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RELATIONSHIP AMONG VARIOUS VIETORIS-TYPE AND MICROSIMPLICIAL HOMOLOGY THEORIES

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Abstract. In this paper, we clarify the relationship among the Vietoris-type homology theories and the microsimplicial homology theories, where the latter are nonstandard homology theories defined by M.C. McCord (for topological spaces), T. Korppi (for completely regular topological spaces) and the author (for uniform spaces). We show that McCord's and our homology are isomorphic for all compact uniform spaces and that Korppi's and our homology are isomorphic for all fine uniform spaces. Our homology shares many good properties with Korppi's homology. As an example, we outline a proof of the continuity of our homology with respect to uniform resolutions. S. Garavaglia proved that McCord's homology is isomorphic to Vietoris homology for all compact topological spaces. Inspired by this result, we prove that our homology is isomorphic to uniform Vietoris homology for all precompact uniform spaces and that Korppi's homology is isomorphic to normal Vietoris homology for all pseudocompact completely regular topological spaces.

1. Introduction

Nonstandard homology theories have been developed for various spaces in the existing literature. M.C. McCord [9] introduced a nonstandard homology of topological spaces, the first investigation of nonstandard homology. By imitating McCord's definition, T. Korppi [5, 7] constructed another nonstandard homology for completely regular spaces, and the author [3] defined a similar one for uniform spaces, called μ-homology. These homology theories are all *microsimplicial*, i.e., based on hyperfinite chains of infinitesimally small (micro) simplices. Other types of nonstandard homology can be found in [3, Section 7].

McCord mentioned a similarity between microsimplicial and Vietoris-type homology theories. Typical examples of Vietoris-type are as follows: Vietoris homology of topological spaces, normal Vietoris homology of completely regular spaces, and uniform Vietoris homology of uniform spaces. *Informally*, Vietoris-type homology is the homology of the Vietoris complex of some "infinitely fine" cover. Formally, Vietoris-type homology is defined as the inverse limit of the homologies of Vietoris complexes, where the limit runs over some directed set of covers with respect to

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$$\begin{tikzcd}[column sep=large]
 & \text{Vietoris-type} & \text{microsimplicial} \\
\text{topological} & \text{Vietoris} \arrow[r, equal, "\text{compact}"] \arrow[d, equal, "\text{paracompact}"'] & \text{McCord} \arrow[d, equal, "\text{compact}"] \\
\text{completely regular} & \text{normal Vietoris} \arrow[r, equal, "\text{pseudocompact}"] \arrow[d, equal, "\text{fine}"'] & \text{Korppi} \arrow[d, equal, "\text{fine}"] \\
\text{uniform} & \text{uniform Vietoris} \arrow[r, equal, "\text{precompact}"] & \mu
\end{tikzcd}$$

Fig. 1.1: Double lines indicate isomorphisms. The label of each line indicates the condition for that isomorphism.

refinement. Note that Vietoris' construction indirectly deals with infinitely fine covers, just like epsilon–delta analysis does with infinitesimals. There is another way to formulate the informal concept "infinitely fine", namely, the use of nonstandard analysis. Nonstandard analysis enables us directly to deal with infinitely fine covers. Microsimplicial homology is *formally* defined as the homology of the Vietoris complex of some infinitely fine cover in the nonstandard sense. Under these circumstances, it is natural to expect that each microsimplicial homology is isomorphic to the corresponding Vietoris-type homology. Indeed, Garavaglia [2] proved that McCord homology is isomorphic to Vietoris homology for all compact spaces. The equivalence, however, does not hold for some non-compact spaces. This means that McCord homology is *not* just a paraphrase of Vietoris homology. Vietoris-type homology theories are inexact even for compact metrisable pairs (see [8, Example 2, p.126]), because they are defined by inverse limits which do not preserve exact sequences. By contrast, microsimplicial homology theories are exact for all pairs of spaces. This is an advantage of microsimplicial-type compared with Vietoris-type.

This paper aims to clarify the relationship among Vietoris-type and microsimplicial homology theories. Figure 1.1 on page 132 illustrates the relationship among the homology theories mentioned so far.

In Section 3, we deal with the microsimplicial homology theories. We show that

- McCord homology and μ-homology are exactly the same for all compact uniform spaces;
- Korppi homology and μ-homology are exactly the same for all fine uniform spaces.

We remark that these equalities do not hold in general. Because of the second equality, we can regard μ-homology as a generalisation of Korppi homology from fine uniform spaces to general uniform spaces. μ-homology inherits many properties from Korppi homology. For example, uniform Vietoris homology with standard coefficients can be embedded into μ-homology with nonstandard coefficients. This is a generalisation of the fact that normal Vietoris homology can be embedded into Korppi homology. As another example, we outline a proof of the continuity of μ-homology with respect to uniform resolutions.

In Section 4, we prove the following two isomorphisms (inspired by Garavaglia's result):

- μ-homology is isomorphic to uniform Vietoris homology for all precompact uniform spaces;
- Korppi homology is isomorphic to normal Vietoris homology for all pseudo-compact completely regular spaces.

To prove the former isomorphism, we introduce the notion of S-homotopy, the singular homotopy on the category of uniform spaces with S-continuous maps (which admit infinitesimal discontinuity), and show that uniform Vietoris homology satisfies the S-homotopy axiom for all precompact uniform spaces. The latter isomorphism immediately follows from the former one. We also prove that these isomorphisms do not hold in general.

μ-homology can be extended to the category of nonstandard subsets of uniform spaces with nonstandardly continuous maps. We briefly discuss this extension in Section 5.

Finally, in Section 6, we remark the necessity and unnecessity of the saturation principle. This remark will be important for considering the microsimplicial homology theories in general nonstandard models.

2. Preliminaries

We use the basic terminology in uniform topology (see the monograph [4] by Isbell). We assume the reader to be familiar with nonstandard mathematics, in particular, nonstandard topology. For basic notions and results in nonstandard topology, we refer to Robinson [10] and Stroyan and Luxemburg [11].

2.1. **Basic settings of nonstandard analysis.** As in [3], we use Robinson-style nonstandard analysis. We fix a transitive universe $\mathbb{U}$, called the standard universe, that satisfies sufficiently many (but only finitely many) axioms of ZFC. All standard objects we consider belong to $\mathbb{U}$. More specifically, using the reflection principle, we pick a large enough ordinal λ and let $\mathbb{U} := V_\lambda$, where V_λ is the cumulative hierarchy of rank λ. We also fix an elementary extension ${}^*\mathbb{U}$ of $\mathbb{U}$, called the internal universe, that is $|\mathbb{U}|^+$-saturated. In particular, ${}^*\mathbb{U}$ is an enlargement of $\mathbb{U}$. By saying the words 'transfer', 'saturation' and 'weak saturation', we indicate the elementary equivalence between $\mathbb{U}$ and ${}^*\mathbb{U}$, the saturation property of ${}^*\mathbb{U}$ and the enlargement property of ${}^*\mathbb{U}$ relative to $\mathbb{U}$, respectively. The map $x \mapsto {}^*x$ denotes the elementary embedding $\mathbb{U} \hookrightarrow {}^*\mathbb{U}$. We omit the star of ${}^*x$ when x is considered to be an atomic object (such as a number and a point). Given a concept X on $\mathbb{U}$ definable by a first-order formula with parameters in $\mathbb{U}$, the associated concept on ${}^*\mathbb{U}$ definable by the same formula is called *internal* X, *hyper* X, ${}^*X$, etc.

2.2. **Vietoris-type homology theories.** We here recall the definition schema of Vietoris-type homology theories [1].

Let X and Y be sets and let R be a subset of $X \times Y$. The *Vietoris complex* of (X, Y, R) is the simplicial set $\mathcal{V}(X, Y, R)$ whose vertices are the points of X and whose vertices $a_0, \dots, a_p$ span a simplex if and only if there exists a $b \in Y$ such that $a_i \; R \; b \;\; (0 \le i \le p)$. Figure 2.1 on page 134 gives an example of a Vietoris complex.

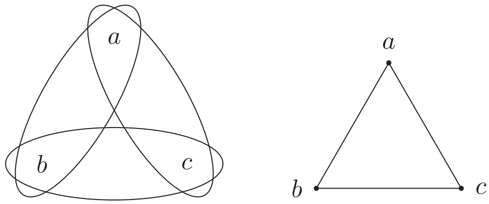


Fig. 2.1: The left depicts the three point space $X := \{ a, b, c \}$ equipped with the cover $\lambda := \{ \{ a, b \}, \{ a, c \}, \{ b, c \} \}$. The right depicts a geometric realisation of $\mathcal{V}(X, \lambda, \in)$ within the plane.

Let X be a set and let λ and μ be covers of X. λ is called a *refinement* of μ if there is a map $\varphi \colon \lambda \to \mu$ such that $U \subseteq \varphi(U)$ for all $U \in \lambda$. The cover $\lambda \wedge \mu := \{ U \cap V \mid U \in \lambda, V \in \mu \}$ is a common refinement of λ and μ. Hence, the set of all covers forms a (downward) directed set under refinement.

Let G be an abelian group. Let (X, A) be a pair of sets (such that $A \subseteq X$) and $\mathcal{D}_X$ a directed set of covers of X. Given $\lambda \in \mathcal{D}_X$, we denote by (X_λ, A_λ) the simplicial pair $(\mathcal{V}(X, \lambda, \in), \mathcal{V}(A, \lambda, \in))$. If λ is a refinement of μ, X_λ and A_λ are simplicial subsets of X_μ and A_μ, respectively. Let $i_{\mu\lambda} \colon (X_\lambda, A_\lambda) \hookrightarrow (X_\mu, A_\mu)$ be the inclusion and let $p_{\mu\lambda} := H_\bullet(i_{\mu\lambda}; G)$. Thus we have an inverse system $(\mathcal{D}_X, H_\bullet(X_\lambda, A_\lambda; G), p_{\mu\lambda})$, called the *Vietoris system* for (X, A). Here $H_\bullet(\cdot; G)$ denotes the ordinary homology functor of simplicial pairs with coefficients in G. The *Vietoris homology* of (X, A) with coefficients in G with respect to $\mathcal{D}_X$ is the inverse limit

$$\check{H}_\bullet^{\mathcal{D}_X}(X, A; G) := \lim_{\lambda \in \mathcal{D}_X} H_\bullet(X_\lambda, A_\lambda; G) \; .$$

Example 2.1. Let $\mathbf{Top}_2$ be the category of topological pairs with continuous maps. We denote by $\mathcal{O}_X$ the directed set of all open covers of a topological space X. The *Vietoris homology* of a topological pair (X, A) is $\check{H}_\bullet(X, A; G) := \check{H}_\bullet^{\mathcal{O}_X}(X, A; G)$. One can extend $\check{H}_\bullet(\cdot; G)$ to a functor from $\mathbf{Top}_2$ to the category of graded abelian groups as follows: let $f \colon (X, A) \to (Y, B)$ be a continuous map. Given $\lambda \in \mathcal{O}_Y$, its pullback $f^{-1}\lambda := \{ f^{-1}(U) \mid U \in \lambda \}$ is in $\mathcal{O}_X$. Hence, f canonically induces a homomorphism $f_\lambda \colon \check{H}_\bullet(X, A; G) \to H_\bullet(Y_\lambda, B_\lambda; G)$. If λ is a refinement of μ, then the diagram

$$\begin{tikzcd} & \check{H}_\bullet(X,A;G) \arrow[dl, "f_\lambda"'] \arrow[dr, "f_\mu"] & \\ H_\bullet(Y_\lambda,B_\lambda;G) \arrow[rr, "p_{\mu\lambda}"] & & H_\bullet(Y_\mu,B_\mu;G) \end{tikzcd}$$

commutes. By the universal property of $\check{H}_\bullet(Y,B;G)$, there exists a unique homomorphism $\check{H}_\bullet(f;G) : \check{H}_\bullet(X,A;G) \to \check{H}_\bullet(Y,B;G)$ that makes the diagram

$$\begin{tikzcd} \check{H}_\bullet(X,A;G) \arrow[rr, "\check{H}_\bullet(f;G)"] \arrow[dr, "f_\lambda"'] & & \check{H}_\bullet(Y,B;G) \arrow[dl, "p_\lambda"] \\ & H_\bullet(Y_\lambda,B_\lambda;G) & \end{tikzcd}$$

commutative for every $\lambda \in \mathcal{O}_Y$, where p_λ is the canonical projection. It is easy to verify that $\check{H}_\bullet(\cdot;G)$ is functorial.

Example 2.2. Let $\mathbf{CR}_2$ be the category of completely regular pairs with continuous maps. We denote by $\mathcal{N}_X$ the directed set of all normal covers of a completely regular space X. The *normal Vietoris homology* of a completely regular pair (X,A) is $\check{H}^n_\bullet(X,A;G) := \check{H}^{\mathcal{N}_X}_\bullet(X,A;G)$. Like above, one can extend $\check{H}_\bullet(\cdot;G)$ to a functor on $\mathbf{CR}_2$.

Example 2.3. Let $\mathbf{Unif}_2$ be the category of uniform pairs with uniformly continuous maps. We denote by $\mathcal{U}_X$ the directed set of all uniform covers of a uniform space X. The *uniform Vietoris homology* of a uniform pair (X,A) is $\check{H}^u_\bullet(X,A;G) := \check{H}^{\mathcal{U}_X}_\bullet(X,A;G)$. One can extend $\check{H}_\bullet(\cdot;G)$ to a functor on $\mathbf{Unif}_2$ as before.

Remark 2.4. Uniform Vietoris homology has another definition in terms of entourages instead of uniform covers. Let (X,A) be a uniform pair. Let $\mathcal{E}_X$ be the set of all entourages of X. $\mathcal{E}_X$ forms a (downward) directed set under inclusion. Given $U \in \mathcal{E}_X$, we denote by (X_U,A_U) the simplicial pair $(\mathcal{V}(X,X,U), \mathcal{V}(A,X,U \cap (A \times X)))$. If $U \subseteq V$, X_U and A_U are simplicial subsets of X_V and A_V, respectively. Let $i_{VU}\colon (X_U,A_U) \hookrightarrow (X_V,A_V)$ be the inclusion and let $p_{VU} := H_\bullet(i_{VU};G)$. Thus we have an inverse system $(\mathcal{E}_X, H_\bullet(X_U,A_U;G), p_{VU})$. The uniform Vietoris homology of (X,Y) can be redefined as the inverse limit

$$\check{H}^u_\bullet(X,A;G) := \lim_{U\in\mathcal{E}_X} H_\bullet(X_U,A_U;G) .$$

2.3. **Microsimplicial homology theories.** We first recall the definition of McCord homology [9]. McCord homology is a homology theory on $\mathbf{Top}_2$ (or, more rigorously, on its small full subcategory $\mathbf{Top}_2 \cap \mathbb{U}$). Let X be a standard topological space and G an internal abelian group. The *monad* of $x \in X$ is the set

$$\mu_X(x) := \bigcap \{ {}^*U \mid x \in U\colon \text{open} \} .$$

We say that a member $(a_0, \ldots, a_p)$ of ${}^*X^{p+1}$ is a *p-microsimplex* if $\{ a_0, \ldots, a_p \} \subseteq \mu_X(x)$ holds for some $x \in X$. We denote by $C_p^M(X; G)$ the G-module that consists of all internal hyperfinite formal sums of p-microsimplices with coefficients in G. The boundary map $\partial_p \colon C_p^M(X; G) \to C_{p-1}^M(X; G)$ is defined by

$$\partial_p(a_0, \ldots, a_p) := \sum_{i=0}^{p} (-1)^i (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p) \ .$$

It is obvious that $C_\bullet^M(X; G)$ is a chain complex, called the *McCord microchain complex*. Every standard continuous map $f \colon X \to Y$ functorially induces a homomorphism $C_\bullet^M(f; G) : C_\bullet^M(X; G) \to C_\bullet^M(Y; G)$ defined by letting

$$C_p^M(f; G)(a_0, \ldots, a_p) := ({}^*f(a_0), \ldots, {}^*f(a_p)) \ .$$

Let (X, A) be a standard topological pair. $C_\bullet^M(A; G)$ is a subchain complex of $C_\bullet^M(X; G)$. We define

$$C_\bullet^M(X, A; G) := \frac{C_\bullet^M(X; G)}{C_\bullet^M(A; G)} \,.$$

Let $f \colon (X, A) \to (Y, B)$ be a standard continuous map. Since $C_\bullet^M(f; G) : C_\bullet^M(X; G) \to C_\bullet^M(Y; G)$ maps $C_\bullet^M(A; G)$ to $C_\bullet^M(B; G)$, f functorially induces a homomorphism $C_\bullet^M(f; G) : C_\bullet^M(X, A; G) \to C_\bullet^M(Y, B; G)$. We define $H_\bullet^M(\cdot, \cdot; G) := H_\bullet C_\bullet^M(\cdot, \cdot; G)$, where $H_\bullet$ is the ordinary homology functor of chain complexes. The resulting functor is called *McCord homology*.

Remark 2.5. Let (X, A) be a standard topological pair. It may happen that

$$C_p^M(A; G) \neq \left\{ \sum_i g_i \sigma_i \in C_p^M(X; G) \mid \sigma_i \subseteq {}^*A \text{ for all } i \right\}.$$

For instance, consider the topological pair $(\mathbb{R}, \mathbb{R}_+)$. Given positive infinitesimals $a_0, \ldots, a_p \in {}^*\mathbb{R}_+$, the p-tuple $(a_0, \ldots, a_p)$ is a microsimplex of $\mathbb{R}$, although not a microsimplex of $\mathbb{R}_+$. Because of this, the short exact sequence

$$0 \longrightarrow C_\bullet^M(A; G) \longrightarrow C_\bullet^M(X; G) \longrightarrow C_\bullet^M(X, A; G) \longrightarrow 0$$

may not split. However, if A is closed in X, the above two sets are equal, and the short exact sequence splits.

Next, recall the definition of Korppi homology [5, 7]. Korppi homology is a homology theory on $\mathbf{CR}_2$ (or, more precisely, on $\mathbf{CR}_2 \cap \mathbb{U}$). Korppi's definition is similar to McCord's. The only difference lies in the definition of the term 'microsimplex'. Let X be a standard completely regular space and G an internal abelian group. Two points $x, y \in {}^*X$ are said to be *infinitely close* (denoted by $x \sim_X y$) if for each normal cover $\lambda \in \mathcal{N}_X$, x and y are ${}^*\lambda$-close, i.e., $\{ x, y \} \subseteq U$ for some $U \in {}^*\lambda$. The *normal monad* of $x \in {}^*X$ is the set

$$\mu_X^n(x) := \{ y \in {}^*X \mid x \sim_X y \} = \bigcap_{\lambda \in \mathcal{N}_X} \mathrm{St}(x, {}^*\lambda) \ ,$$

where $\mathrm{St}\,(-)$ denotes the star-neighbourhood. For each $x \in X$, the normal monad of x is equal to the monad of x [7, Lemma 6-(2)], while the normal monad makes sense for $x \in {}^*X \setminus X$. We say that a member $(a_0, \dots, a_p)$ of ${}^*X^{p+1}$ is a *p-microsimplex* if $\{ a_0, \dots, a_p \} \subseteq \mu^n_X(x)$ holds for some $x \in {}^*X$, or equivalently, if $a_i \sim_X a_j$ for all $0 \leq i, j \leq p$. The rest of the definition is the same as McCord's one. We denote by $C^K_\bullet$ the Korppi microchain complex functor and by $H^K_\bullet$ the Korppi homology functor.

Remark 2.6. Let (X, A) be a standard completely regular pair. It is possible that $x \sim_X y$ but $x \nsim_A y$ for some $x, y \in {}^*A$ (see [7, Remark 8]). The short exact sequence

$$0 \longrightarrow C^K_\bullet(A; G) \longrightarrow C^K_\bullet(X; G) \longrightarrow C^K_\bullet(X, A; G) \longrightarrow 0$$

may not split. We say that A is *normally embedded* in X if for every normal cover μ of A, there exists a normal cover λ of X such that $\{ U \cap A \mid U \in \lambda \}$ refines μ. If A is normally embedded in X, then $\sim_A$ agrees with $\sim_X$ on ${}^*A$ [7, Lemma 9], and the above sequence splits.

Finally, we recall the definition of μ-homology [3]. μ-homology is a homology theory on $\mathbf{Unif}_2$ (or, more precisely, on $\mathbf{Unif}_2 \cap \mathbb{U}$). The definition of μ-homology is similar to that of Korppi homology. Let X be a standard uniform space and G an internal abelian group. Two points $x, y \in {}^*X$ are said to be *infinitely close* (denoted by $x \approx_X y$) if for each uniform cover $\lambda \in \mathcal{U}_X$, x and y are ${}^*\lambda$-close, or equivalently, if $x \; {}^*U \; y$ holds for any entourage $U \in \mathcal{E}_X$. The *uniform monad* of $x \in {}^*X$ is the set

$$\begin{aligned} \mu^u_X(x) &:= \{ y \in {}^*X \mid x \approx_X y \} \\ &= \bigcap_{\lambda \in \mathcal{U}_X} \mathrm{St}\,(x, {}^*\lambda) \\ &= \bigcap_{U \in \mathcal{E}_X} {}^*U[x], \end{aligned}$$

where $U[x] := \{ y \in X \mid (x, y) \in U \}$. (We will also use the notation $U[A] := \{ y \in X \mid \exists x \in A. (x, y) \in U \}$ throughout.) The uniform monad of x coincides with the monad of x for $x \in X$. We say that a member $(a_0, \dots, a_p)$ of ${}^*X^{p+1}$ is a *p-microsimplex* if $\{ a_0, \dots, a_p \} \subseteq \mu^u_X(x)$ holds for some $x \in {}^*X$, or equivalently, if $a_i \approx_X a_j$ for all $0 \leq i, j \leq p$. The rest of the definition is the same as McCord's and Korppi's. We denote by $C^\mu_\bullet$ the μ-microchain complex functor and by $H^\mu_\bullet$ the μ-homology functor.

Remark 2.7. In contrast to the previous case (Remark 2.6), $\approx_A$ agrees with $\approx_X$ on ${}^*A$ for each standard uniform pair (X, A). Hence the short exact sequence

$$0 \longrightarrow C^\mu_\bullet(A; G) \longrightarrow C^\mu_\bullet(X; G) \longrightarrow C^\mu_\bullet(X, A; G) \longrightarrow 0$$

always splits (see [3, Proposition 2]).

The excision property of μ-homology is not proved in the preceding paper [3], whilst a weak form of the excision is (see [3, Proposition 3]). In the rest of this section, we prove the *full* excision property of μ-homology.

Definition 2.8. Let X be a uniform space. Let A and B be subsets of X. We say that A is *strongly contained in* B (and we write $A \Subset B$) if $\mathrm{St}(A, \lambda) \subseteq B$ for some $\lambda \in \mathcal{U}_X$.

Lemma 2.9. *Let X be a standard uniform space. Let A and B be subsets of X. Then $A \Subset B$ if and only if $\mu^u_X({}^*A) \subseteq {}^*B$, where $\mu^u_X({}^*A) := \bigcup_{a \in {}^*A} \mu^u_X(a)$.*

Proof. Suppose that $\mu^u_X({}^*A) \subseteq {}^*B$. By weak saturation, we can find an $\lambda \in {}^*\mathcal{U}_X$ such that λ refines ${}^*\nu$ for all $\nu \in \mathcal{U}_X$. Then $\mathrm{St}({}^*A, \lambda) \subseteq \mu^u_X({}^*A) \subseteq {}^*B$. By transfer, we see that $A \Subset B$. Conversely, suppose $A \Subset B$. Let $\lambda \in \mathcal{U}_X$ be with $\mathrm{St}(A, \lambda) \subseteq B$. Then, by transfer, we have that $\mu^u_X({}^*A) \subseteq {}^*(\mathrm{St}(A, \lambda)) \subseteq {}^*B$. $\square$

Proposition 2.10. *Let X be a standard uniform space. Let A and B be subsets of X. If $X \setminus A \Subset B$ (or $X \setminus B \Subset A$), then the inclusion map $i\colon (A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $H^\mu_\bullet(i; G) : H^\mu_\bullet(A, A \cap B; G) \cong H^\mu_\bullet(X, B; G)$.*

Proof. The proof is a slight modification of the proof of [3, Proposition 3]. It suffices to show that the following two inclusions hold:

(1) $C^\mu_p(A; G) \cap C^\mu_p(B; G) \subseteq C^\mu_p(A \cap B; G)$,

(2) $C^\mu_p(X; G) \subseteq C^\mu_p(A; G) + C^\mu_p(B; G)$.

The first inclusion is trivial. We only need to prove that each microsimplex σ of X is contained in either ${}^*A$ or ${}^*B$. Suppose that σ is not contained in ${}^*A$. Therefore, σ intersects ${}^*(X \setminus A)$. All the vertices of σ are in $\mu^u_X\left({}^*(X \setminus A)\right)$. By Lemma 2.9, σ is contained in ${}^*B$.

$\square$

Corollary 2.11. *Let X be a standard uniform space. Let A and U be subsets of X. If $U \Subset A$, then the inclusion map $i\colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism $H^\mu_\bullet(i; G) : H^\mu_\bullet(X \setminus U, A \setminus U; G) \cong H^\mu_\bullet(X, A; G)$.*

Proof. By applying Proposition 2.10 to the triad $(X, X \setminus U, A)$, we obtain the desired result. $\square$

3. Relationship among the microsimplicial homology theories

In the first half of this section, we discuss the relationship among the microsimplicial homology theories. In Subsection 3.1, we prove the equality of McCord homology and μ-homology on the category of compact topological spaces. In Subsection 3.2, we prove the equality of Korppi homology and μ-homology on the category of fine uniform spaces. μ-homology inherits many properties from Korppi homology. In the rest of this section, we present two examples of such properties, the embeddability of the Vietoris-type homology (Subsection 3.3) and the continuity with respect to resolutions (Subsection 3.4).

3.1. **McCord homology and μ-homology.** In [3], the author stated (without detailed proof) that μ-homology and McCord homology are completely the same for all compact uniform spaces. Here we give a detailed proof.

Let $U\colon \mathbf{Unif} \to \mathbf{CR}$ be the forgetful (topologisation) functor, where **Unif** is the category of uniform spaces, and **CR** is the category of completely regular spaces.

Lemma 3.1. *Let G be an internal abelian group. Let X be a standard uniform space. Then $C^{M}_{\bullet}(UX;G) \subseteq C^{\mu}_{\bullet}(X;G)$. If X is compact, $C^{\mu}_{\bullet}(X;G) \subseteq C^{M}_{\bullet}(UX;G)$.*

Proof. Let $(a_0,\dots,a_p) \in {}^*X^{p+1}$ be a McCord microsimplex. Choose an $x \in X$ such that $\{a_0,\dots,a_p\} \subseteq \mu_{UX}(x)$. Since $\mu_{UX}(x) = \mu^{u}_{X}(x)$, $(a_0,\dots,a_p)$ is a μ-microsimplex. It follows that $C^{M}_{\bullet}(UX;G) \subseteq C^{\mu}_{\bullet}(X;G)$.

Assume that X is compact. Let $(a_0,\dots,a_p) \in {}^*X^{p+1}$ be a μ-microsimplex. By the nonstandard characterisation of compactness [10, Theorem 4.1.13], there exists an $x \in X$ such that $a_0 \in \mu_{UX}(x) = \mu^{u}_{X}(x)$. Since $a_i \approx_X a_0 \approx_X x$, it follows that $a_i \in \mu_{UX}(x)$ for all $0 \le i \le p$. Therefore, $(a_0,\dots,a_p)$ is a McCord microsimplex. Hence $C^{\mu}_{\bullet}(X;G) \subseteq C^{M}_{\bullet}(UX;G)$. □

Proposition 3.2. *Let G be an internal abelian group. Let (X,A) be a standard compact uniform pair. Then, we have that $C^{M}_{\bullet}(UX,A;G) = C^{\mu}_{\bullet}(X,A;G)$ and $H^{M}_{\bullet}(UX,A;G) = H^{\mu}_{\bullet}(X,A;G)$.*

This equality does not hold for some non-compact uniform spaces (see [3, Example 18]).

3.2. **Korppi homology and μ-homology.** Let $F\colon \mathbf{CR} \to \mathbf{Unif}$ be the left adjoint functor of $U\colon \mathbf{Unif} \to \mathbf{CR}$. More specifically, given a completely regular space X, FX is the uniform space whose underlying set is the same as X and whose uniformity is the finest uniformity compatible with the topology of X.

Lemma 3.3. *Let X be a standard completely regular space. For any $x,y \in {}^*X$, $x \sim_X y$ if and only if $x \approx_{FX} y$. Hence $\mu^{n}_{X} = \mu^{u}_{FX}$.*

Proof. Immediately from the fact that every uniform (open) cover of FX is a normal cover of X, and vice versa (see [4, Theorem 20]). □

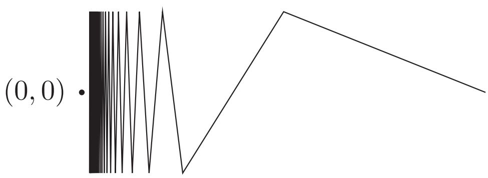
The definition of μ-homology is very similar to that of Korppi homology. Recall that the only difference lies in the definition of the term 'infinitely close to'. The following equivalence is now obvious.

Proposition 3.4. *Let G be an internal abelian group. Let (X,A) be a standard completely regular pair such that A is normally embedded in X. Then, we have that $C^{K}_{\bullet}(X,A;G) = C^{\mu}_{\bullet}(FX,A;G)$ and $H^{K}_{\bullet}(X,A;G) = H^{\mu}_{\bullet}(FX,A;G)$.*

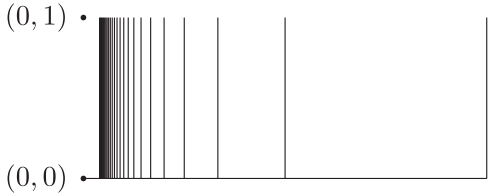
Corollary 3.5 ([7, Remark 46]). *Let G be an internal abelian group. Let (X,A) be a standard compact completely regular pair. Then, we have that $C^{K}_{\bullet}(X,A;G) = C^{M}_{\bullet}(X,A;G)$ and $H^{K}_{\bullet}(X,A;G) = H^{M}_{\bullet}(X,A;G)$.*

Because of Proposition 3.4, μ-homology is a generalisation of Korppi homology from fine uniform spaces to general uniform spaces. μ-homology inherits many properties from Korppi homology (see the next two subsections).

Remark 3.6. The equality between Korppi and μ-homology does not hold for some non-fine uniform spaces (e.g. the real line without the origin equipped with the usual uniformity). In addition, the equality between McCord and Korppi homology does not hold for some non-compact completely regular spaces. The topologist's sine curve and the deleted comb space (Figure 3.1 on page 140) give counterexamples.



(A) The topologist's sine curve



(B) The deleted comb space

FIG. 3.1:

3.3. **Natural embeddings of Vietoris-type into microsimplicial homology.** Korppi proved that Vietoris homology with standard coefficients can be embedded into Korppi homology with nonstandard coefficients for all paracompact spaces [7, Theorem 76]. Actually, Korppi homology is better related with *normal* Vietoris homology than with Vietoris homology. For example, normal Vietoris homology can be embedded into Korppi homology without any extra condition.

Theorem 3.7. *Let G be a standard abelian group. Let (X, A) be a standard completely regular pair such that A is normally embedded in X. Then there exists a monomorphism $\check{H}^n_\bullet(X, A; G) \to H^K_\bullet(X, A; {}^*G)$ natural in (X, A).*

Proof. Let $C_\bullet$ denote the ordinary chain complex functor of simplicial pairs. For each $\lambda \in \mathcal{N}_X$, let $i_\lambda \colon H_\bullet(X_\lambda, A_\lambda; G) \to {}^*(H_\bullet(X_\lambda, A_\lambda; G))$ be the elementary embedding. These maps induce a limiting map
$i_{X,A} := \lim_{\lambda\in\mathcal{N}_X} i_\lambda \colon \lim_{\lambda\in\mathcal{N}_X} H_\bullet(X_\lambda, A_\lambda; G) \to \lim_{\lambda\in\mathcal{N}_X} {}^*(H_\bullet(X_\lambda, A_\lambda; G))$. It is easy to see that $i_{X,A}$ is a monomorphism natural in (X, A). The domain $\lim_{\lambda\in\mathcal{N}_X} H_\bullet(X_\lambda, A_\lambda; G)$ is precisely the same as $\check{H}^n_\bullet(X, A; G)$. By [7, Theorem

48], the codomain is

$$\begin{aligned}
\lim_{\lambda \in \mathcal{N}_X} {}^*(H_\bullet(X_\lambda, A_\lambda; G)) &= \lim_{\lambda \in \mathcal{N}_X} {}^*(H_\bullet(C_\bullet(X_\lambda, A_\lambda; G))) \\
&= H_\bullet \lim_{\lambda \in \mathcal{N}_X} {}^*(C_\bullet(X_\lambda, A_\lambda; G)) \\
&= H_\bullet \bigcap_{\lambda \in \mathcal{N}_X} {}^*(C_\bullet(X_\lambda, A_\lambda; G)) \\
&= H_\bullet C^K_\bullet(X, A; {}^*G) \\
&= H^K_\bullet(X, A; {}^*G) \ .
\end{aligned}$$

Hence $i_{X,A}$ is a desired natural monomorphism $\check{H}^n_\bullet(X, A; G) \to H^K_\bullet(X, A; {}^*G)$. $\square$

One can prove the following uniform analogue in the same way.

Theorem 3.8. *Let G be a standard abelian group. Let (X, A) be a standard uniform pair. Then there exists a monomorphism $\check{H}^u_\bullet(X, A; G) \to H^\mu_\bullet(X, A; {}^*G)$ natural in (X, A).*

3.4. **Continuity of μ-homology with respect to uniform resolutions.** Korppi homology is continuous with respect to resolutions ([7, Theorem 71]). Analogously, μ-homology is continuous with respect to uniform resolutions.

Definition 3.9. Let $\boldsymbol{X} := (I, X_i, \pi_{ij})$ be an inverse system of uniform spaces and let $\pi\colon X \to \boldsymbol{X}$ be a cone over $\boldsymbol{X}$. Then $\boldsymbol{X}$ (together with π) is called a *uniform resolution* of X if the following conditions hold:

- **(UR1):** for each $\lambda \in \mathcal{U}_X$ there exist an $i \in I$ and a $\mu \in \mathcal{U}_{X_i}$ such that $\pi_i^{-1}\mu$ refines λ;
- **(UR2):** for each $i \in I$ and each $\lambda \in \mathcal{U}_{X_i}$ there exists an $j \in I$ such that $\pi_{ij}(X_j) \subseteq \mathrm{St}(\pi_i(X), \lambda)$.

Let $(\boldsymbol{X}, \boldsymbol{A}) := (I, X_i, A_i, \pi_{ij})$ be an inverse system of uniform pairs and let $\pi\colon (X, A) \to (\boldsymbol{X}, \boldsymbol{A})$ be a cone over $(\boldsymbol{X}, \boldsymbol{A})$. Then $(\boldsymbol{X}, \boldsymbol{A})$ is called a uniform resolution of (X, A) if (I, X_i, π_{ij}) and (I, A_i, π_{ij}) are uniform resolutions of X and A, respectively.

Theorem 3.10. *Let $(\boldsymbol{X}, \boldsymbol{A}) := (I, X_i, A_i, \pi_{ij})$ be a standard inverse system of uniform pairs. Let $\pi\colon (X, A) \to (\boldsymbol{X}, \boldsymbol{A})$ be a standard cone over $(\boldsymbol{X}, \boldsymbol{A})$. If $(\boldsymbol{X}, \boldsymbol{A})$ is a uniform resolution of (X, A), then π induces an isomorphism $H^\mu_\bullet(X, A; G) \cong \lim_{i \in I} H^\mu_\bullet(X_i, A_i; G)$.*

The proof is completely analogous to Korppi's. For example, we should replace [7, Lemma 60] and [7, Lemma 61] with the following two lemmas.

Definition 3.11. Let $\boldsymbol{X} := (I, X_i, \pi_{ij})$ be a standard inverse system of uniform spaces. We denote $J := \{ j \in {}^*I \mid i \le j \text{ for all } i \in I \}$. Note that J is nonempty by weak saturation. For $x, y \in {}^*X_j$ $(j \in J)$, we write $x \approx_{\boldsymbol{X}} y$ if ${}^*\pi_{ij}(x) \approx_{X_i} {}^*\pi_{ij}(y)$ holds for any $i \in I$.

Lemma 3.12. *Let $\boldsymbol{X} := (I, X_i, \pi_{ij})$ be a standard inverse system of uniform spaces and let $\pi\colon X \to \boldsymbol{X}$ be a standard cone over $\boldsymbol{X}$. The following are equivalent:*

(1) (UR1);

(2) *for any* $x, y \in {}^*X$, $x \approx_X y$ *if and only if* ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ *for all* $i \in I$;

(3) *for any* $x, y \in {}^*X$, $x \approx_X y$ *if and only if* ${}^*\pi_j(x) \approx_{\boldsymbol{X}} {}^*\pi_j(y)$ *for all* $j \in J$;

(4) *for any* $x, y \in {}^*X$, $x \approx_X y$ *if and only if* ${}^*\pi_j(x) \approx_{\boldsymbol{X}} {}^*\pi_j(y)$ *for some* $j \in J$.

Proof.

(1) $\Rightarrow$ (2)**:** Suppose that $x \not\approx_X y$. We can find a $\lambda \in \mathcal{U}_X$ such that x and y are not ${}^*\lambda$-near. There exist an $i \in I$ and a $\mu \in \mathcal{U}_{X_i}$ such that $\pi_i^{-1}\mu$ refines λ. x and y are not ${}^*\left(\pi_i^{-1}\mu\right)$-near. Therefore ${}^*\pi_i(x)$ and ${}^*\pi_i(y)$ are not ${}^*\mu$-near. Hence ${}^*\pi_i(x) \not\approx_{X_i} {}^*\pi_i(y)$. The reverse direction immediately follows from the uniform continuity of π_i.

(2) $\Rightarrow$ (1)**:** Suppose, on the contrary, that there exists a $\lambda \in \mathcal{U}_X$ such that for any $i \in I$ and any uniform cover $\mu \in \mathcal{U}_{X_i}$, $\pi_i^{-1}\mu$ does not refine λ. Let λ' be a uniform star-refinement of λ.

Let $i \in I$ and $\mu \in \mathcal{U}_{X_i}$. Since $\pi_i^{-1}\mu$ does not refine λ, there exists a $V \in \pi_i^{-1}\mu$ such that V is not contained in any member of λ. Clearly $V \neq \varnothing$. Choose an $x_{i,\mu} \in V$. We have that $V \nsubseteq \mathrm{St}(x_{i,\mu}, \lambda')$. Choose a $y_{i,\mu} \in V \setminus \mathrm{St}(x_{i,\mu}, \lambda')$. Then, $x_{i,\mu}$ and $y_{i,\mu}$ are $\pi_i^{-1}\mu$-near but not λ'-near.

Now, let P be a finite set of all pairs (i, μ) such that $i \in I$ and $\mu \in \mathcal{U}_{X_i}$. Fix an upper bound $i' \in I$ of all i with $(i, \mu) \in P$, and a common refinement μ' of all $\pi_{ii'}^{-1}\mu$ with $(i, \mu) \in P$. We can find $x_P, y_P \in X$ such that x_P and y_P are $\pi_{i'}^{-1}\mu'$-near but not λ'-near. x_P and y_P are $\pi_i^{-1}\mu$-near for all $(i, \mu) \in P$. By weak saturation, there exist $x, y \in {}^*X$ such that x and y are ${}^*\left(\pi_i^{-1}\mu\right)$-near for all $i \in I$ and $\mu \in \mathcal{U}_{X_i}$ but not ${}^*\lambda'$-near. Hence ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ but $x \not\approx_X y$.

(2) $\Rightarrow$ (3)**:** Suppose $x \approx_X y$. Let $j \in J$. Since π_i is uniformly continuous, we have that ${}^*\pi_{ij}\left({}^*\pi_j(x)\right) = {}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y) = {}^*\pi_{ij}\left({}^*\pi_j(y)\right)$ for all $i \in I$. Hence ${}^*\pi_j(x) \approx_{\boldsymbol{X}} {}^*\pi_j(y)$. Next, suppose that ${}^*\pi_j(x) \approx_{\boldsymbol{X}} {}^*\pi_j(y)$ for all $j \in J$. Fix a $j_0 \in J$. By the definition of $\approx_{\boldsymbol{X}}$, for any $i \in I$ we have that

$$\begin{aligned}
{}^*\pi_i(x) &= {}^*\pi_{ij_0}\left({}^*\pi_{j_0}(x)\right) \\
&\approx_{X_i} {}^*\pi_{ij_0}\left({}^*\pi_{j_0}(y)\right) \\
&= {}^*\pi_i(y).
\end{aligned}$$

By (2) we have $x \approx_X y$.

(3) ⇒ (4)**:** Suppose that ${}^*\pi_{j_0}(x) \approx_{\boldsymbol{X}} {}^*\pi_{j_0}(y)$ for some $j_0 \in J$. Let $j \in J$. For any $i \in I$ we have that

$$\begin{aligned} {}^*\pi_{ij}\left({}^*\pi_j(x)\right) &= {}^*\pi_i(x) \\ &= {}^*\pi_{ij_0}\left({}^*\pi_{j_0}(x)\right) \\ &\approx_{X_i} {}^*\pi_{ij_0}\left({}^*\pi_{j_0}(y)\right) \\ &= {}^*\pi_i(y) \\ &= {}^*\pi_{ij}\left({}^*\pi_j(y)\right). \end{aligned}$$

Hence ${}^*\pi_j(x) \approx_{\boldsymbol{X}} {}^*\pi_j(y)$. By (3) we have $x \approx_X y$. The reverse direction is trivial.

(4) ⇒ (2)**:** Suppose that ${}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y)$ for all $i \in I$. Fix a $j_0 \in J$. We have that ${}^*\pi_{j_0 i}\left({}^*\pi_{j_0}(x)\right) = {}^*\pi_i(x) \approx_{X_i} {}^*\pi_i(y) = {}^*\pi_{j_0 i}\left({}^*\pi_{j_0}(y)\right)$ for all $i \in I$. Hence ${}^*\pi_{j_0}(x) \approx_{\boldsymbol{X}} {}^*\pi_{j_0}(y)$. By (4) we have $x \approx_X y$. The reverse direction immediately follows from the uniform continuity of π_i. □

Lemma 3.13. *Let $\boldsymbol{X} := (I, X_i, \pi_{ij})$ be a standard inverse system of uniform spaces and let $\pi\colon X \to \boldsymbol{X}$ be a standard cone over $\boldsymbol{X}$. The following are equivalent:*

1. (UR2)*;*
2. *for any $j \in J$ and any $i \in I$, every $x \in {}^*\pi_{ij}\left({}^*X_j\right)$ is $\approx_{X_i}$-near to some $x' \in {}^*\pi_i({}^*X)$;*
3. *for some $j \in J$ and any $i \in I$, every $x \in {}^*\pi_{ij}\left({}^*X_j\right)$ is $\approx_{X_i}$-near to some $x' \in {}^*\pi_i({}^*X)$;*
4. *for any $j \in J$, every $x \in {}^*X_j$ is $\approx_{\boldsymbol{X}}$-near to some $x' \in {}^*\pi_j({}^*X)$;*
5. *for some $j \in J$, every $x \in {}^*X_j$ is $\approx_{\boldsymbol{X}}$-near to some $x' \in {}^*\pi_j({}^*X)$.*

Proof. (2) ⇒ (3) and (4) ⇒ (5) are trivial.

(1) ⇒ (2)**:** Let $j \in J$, $i \in I$ and $x \in {}^*\pi_{ji}\left({}^*X_j\right)$. Let P be a finite subset of $\mathcal{U}_{X_i}$. Let λ' be a common refinement of the members of P. Choose an $i' \geq i$ such that $\pi_{ii'}(X_{i'}) \subseteq \mathrm{St}(\pi_i(X), \lambda')$. Since ${}^*\pi_{ij}\left({}^*X_j\right) \subseteq {}^*(\pi_{ii'}(X_{i'})) \subseteq {}^*(\mathrm{St}(\pi_i(X), \lambda'))$, x is ${}^*\lambda'$-near to some $x_P \in {}^*\pi_i({}^*X)$. Such an x_P is ${}^*\lambda$-near to x for all $\lambda \in P$. By saturation, there exists an $x' \in {}^*\pi_i({}^*X)$ such that x and x' are ${}^*\lambda$-near for all $\lambda \in \mathcal{U}_{X_i}$, i.e., $x \approx_{X_i} x'$.

(2) ⇒ (4) and (3) ⇒ (5)**:** Let $j \in J$. Suppose that for any $i \in I$ every $x \in {}^*\pi_{ij}\left({}^*X_j\right)$ is $\approx_{X_i}$-near to some $x' \in {}^*\pi_i({}^*X)$. Let $x \in {}^*X_j$. Given a finite set P of all pairs (i, λ) such that $i \in I$ and $\lambda \in \mathcal{U}_{X_i}$, choose an upper bound $i' \in I$ of all i with $(i, \lambda) \in P$ and a common refinement λ' of all $\pi_{ii'}^{-1}\lambda$ with $(i, \lambda) \in P$. Since ${}^*\pi_{i'j}(x)$ is ${}^*\lambda'$-near to some points in ${}^*\pi_{i'}({}^*X)$, x is ${}^*\pi_{i'j}^{-1}({}^*\lambda')$-near to some $x_P \in {}^*\pi_j({}^*X)$. Such an x_P is ${}^*\pi_{ij}^{-1}({}^*\lambda)$-near to x for all $(i, \lambda) \in P$. By saturation, there exists an $x' \in {}^*\pi_j({}^*X)$ such that x and x' are ${}^*\pi_{ij}{}^{-1}({}^*\lambda)$-near for all $\lambda \in \mathcal{U}_{X_i}$ and $i \in I$, i.e., $x \approx_{\boldsymbol{X}} x'$.

(5) ⇒ (1)**:** Suppose that (1) does not hold, i.e., there exist an $i \in I$ and a $\lambda \in \mathcal{U}_{X_i}$ such that $\pi_{ii'}(X_{i'}) \nsubseteq \mathrm{St}(\pi_i(X), \lambda)$ for all $i' \in I$. For each $i' \geq i$

choose an $x_{i'} \in X_{i'}$ such that $\pi_{ii'}(x_{i'}) \notin \mathrm{St}(\pi_i(X), \lambda)$. Let $j \in J$. By transfer, we have that ${}^*x_j \in {}^*X_j$ and ${}^*\pi_{ij}\left({}^*x_j\right) \notin {}^*(\mathrm{St}(\pi_i(X), \lambda))$. It follows that ${}^*x_j$ is not $\approx_{\mathbf{X}}$-near to any $x' \in {}^*\pi_j({}^*X)$. □

Observe that the modified lemmas and their proofs are completely similar to the original ones. The remaining part of [7, Chapter 10, 11, 13] can be transferred as well. Thus we obtain the proof of Theorem 3.10.

4. Relationship between the Vietoris-type and microsimplicial homology theories

In this section, we show that the microsimplicial homology theories are isomorphic to the Vietoris-type homology theories under certain compactness conditions.

4.1. The compact case.

Garavaglia [2] showed that McCord homology is isomorphic to Vietoris homology for all standard compact spaces. Using this result, we can prove the following.

Proposition 4.1. *Let G be a standard abelian group. Let (X, A) be a standard compact uniform pair. Then, $H^{\mu}_{\bullet}(X, A; {}^*G) \cong \check{H}^{u}_{\bullet}(X, A; {}^*G)$.*

Proof. We know that $H^{\mu}_{\bullet}(X, A; {}^*G) = H^{M}_{\bullet}(UX, A; {}^*G)$. By [6, Theorem 9], $H^{M}_{\bullet}(UX, A; {}^*G) \cong \check{H}_{\bullet}(UX, A; {}^*G)$. By Lebesgue's number lemma, $\check{H}_{\bullet}(UX, A; {}^*G)$ $\cong \check{H}^{u}_{\bullet}(X, A; {}^*G)$.
Combining these isomorphisms gives $H^{\mu}_{\bullet}(X, A; {}^*G) \cong \check{H}^{u}_{\bullet}(X, A; {}^*G)$. □

Corollary 4.2. *Let G be a standard abelian group. Let (X, A) be a standard compact completely regular pair. Then, $H^{K}_{\bullet}(X, A; {}^*G) \cong \check{H}^{n}_{\bullet}(X, A; {}^*G)$.*

This proof depends on the fact that uniform Vietoris homology is isomorphic to Vietoris homology for all compact uniform spaces. In the non-compact case, uniform Vietoris homology may not be isomorphic to Vietoris homology. For instance, consider the quotient space $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ has disjoint open covers as fine as one likes, the 1-st Vietoris homology of $\mathbb{Q}/\mathbb{Z}$ vanishes. On the other hand, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ have the same uniform Vietoris homology. In particular, the 1-st uniform Vietoris homology of $\mathbb{Q}/\mathbb{Z}$ is isomorphic to G, and hence does not vanish (except for the trivial case $G = 0$). We note that $\mathbb{Q}/\mathbb{Z}$ is a dense subspace of $\mathbb{R}/\mathbb{Z}$.

4.2. Nonstandard continuity and homotopy.

Definition 4.3. Let X and Y be standard uniform spaces. A map $f\colon {}^*X \to {}^*Y$ is said to be *S-continuous* at $x \in {}^*X$ if $f(\mu^u_X(x)) \subseteq \mu^u_Y(f(x))$, or equivalently, if $x \approx_X y$ implies $f(x) \approx_Y f(y)$ for all $y \in {}^*X$.

It is clear that the collection of standard uniform pairs and internal S-continuous maps forms a category $\mathbf{Unif}_{2,S}$. The category $\mathbf{Unif}_2$ can be regarded as a subcategory of $\mathbf{Unif}_{2,S}$ under the identification of standard uniformly continuous maps $f\colon (X, A) \to (Y, B)$ with their nonstandard extensions ${}^*f\colon {}^*(X, A) \to {}^*(Y, B)$. Consider the singular homotopy equivalence within $\mathbf{Unif}_{2,S}$ defined as follows.

Definition 4.4. Let (X, A) and (Y, B) be standard uniform pairs. We say that two internal S-continuous maps $f, g\colon {}^*(X, A) \to {}^*(Y, B)$ are *S-homotopic* if there exists an internal S-continuous homotopy $h\colon {}^*(X, A) \times {}^*[0, 1] \to {}^*(Y, B)$ between f and g.

The S-continuous maps are precisely the maps that send microsimplices to microsimplices. Hence every internal S-continuous map $f\colon {}^*(X, A) \to {}^*(Y, B)$ functorially induces homomorphisms $C^{\mu}_{\bullet}(f; G) : C^{\mu}_{\bullet}(X, A; G) \to C^{\mu}_{\bullet}(Y, B; G)$ and hence $H^{\mu}_{\bullet}(f; G) : H^{\mu}_{\bullet}(X, A; G) \to H^{\mu}_{\bullet}(Y, B; G)$. In this setting, μ-homology satisfies the S-homotopy axiom.

Theorem 4.5 ([3, Theorem 11])**.** *Let G be an internal abelian group. Let (X, A) and (Y, B) be standard uniform pairs. If two internal S-continuous maps $f, g\colon {}^*(X, A) \to {}^*(Y, B)$ are S-homotopic, then $C^{\mu}_{\bullet}(f; G)$ and $C^{\mu}_{\bullet}(g; G)$ are chain-homotopic. Hence $H^{\mu}_{\bullet}(f; G) = H^{\mu}_{\bullet}(g; G)$.*

Next, we consider the relationship between uniform Vietoris homology and S-homotopy equivalence.

Definition 4.6. Let X and Y be uniform spaces. Let V be an entourage of Y. We say that a map $f\colon X \to Y$ is *V-continuous* at $x \in X$ if there is an entourage U of X such that $f(U[x]) \subseteq V[f(x)]$. We say that f is *uniformly V-continuous* if there is an entourage U of X such that $f(U[x]) \subseteq V[f(x)]$ for all $x \in X$.

Lemma 4.7. *Let (X, A) and (Y, B) be standard precompact uniform pairs. Let $f\colon {}^*(X, A) \to {}^*(Y, B)$ be an internal S-continuous map. For each entourage V of Y, there exists a (standard) uniformly V-continuous map $f^V\colon (X, A) \to (Y, B)$ (called a V-preshadow of f) such that ${}^*f^V(x)\ {}^*V\ f(x)$ holds for all $x \in {}^*X$.*

Proof. Let $V \in \mathcal{E}_Y$. Fix a symmetric $W \in \mathcal{E}_Y$ such that $W^5 \subseteq V$, where W^n refers to the n-fold composition $W \circ W \circ \cdots \circ W$. By the nonstandard characterisation of precompactness [11, Theorem 8.4.34], for each $x \in X$, we can choose a $y \in Y$ such that $y\ {}^*W\ f(x)$. Similarly, for each $a \in A$, we can find a $b \in B$ such that $b\ {}^*W\ f(a)$. Thus, we can define a (standard) map $f^V\colon (X, A) \to (Y, B)$ that satisfies $f^V(x)\ {}^*W\ f(x)$ on X.

According to the equivalent condition for S-continuity [11, Theorem 8.4.23], there exists an entourage U of X such that $f({}^*U[x]) \subseteq {}^*W[f(x)]$ for all $x \in X$. Let $x \in X$ and $y \in U[x]$. Then, since $f^V(x)\ {}^*W\ f(x)\ {}^*W\ f(y)\ {}^*W\ f^V(y)$, it follows that $f^V(x)\ {}^*V\ f^V(y)$. By transfer, we have $f^V(x)\ V\ f^V(y)$. Hence $f^V(U[x]) \subseteq V\left[f^V(x)\right]$ for all $x \in X$. Therefore, f^V is uniformly V-continuous.

Let $x \in {}^*X$. By [11, Theorem 8.4.34], there exists a $\xi \in X$ such that $\xi\ {}^*U\ x$. From the previous paragraph, we have that ${}^*f^V(x)\ {}^*W^3\ f^V(\xi)$ by transfer. Since $f^V(\xi)\ {}^*W\ f(\xi)\ {}^*W\ f(x)$, we conclude that ${}^*f^V(x)\ {}^*V\ f(x)$. □

Remark 4.8. This lemma can be easily refined as follows: let (X, A) and (Y, B) be standard uniform pairs. Let $f\colon {}^*(X, A) \to {}^*(Y, B)$ be an internal map such that $f({}^*X) \subseteq \operatorname{pns}(Y)$ and $f({}^*A) \subseteq \operatorname{pns}(B)$, where $\operatorname{pns}(-)$ denotes the set of all prenearstandard points. If f is S-continuous on X, then for each entourage V of

Y, there exists a (non-uniformly) V-continuous map $f^V\colon (X,A)\to(Y,B)$ such that $f^V(x)\ {}^*V\ f(x)$ holds for all $x\in X$. Moreover, if f is S-continuous also on ${}^*X\setminus X$, f^V can be uniformly V-continuous. Furthermore, if X is precompact, we can ensure that ${}^*f^V(x)\ {}^*V\ f(x)$ holds also for all $x\in {}^*X\setminus X$.

Theorem 4.9. *Let G be a standard abelian group. Let (X,A) and (Y,B) be standard precompact uniform pairs. Every internal S-continuous map $f\colon {}^*(X,A)\to {}^*(Y,B)$ functorially induces a homomorphism $\check{H}^u_\bullet(f;G):\check{H}^u_\bullet(X,A;G)\to\check{H}^u_\bullet(Y,B;G)$.*

Proof. For each entourage V, fix a symmetric entourage $\sqrt{V}$ with $\sqrt{V}^2\subseteq V$. By Lemma 4.7, for each entourage V of Y, there exists a $\sqrt{V}$-preshadow $f^{\sqrt{V}}\colon (X,A)\to(Y,B)$. For a sufficiently small entourage U of X, $f^{\sqrt{V}}$ is a simplicial map from (X_U,A_U) to (Y_V,B_V). It induces a homomorphism $f_V:=H_\bullet\left(f^{\sqrt{V}};G\right)\circ p_U\colon \check{H}^u_\bullet(X,A;G)\to H_\bullet(Y_V,B_V;G)$.

On the other hand, by [11, Theorem 8.4.23], there exists a symmetric entourage U of X such that $f({}^*U[x])\subseteq {}^*\sqrt{V}[f(x)]$ for all $x\in{}^*X$. For simplicity, we abbreviate $f^{\sqrt{V}}={}^*f^{\sqrt{V}}$. Taking U to be sufficiently small, we may assume that $f^{\sqrt{V}}({}^*U[x])\subseteq {}^*\sqrt{V}\left[f^{\sqrt{V}}(x)\right]$ holds for all $x\in{}^*X$. Let us prove that f and $f^{\sqrt{V}}$ are internally contiguous as internal simplicial maps from ${}^*(X_U,A_U)$ to ${}^*(Y_V,B_V)$. Let $(a_0,\dots,a_p)$ be an internal p-simplex of ${}^*X_U$ (resp. ${}^*A_U$). There exists an $x\in{}^*X$ such that $a_k\ {}^*U\ x$ for all $0\le k\le p$. Since $f(a_k)\ {}^*\sqrt{V}\ f(x)\ {}^*\sqrt{V}\ f^{\sqrt{V}}(x)$, we have $f(a_k)\ {}^*V\ f^{\sqrt{V}}(x)$. Moreover $f^{\sqrt{V}}(a_k)\ {}^*V\ f^{\sqrt{V}}(x)$ holds. Hence $\left(f(a_0),\dots,f(a_p),f^{\sqrt{V}}(a_0),\dots,f^{\sqrt{V}}(a_p)\right)$ is an internal $(2p+1)$-simplex of ${}^*Y_V$ (resp. ${}^*B_V$).

Now, we will prove that the following diagram commutes for all $V\supseteq W$:

$$\begin{tikzcd}
 & \check{H}^u_\bullet(X,A;G) \arrow[dl, "f_V"'] \arrow[dr, "f_W"] & \\
H_\bullet(Y_V,B_V;G) \arrow[rr, "p_{WV}"] & & H_\bullet(Y_W,B_W;G)
\end{tikzcd}$$

As we proved above, f and $f^{\sqrt{V}}$ are internally contiguous, and so are f and $f^{\sqrt{W}}$. It follows that ${}^*\left(H_\bullet\left(f^{\sqrt{W}};G\right)\right)=({}^*H_\bullet)(f;{}^*G)={}^*\left(p_{WV}\circ H_\bullet\left(f^{\sqrt{V}};G\right)\right)$. By transfer, we have that $H_\bullet\left(f^{\sqrt{W}};G\right)=p_{WV}\circ H_\bullet\left(f^{\sqrt{V}};G\right)$, and hence $f_W=p_{WV}\circ f_V$. By the universal property of $\check{H}^u_\bullet(Y,B;G)$, we obtain a homomorphism $\check{H}^u_\bullet(f;G):\check{H}^u_\bullet(X,A;G)\to\check{H}^u_\bullet(Y,B;G)$. We can also show that $\check{H}^u_\bullet(f;G)$ is independent of the choice of preshadows.

Suppose that $f\colon {}^*(X,A)\to{}^*(Y,B)$ and $g\colon {}^*(Y,B)\to{}^*(Z,C)$ are internal S-continuous maps, where (X,A), (Y,B) and (Z,C) are standard precompact uniform pairs. Let V be an entourage of Z. Let $g^{\sqrt{V}}$ be a $\sqrt{V}$-preshadow of g. There is an entourage U of Y such that $g^{\sqrt{V}}(U[x])\subseteq\sqrt{V}\left[g^{\sqrt{V}}(x)\right]$. Let $f^{\sqrt{U}}$ be a $\sqrt{U}$-preshadow of f. Then, $g^{\sqrt{V}} \circ f^{\sqrt{U}}$ is a $\sqrt{V}$-preshadow of $g \circ f$. There is an entourage W of X such that $f^{\sqrt{U}}(W[x]) \subseteq \sqrt{U}\left[f^{\sqrt{U}}(x)\right]$ and $g^{\sqrt{V}} \circ f^{\sqrt{U}}(W[x]) \subseteq \sqrt{V}\left[g^{\sqrt{V}} \circ f^{\sqrt{U}}(x)\right]$ for all $x \in X$. The following diagram is commutative:

$$\begin{tikzcd}
\check{H}^u_\bullet(X,A;G) \arrow[r, "\check{H}^u_\bullet(f;G)"] \arrow[d] \arrow[rr, bend left, "\check{H}^u_\bullet(g;G)\circ\check{H}^u_\bullet(f;G)"] & \check{H}^u_\bullet(Y,B;G) \arrow[r, "\check{H}^u_\bullet(g;G)"] \arrow[d] & \check{H}^u_\bullet(Z,C;G) \arrow[d] \\
H_\bullet(X_W,A_W;G) \arrow[r, "H_\bullet\left(f^{\sqrt{U}};G\right)"] \arrow[rr, bend right, "H_\bullet\left(g^{\sqrt{V}}\circ f^{\sqrt{U}};G\right)"'] & H_\bullet(Y_U,B_U;G) \arrow[r, "H_\bullet\left(g^{\sqrt{V}};G\right)"] & H_\bullet(Z_V,C_V;G)
\end{tikzcd}$$

We have that $\check{H}^u_\bullet(g \circ f;G) = \check{H}^u_\bullet(g;G) \circ \check{H}^u_\bullet(f;G)$. □

Theorem 4.10. *Let G be a standard abelian group. Let (X,A) and (Y,B) be standard precompact uniform pairs. If two internal S-continuous maps $f,g\colon {}^*(X,A) \to {}^*(Y,B)$ are S-homotopic, then $\check{H}^u_\bullet(f;G) = \check{H}^u_\bullet(g;G)$.*

Proof. Let $h\colon {}^*(X,A) \times {}^*[0,1] \to {}^*(Y,B)$ be an internal S-homotopy between f and g. Fix an infinite $N \in {}^*\mathbb{N}$. Define $h_i := h(\cdot, i/N)$ for $i = 0,1,\dots,N$. Let V be an entourage of Y. Let $\sqrt{V}$ be a symmetric entourage of Y with $\sqrt{V}^2 \subseteq V$. By [11, Theorem 8.4.23], there exists a symmetric entourage U of X such that $h_i({}^*U[x]) \subseteq {}^*\sqrt{V}[h_i(x)]$ for all $x \in {}^*X$ and $0 \le i \le N$.

All h_is are internal simplicial maps from ${}^*(X_U,A_U)$ to ${}^*(Y_V,B_V)$. Let us prove that h_i and h_{i+1} are internally contiguous for all $0 \le i < N$. Let $(a_0,\dots,a_p)$ be a p-simplex of ${}^*X_U$ (resp. ${}^*A_U$). There exists an $x \in {}^*X$ with $(a_k,x) \in {}^*U$ for all $0 \le k \le n$. Since $h_i(a_k)\ {}^*\sqrt{V}\ h_i(x)\ {}^*\sqrt{V}\ h_{i+1}(x)$, we have $h_i(a_k)\ {}^*V\ h_{i+1}(x)$. Moreover $h_{i+1}(a_k)\ {}^*V\ h_{i+1}(x)$ holds. Hence $(h_i(a_0),\dots,h_i(a_p),h_{i+1}(a_0),\dots,h_{i+1}(a_p))$ is a $(2p+1)$-simplex of ${}^*Y_V$ (resp. ${}^*B_V$). It follows that $({}^*H_\bullet)(f;{}^*G) = ({}^*H_\bullet)(g;{}^*G)$ and ${}^*f_V = {}^*g_V$. By transfer, we have $f_V = g_V$ and therefore $\check{H}^u_\bullet(f;G) = \check{H}^u_\bullet(g;G)$. □

Theorem 4.11 ([3, Theorem 12])**.** *Let (X,A) be a standard uniform pair and let (Y,B) be a subpair of (X,A). Suppose that Y and B are dense in X and A, respectively. Then there exists an internal S-deformation retraction $r\colon {}^*(X,A) \to {}^*(Y,B)$.*

Combining with the S-homotopy axiom (Theorem 4.5 and Theorem 4.10), we obtain the following two results.

Corollary 4.12. *Let G be an internal abelian group. Let (X,A) be a standard uniform pair and let (Y,B) be a subpair of (X,A). Suppose that Y and B are dense in X and A, respectively. Then the inclusion map $i\colon (Y,B) \hookrightarrow (X,A)$ induces an isomorphism $H^\mu_\bullet(i;G) : H^\mu_\bullet(Y,B;G) \cong H^\mu_\bullet(X,A;G)$.*

Corollary 4.13. *Let G be an abelian group. Let (X, A) be a precompact uniform pair and let (Y, B) be a subpair of (X, A). Suppose that Y and B are dense in X and A, respectively. Then the inclusion map $i\colon (Y, B) \hookrightarrow (X, A)$ induces an isomorphism $\check{H}^u_\bullet(i; G) : \check{H}^u_\bullet(Y, B; G) \cong \check{H}^u_\bullet(X, A; G)$.*

Notice that the latter generalises the fact that $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ have the same uniform Vietoris homology.

4.3. **The precompact case.** Based on these preparations, we now prove the main results in this section.

Theorem 4.14. *Let G be a standard abelian group. Let (X, A) be a standard precompact uniform pair. Then, $H^\mu_\bullet(X, A; {}^*G) \cong \check{H}^u_\bullet(X, A; {}^*G)$.*

Proof. Let $\bar{X}$ be a (standard) uniform completion of X. Let $\bar{A}$ be the closure of A in $\bar{X}$. Since X is precompact, $\bar{X}$ is compact, and so is $\bar{A}$. By Corollary 4.12, $H^\mu_\bullet(X, A; {}^*G) \cong H^\mu_\bullet\left(\bar{X}, \bar{A}; {}^*G\right)$. By Proposition 4.1, $H^\mu_\bullet\left(\bar{X}, \bar{A}; {}^*G\right) \cong \check{H}^u_\bullet\left(\bar{X}, \bar{A}; {}^*G\right)$. Finally, by Corollary 4.13, $\check{H}^u_\bullet\left(\bar{X}, \bar{A}; {}^*G\right) \cong \check{H}^u_\bullet(X, A; {}^*G)$. The proof is completed. □

Corollary 4.15. *Let G be a standard abelian group. Let (X, A) be a standard pseudocompact completely regular pair such that A is normally embedded in X. Then, $H^K_\bullet(X, A; {}^*G) \cong \check{H}^n_\bullet(X, A; {}^*G)$.*

Proof. By Proposition 3.4, we have that $H^K(X, A; {}^*G) = H^\mu_\bullet(FX, A; {}^*G)$. Since X is pseudocompact, FX is precompact. Hence $H^\mu_\bullet(FX, A; {}^*G) \cong \check{H}^u_\bullet(FX, A; {}^*G)$. By [4, Theorem 20], we have that $\check{H}^u_\bullet(FX, A; {}^*G) \cong \check{H}^n_\bullet(X, A; {}^*G)$. □

4.4. **Counterexample in the general case.** The above-mentioned isomorphisms do not hold in general. The cause is that the elementary embedding $\mathbb{U} \hookrightarrow {}^*\mathbb{U}$ does not preserve infinitary operations such as infinite direct sums.

Proposition 4.16. *There exist a standard uniform space X and a standard abelian group G such that $H^\mu_\bullet(X; {}^*G) \ncong \check{H}^u_\bullet(X; {}^*G)$.*

Proof. Let X be the discrete uniform space $\mathbb{N}$. Let G be any (standard) nontrivial finite abelian group. Then, by transfer, ${}^*G = G$. It is easy to see that $\check{H}^u_0(X; G) \cong G^{\oplus\mathbb{N}}$. The cardinality of $\check{H}^u_0(X; G)$ is $\aleph_0$. On the other hand, we have $H^\mu_0(X; G) \cong {}^*\left(G^{\oplus\mathbb{N}}\right)$. By weak saturation, each element of $G^{\mathbb{N}}$ can be extended to an element of ${}^*\left(G^{\oplus\mathbb{N}}\right)$. Since the cardinality of $G^{\mathbb{N}}$ is $2^{\aleph_0}$, the cardinality of $H^\mu_0(X; G)$ must be at least $2^{\aleph_0}$. Hence $\check{H}^u_0(X; G)$ and $H^\mu_0(X; G)$ cannot be isomorphic. □

Corollary 4.17. *There exist a standard completely regular space X and a standard abelian group G such that $H^K_\bullet(X; {}^*G) \ncong \check{H}^n_\bullet(X; {}^*G)$.*

Proof. Let X and G be the same as in Proposition 4.16. Since X is fine, we have that $H^K_\bullet(UX; G) = H^\mu_\bullet(X; G) \ncong \check{H}^u_\bullet(X; G) = \check{H}^n_\bullet(UX; G)$. □

5. Homology of nonstandard subsets

Let X be a standard uniform space. We call a (possibly external) subset A of ${}^*X$ a *uniform $\mathcal{E}$-set*, and denote the whole space X by $\mathrm{Amb}\,(A)$. The notion of uniform $\mathcal{E}$-set is the uniform analogue of $\mathcal{E}$-set (see Wattenberg [12]). We say that a map $f\colon A \to B$ between uniform $\mathcal{E}$-sets is an *S-map* if f has an internal extension $F\colon {}^*\mathrm{Amb}\,(A) \to {}^*\mathrm{Amb}\,(B)$ that is S-continuous on A. The collection of uniform $\mathcal{E}$-sets and S-maps forms a category $\mathbf{uE}_S$. We denote by $\mathbf{uE}_{2,S}$ the category of pairs of uniform $\mathcal{E}$-sets with S-maps.

μ-homology can be extended to $\mathbf{uE}_S$ (and to $\mathbf{uE}_{2,S}$) as follows. First, define the *microchain complex* of $A \in \mathbf{uE}_S$ by

$$C_p^\mu\,(A;G) := \Big\{ \sum_i g_i\sigma_i \in C_p^\mu\,(\mathrm{Amb}\,(A)\,;G) \mid \sigma_i \subseteq A \text{ for all } i \Big\},$$

where $C_p^\mu\,(\mathrm{Amb}\,(A))$ in the right hand side is defined as in Section 2. Given an S-map $f\colon A \to B$, choose an internal extension $F\colon {}^*\mathrm{Amb}\,(A) \to {}^*\mathrm{Amb}\,(B)$ that is S-continuous on A, and define a homomorphism $C_\bullet^\mu\,(f;G) : C_\bullet^\mu\,(A;G) \to C_\bullet^\mu\,(B;G)$ by letting

$$C_p^\mu\,(f;G)\,(a_0,\ldots,a_p) := ({}^*F\,(a_0)\,,\ldots,{}^*F\,(a_p))\,.$$

It is independent of the choice of F. Obviously $C_\bullet^\mu$ is a functor on $\mathbf{uE}_S$. Finally, let $H_\bullet^\mu\,(\cdot;G) := H_\bullet C_\bullet^\mu\,(\cdot;G)$.

We can show (in the same way as in [3]) that the extended μ-homology satisfies the homotopy, exactness, excision, dimension and finite additivity axioms. The excision axiom is formulated as follows.

Definition 5.1. Let X be a uniform $\mathcal{E}$-set. Let A and B be subsets of X. We say that A is *strongly contained in B* (and we write $A \Subset B$) if $\mu^u_{\mathrm{Amb}(X)}\,(A) \cap X \subseteq B$.

Proposition 5.2. *Let X be a uniform $\mathcal{E}$-set. Let A, B be subsets of X such that either of them is internal. If $X \setminus A \Subset B$ (or $X \setminus B \Subset A$), then the inclusion map $i\colon (A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $H_\bullet^\mu\,(i;G) : H_\bullet^\mu\,(A, A\cap B;G) \cong H_\bullet^\mu\,(X,B;G)$.*

Proof. Similarly to Proposition 2.10, we can prove that each microsimplex of X is contained in either A or B. Given $u := \sum_i g_i\sigma_i \in C_p^\mu\,(X;G)$, if A is internal, then u can be decomposed into $\sum_{\sigma_i \subseteq A} g_i\sigma_i \in C_p^\mu\,(A;G)$ and $\sum_{\sigma_i \nsubseteq A} g_i\sigma_i \in C_p^\mu\,(B;G)$. If B is internal, then u can be decomposed into $\sum_{\sigma_i \subseteq B} g_i\sigma_i \in C_p^\mu\,(B;G)$ and $\sum_{\sigma_i \nsubseteq B} g_i\sigma_i \in C_p^\mu\,(A;G)$. □

Remark 5.3. The internality condition of the excision axiom cannot be omitted. Otherwise, the excision axiom causes a contradiction as follows. Let G be any standard nontrivial finite abelian group. (Note that $G = {}^*G$ by transfer.) By applying the Mayer-Vietoris theorem to the triad $({}^*\mathbb{R}; \mu_{\mathbb{R}}\,(0)\,, {}^*\mathbb{R} \setminus \mu_{\mathbb{R}}\,(0))$, we obtain the following exact sequence:

$$0 \longrightarrow H_0^\mu\,(\mu_{\mathbb{R}}\,(0)\,;G) \oplus H_0^\mu\,({}^*\mathbb{R} \setminus \mu_{\mathbb{R}}\,(0)\,;G) \longrightarrow H_0^\mu\,({}^*\mathbb{R};G) \longrightarrow 0$$

It is not difficult to see that $H_0^\mu(\mu_{\mathbb{R}}(0);G) \cong G$, $H_0^\mu({}^*\mathbb{R} \setminus \mu_{\mathbb{R}}(0);G) \cong G \oplus G$ and $H_0^\mu({}^*\mathbb{R};G) \cong G$. However, by exactness $G \oplus G \oplus G \cong H_0^\mu(\mu_{\mathbb{R}}(0);G) \oplus H_0^\mu({}^*\mathbb{R} \setminus \mu_{\mathbb{R}}(0);G) \cong H_0^\mu({}^*\mathbb{R};G) \cong G$, which makes a contradiction.

6. Use of the saturation principle

The saturation principle is not necessary to construct the microsimplicial homology theories. The satisfaction of the Eilenberg-Steenrod axioms can be proved without the *full* saturation principle (only using the *weak* saturation principle, i.e. the enlargement property of the nonstandard universe). The equivalences among the microsimplicial homology theories can also be proved without saturation. On the other hand, the equivalences between the Vietoris-type and microsimplicial homology theories depend on saturation (see [2, Theorem 4]). The proofs of the continuities of Korppi and μ-homology also depend on saturation.

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