 < 32B_j + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k < B_{2k+1}\,.$$

We have

$$\begin{aligned}
&32B_j + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k \\
&\quad \ge 32 + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k > 0\,. \qquad (6.15)
\end{aligned}$$

On the other hand, since $j \le 2k-2$, we have

$$\begin{aligned}
&B_{2k+1} - 32B_j - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k \\
&\ge B_{2k+1} - 32B_{2k-2} - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k \\
&= B_{2k+1} - 32\frac{B_{2k-1} + B_{2k-3}}{6} - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k \\
&= B_{k+1}^2 - B_k^2 - \frac{16}{3}(B_k^2 - B_{k-1}^2 + B_{k-1}^2 - B_{k-2}^2) - 6k(k+1)B_{k+1} \\
&\quad + [2k(k+1)+1]\,B_k \\
&\ge B_{k+1}^2 - B_k^2 - 6(B_k^2 - B_{k-2}^2) - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k \qquad (6.16)\\
&= B_k\,[36B_k - 6B_{k-1} - 36k(k+1)] - B_{k-1}\,[6B_k - B_{k-1} - 6k(k+1)] \\
&\quad - B_k\,[7B_k - 2k(k+1) - 1] + 6B_{k-2}^2 \\
&> B_k\,[29B_k - 6B_{k-1} - 34k(k+1) + 1] \\
&\quad - B_{k-1}\,[6B_k - B_{k-1} - 6k(k+1)]\,.
\end{aligned}$$

Observe that

$$\begin{aligned} &29B_k - 6B_{k-1} - 34k(k+1) + 1 - 6B_k + B_{k-1} + 6k(k+1) \\ &= 23B_k - 5B_{k-1} - 28k(k+1) + 1 > 18B_k - 28k(k+1) + 1 > 0\,, \end{aligned} \tag{6.17}$$

since $18B_k = 9P_{2k} \geq 9\alpha^{2k-2} > 28k(k+1)$, for $k \geq 4$. Now, (6.17) implies that

$$29B_k - 6B_{k-1} - 34k(k+1) + 1 > 6B_k - B_{k-1} - 6k(k+1)\,.$$

Then, one obtains

$$B_k\left[29B_k - 6B_{k-1} - 34k(k+1) + 1\right] - B_{k-1}\left[6B_k - B_{k-1} - 6k(k+1)\right] > 0\,. \tag{6.18}$$

Inequalities (6.16) and (6.18) finally imply that

$$32B_j + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k < B_{2k+1}\,. \tag{6.19}$$

From (6.15) and (6.19), we deduce that

$$0 < 32B_j + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k < B_{2k+1}\,,$$

which contradicts (6.14). Then, $j \in \{2k-1, 2k\}$.

- Suppose that $j = 2k-1$. Then, (6.14) becomes

$$32B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k \equiv 0 \pmod{B_{2k+1}}\,. \tag{6.20}$$

Using the recurrence formula, we check that $32B_{2k-1} = B_{2k+1} + B_{2k-3} - 2B_{2k-1}$. Then, (6.20) becomes

$$B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k \equiv 0 \pmod{B_{2k+1}}\,. \tag{6.21}$$

We will prove that

$$0 < B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k < B_{2k+1}\,.$$

It is obvious that

$$B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k > 0\,. \tag{6.22}$$

Furthermore, one sees that

$$\begin{aligned} &B_{2k+1} - B_{2k+1} - B_{2k-3} + 2B_{2k-1} - 6k(k+1)B_{k+1} + \left[2k(k+1) + 1\right]B_k \\ &\quad = 2B_k^2 - 3B_{k-1}^2 + B_{k-2}^2 - 36k(k+1)B_k + 6k(k+1)B_{k-1} + \left[2k(k+1) + 1\right]B_k \\ &\quad > B_k^2 + B_{k-2}^2 - 36k(k+1)B_k + 6k(k+1)B_{k-1} + \left[2k(k+1) + 1\right]B_k \\ &\quad = B_k\left[B_k - 34k(k+1) + 1\right] + B_{k-2}^2 + 6k(k+1)B_{k-1} > 0\,, \end{aligned}$$

where we used Lemma 2 (i), the fact that $B_k^2 - 3B_{k-1}^2 > 0$ and at the end the fact that $B_k = \frac{P_{2k}}{2} \geq \frac{1}{2}\alpha^{2k-2} > 34k(k+1) - 1$, for $k \geq 6$, which is the case for us. The above inequality implies that

$$B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k < B_{2k+1}\,. \tag{6.23}$$

From inequalities (6.22) and (6.23), we have

$$0 < B_{2k+1} + B_{2k-3} - 2B_{2k-1} + 6k(k+1)B_{k+1} - \left[2k(k+1) + 1\right]B_k < B_{2k+1}\,,$$

which contradicts (6.21).

• Suppose that $j = 2k$. Then, (6.14) becomes

$$32B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k \equiv 0 \pmod{B_{2k+1}}. \tag{6.24}$$

Observe that

$$32B_{2k} = 5B_{2k+1} + 5B_{2k-1} + 2B_{2k} \equiv 5B_{2k-1} + 2B_{2k} \pmod{B_{2k+1}}.$$

Then, (6.24) implies

$$5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k \equiv 0 \pmod{B_{2k+1}}. \tag{6.25}$$

We prove that $0 < 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k < B_{2k+1}$. It is obvious that

$$5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k > 0. \tag{6.26}$$

On the other hand, using Lemma 2 (i), we get that

$$\begin{aligned} &B_{2k+1} - 5B_{2k-1} - 2B_{2k} - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k \\ &\quad = B_{2k+1} - 5B_{2k-1} - 2\frac{B_{2k+1}+B_{2k-1}}{6} \\ &\quad = \frac{1}{3}\left[72B_k^2 - 24B_kB_{k-1} + 2B_{k-1}^2 - 18B_k^2 + 16B_{k-1}^2\right] \\ &\qquad + \frac{1}{3}\left[-108k(k+1)B_k + 18k(k+1)B_{k-1} + [6k(k+1)+3]\,B_k\right] \\ &\quad = \frac{1}{3}\left[B_k\left[50B_k - 102k(k+1)+3\right] + B_{k-1}\left[14B_{k-1} + 18k(k+1)\right]\right], \end{aligned} \tag{6.27}$$

where we used Lemma 3 (i) to get $24B_kB_{k-1} = 4B_k^2 + 4B_{k-1}^2 - 4$. Moreover,

$$\begin{aligned} &B_k[50B_k - 102k(k+1)+3] + B_{k-1}\left[14B_{k-1} + 18k(k+1)\right] \\ &\qquad > B_k\left[50B_k - 102k(k+1)+3\right] > 0, \end{aligned} \tag{6.28}$$

since $50B_k = 25P_{2k} \geq 25\alpha^{2k-2} > 102k(k+1)$, for $k \geq 4$. Now, (6.27) and (6.28) imply that $B_{2k+1} - 5B_{2k-1} - 2B_{2k} - 6k(k+1)B_{k+1} + [2k(k+1)+1]\,B_k > 0$, i.e.

$$5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k < B_{2k+1}. \tag{6.29}$$

From inequalities (6.26) and (6.29), we deduce that

$$0 < 5B_{2k-1} + 2B_{2k} + 6k(k+1)B_{k+1} - [2k(k+1)+1]\,B_k < B_{2k+1},$$

which contradicts (6.25). Then, the proof of Theorem 4 is complete.

References

[1] Altassan, A., Luca, F., *On the Diophantine equation* $\sum_{j=1}^{k} jF_j^p = F_n^q$, J. Number Theory **217** (2020), 256–277.

[2] Behera, A., Panda, G.K., *On the square roots of triangular numbers*, Fibonacci Quart. **37** (2) (1999), 98–105.

[3] Catarino, P., Campos, H., Vasco, P., *On some identities for balancing and cobalancing numbers*, Ann. Math. Inform. **45** (2015), 11–24.

[4] Gueth, K., Luca, F., Szalay, L., *Solutions to* $F_1^p + 2F_2^p + \cdots + kF_k^p = F_n^q$ *with small given exponents*, Proc. Japan Acad. Ser. A, Math. Sci. **96** (4) (2020), 33–37.

[5] Horadam, A.F., *Pell identities*, Fibonacci Quart. **9** (3) (1971), 245–263.

[6] Niven, I., Zuckerman, H.S., Montgomery, H.L., *An Introduction to the Theory of Numbers*, fifth edition ed., John Wiley & Sons, Inc., New York, 1991.

[7] Olajos, P., *Properties of Balancing, Cobalancing and Generalized Balancing Numbers*, Ann. Math. Inform. **37** (2010), 125–138.

[8] Panda, G.K., *Some Fascinating Properties of Balancing Numbers*, Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications, Cong. Numer., vol. 194, 2009, pp. 185–189.

[9] Panda, G.K., Ray, P.K., *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bul. Inst. Math. Acad. Sinica **6** (2011), 41–72.

[10] Ray, P.K., *Balancing and cobalancing numbers*, Ph.D. thesis, National Institute of Technology, Rourkela, India, 2009.

[11] Soydan, G., Németh, L., Szalay, L., *On the diophantine equation* $\sum_{j=1}^{k} jF_j^p = F_n^q$, Arch. Math. (Brno) **54** (2008), 177–188.

[12] Tchammou, E., Togbé, A., *On the diophantine equation* $\sum_{j=1}^{k} jP_j^p = P_n^q$, Acta Math. Hungar. **162** (2) (2020), 647–676.

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