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GENERALIZED PRIME D-FILTERS OF DISTRIBUTIVE LATTICES

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Abstract. The concept of generalized prime D-filters is introduced in distributive lattices. Generalized prime D-filters are characterized in terms of principal filters and ideals. The notion of generalized minimal prime D-filters is introduced in distributive lattices and properties of minimal prime D-filters are then studied with respect to congruences. Some topological properties of the space of all prime D-filters of a distributive lattice are also studied.

Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker [12] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [11] and W.H. Cornish [3, 4, 5] made an extensive study of annihilators in distributive lattices and characterized some algebraic structures like normal lattices and quasicomplemented lattices. Recently in 2013, the concepts of e-filters and D-filters [10] are introduced and characterized in MS-algebras. Some topological properties of the class of all prime e-filters of MS-algebras are studied. In 2016, Rao and Badawy studied the properties of co-annihilator filters of distributive lattices. In [9], authors generalized the concept of D-filters in distributive lattices and studied their properties.

In this paper, the notion of generalized prime D-filters is introduced in the form of F-filters of distributive lattices. Some properties of generalized prime D-filters are investigated and the prime filter theorem is generalized to the case of prime F-filters. The concept of generalized minimal prime D-filters is introduced in the form of minimal prime F-filters of distributive lattices. A characterization theorem of minimal prime D-filters is derived. A set of equivalent conditions is established for any two distinct minimal prime F-filters to become co-maximal. An ideal congruence is introduced in lattices and then some properties of prime D-filters

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are studied with respect to this ideal congruence. A one-to-one correspondence is obtained between the class of all minimal prime D-filters of a distributive lattice and the class of all minimal prime D-filters of the corresponding quotient algebra with respect to this congruence. Topological properties of the space prime D-filters are investigated. Finally, a set of equivalent conditions is derived for the space of all prime D-filters to become a Hausdorff space.

1. Preliminaries

The reader is referred to [1] and [2] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results of [3], [5], [7], [9] and [12] are presented for the ready reference of the reader.

Definition 1.1 ([2]). An algebra $(L, \wedge, \vee)$ of type $(2,2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of L is called an ideal(filter) of L if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever a, $b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal(filter) M of a lattice is called maximal if there exists no proper ideal(filter) N such that $M \subset N$.

Definition 1.2 ([2]). Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. In this case, the pair $(L, \leq)$ is called a partially ordered set. If $x \leq y$ or $y \leq x$ for all x, $y \in L$, then $(L, \leq)$ is called a totally ordered set.

The set $(a] = \{x \in L \mid x \leq a\}$ is called a principal ideal generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually the set $[a) = \{x \in L \mid a \leq x\}$ is called a principal filter generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called prime if for all $a, b \in L$, $a \wedge b \in P$ $(a \vee b \in P)$ then $a \in P$ or $b \in P$. Every maximal ideal (maximal filter) is prime. A prime ideal P of L is called *minimal* [7] if there exists no prime ideal Q of L such that $Q \subset P$.

Theorem 1.3 ([1]). *Let F be a filter and I an ideal of a distributive lattice L such that $F \cap I = \emptyset$. Then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

For any element a of a distributive lattice $(L, \vee, \wedge, 0)$, the *annihilator* of a is defined as the set $(a)^* = \{x \in L \mid x \wedge a = 0\}$. Properties of these annihilators are extensively studied by Cornish [3, 4], T.P Speed [12]. Normal lattices are characterized in terms of annihilators in [3].

Lemma 1.4 ([12]). *For any two elements a, b of a distributive lattice L with 0, we have*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,
(2) $(a \vee b)^* = (a)^* \cap (b)^*$,
(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,
(4) $(a)^* = L$ *if and only if* $a = 0$.

An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a distributive lattice forms a filter. A lattice L with 0 is called quasicomplemented [5] if for each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense.

Definition 1.5 ([9]). A filter F of a distributive lattice L is called a D-filter if $D \subseteq F$.

The set D of all dense elements of a distributive lattice is the smallest D-filter of the lattice.

2. Generalized prime D-filters

In this section, the notion of generalized prime D-filters is introduced in lattices in the form of prime F-filters. A relation between maximal filters and prime D-filters is established. Stone's theorem of prime filters of lattices is generalized to the case of prime F-filters. A set of equivalent conditions is derived for every proper F-filter of a lattice to become a prime F-filter.

Definition 2.1. Let F be an arbitrary filter of a distributive lattice L. A filter G of L is called an F-filter of L if G contains F (i.e. $F \subseteq G$).

For any subset S of a distributive lattice L, define $\langle S\rangle_F = [S) \vee F$, where $[S)$ is the smallest filter containing S. Clearly $\langle S\rangle_F$ is an F-filter of L and, in fact the smallest F-filter containing S. For $S = \{x\}$, we denote simply $\langle x\rangle_F$ for $\langle\{x\}\rangle_F$. Then clearly $\langle 0\rangle_F = L$ and $\langle 1\rangle_F = F$. Moreover, for any $x \in L$, $\langle x\rangle_F = [x) \vee F$ is the smallest F-filter containing x which is known as the principal F-filter generated by x. The following result is an easy consequence of the above facts:

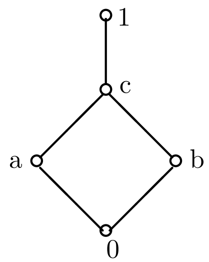
Lemma 2.2. *For any two elements a, b of a distributive lattice L, we have*

(1) $a \leq b$ *implies* $\langle b\rangle_F \subseteq \langle a\rangle_F$,
(2) $\langle a \wedge b\rangle_F = \langle a\rangle_F \vee \langle b\rangle_F$,
(3) $\langle a \vee b\rangle_F = \langle a\rangle_F \cap \langle b\rangle_F$,
(4) $\langle a\rangle_F = F$ *if and only if* $a \in F$.

It is easy to verify that the set $\mathcal{F}_F(L)$ of all F-filters of a lattice L forms a distributive lattice. Further, the set $\mathcal{PF}_F(L)$ of all principal F-filters of L forms a sublattice to $\mathcal{F}_F(L)$.

Definition 2.3. A proper F-filter P of a distributive lattice L is called a *prime F-filter* if for any $a, b \in L$, $a \vee b \in P$ implies that either $a \in P$ or $b \in P$. A proper D-filter P of a lattice L is called a *prime D-filter* if for any $a, b \in L$, $a \vee b \in P$ implies that either $a \in P$ or $b \in P$.

Example 2.4. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure:



Consider the filters $F_1 = \{a, c, 1\}$, $F_2 = \{b, c, 1\}$ and $F_3 = \{c, 1\}$. Then F_1, F_2 and F_3 are D-filters of L. Clearly F_1 and F_2 are prime D-filters but F_3 is not a prime D-filter.

Theorem 2.5. *Let F be an arbitrary filter of a distributive lattice L. Suppose P is a proper F-filter of L. Then the following assertions are equivalent:*

(1) *P is prime;*

(2) *for any two F-filters G_1, G_2 of L, $G_1 \cap G_2 \subseteq P$ implies $G_1 \subseteq P$ or $G_2 \subseteq P$;*

(3) *for any a, $b \in L$, $\langle a\rangle_F \cap \langle b\rangle_F \subseteq P$ implies $a \in P$ or $b \in P$.*

Proof. (1) $\Rightarrow$ (2): Assume that P is a prime F-filter of L. Let G_1 and G_2 be two F-filters of L such that $G_1 \cap G_2 \subseteq P$. Suppose $G_1 \nsubseteq P$ and $G_2 \nsubseteq P$. Then there exist two elements $a, b \in L$ such that $a \in G_1$ and $b \in G_2$ such that $a \notin P$ and $b \notin P$. Since P is prime, we get $a \vee b \notin P$. Since $a \in G_1$ and $b \in G_2$, we get $a \vee b \in G_1 \cap G_2 \subseteq P$. Hence $a \vee b \in P$, which is a contradiction. Therefore $G_1 \subseteq P$ or $G_2 \subseteq P$.

(2) $\Rightarrow$ (3): Since each $\langle x\rangle_F$ is an F-filter of L, it is clear.

(3) $\Rightarrow$ (1): Assume that condition (3) holds. Let $a, b \in L$ be such that $a \vee b \in P$. Since P is an F-filter, we get $\langle a\rangle_F \cap \langle b\rangle_F = \langle a \vee b\rangle_F \subseteq P$. By the condition (3), we get $a \in P$ or $b \in P$. Therefore P is a prime F-filter of L. $\square$

Corollary 2.6. *Let P be a proper D-filter of a distributive lattice L. Then the following assertions are equivalent:*

(1) *P is prime;*

(2) *for any two D-filters G_1, G_2 of L, $G_1 \cap G_2 \subseteq P$ implies $G_1 \subseteq P$ or $G_2 \subseteq P$;*

(3) *for any* a, $b \in L$, $\langle a\rangle_D \cap \langle b\rangle_D \subseteq P$ *implies* $a \in P$ *or* $b \in P$.

Theorem 2.7. *Let F be an arbitrary filter of a distributive lattice L. Then every maximal F-filter of L is a prime F-filter.*

Proof. Let M be a maximal F-filter of a distributive lattice L. Let $x, y \in L$ be such that $x \notin M$ and $y \notin M$. Then $M \vee \langle x\rangle_F = L$ and $M \vee \langle y\rangle_F = L$. Hence we have the following consequence:

$$\begin{aligned} L &= L \cap L \\ &= \{M \vee \langle x\rangle_F\} \cap \{M \vee \langle y\rangle_F\} \\ &= M \vee \{\langle x\rangle_F \cap \langle y\rangle_F\} \\ &= M \vee \langle x \vee y\rangle_F\,. \end{aligned}$$

If $x \vee y \in M$ then $M = L$, which is a contradiction. Hence $x \vee y \notin M$. Thus M is prime. □

Corollary 2.8. *Every maximal filter of a distributive lattice is a prime D-filter.*

Proof. Let M be a maximal filter of a distributive lattice L. We first prove that M is a D-filter of L. Suppose there exists $d \in D$ such that $d \notin M$. Then $M \vee [d) = L$, because of M is maximal. Then $0 \in M \vee [d)$, which gives $x \wedge d = 0$ for some $x \in M$. Since d is dense, we must have $x = 0$. Hence $0 = x \in M$, which is a contradiction to that M is a proper filter. Therefore $D \subseteq M$, which gives that M is a D-filter. By taking $F = D$ in the main theorem, the remaining proof is an easy consequence. □

Corollary 2.9. *If $M_1, M_2, M_3, \dots, M_n$ and M are maximal F-filters of a distributive lattice L such that $\bigcap_{i=1}^{n} M_i \subseteq M$, then there exists $j \in \{1, 2, 3, \dots, n\}$ such that $M_j \subseteq M$.*

Proof. Let $M_1, M_2, M_3, \dots, M_n$ and M be maximal F-filters of L such that $\bigcap_{i=1}^{n} M_i \subseteq M$. By the above theorem M is a prime F-filter of L. Then there exist $j \in \{1, 2, 3, \dots, n\}$ such that $M_j \subseteq M$. □

Definition 2.10. Let F be a filter of a distributive lattice L. Then the lattice L is called *F-quasicomplemented* if to each $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in F$.

Theorem 2.11. *In an F-quasicomplemented lattice, every prime F-filter is a maximal F-filter.*

Proof. Let P be a prime F-filter of L. Suppose there exist a proper F-filter Q of L such that $P \subset Q$. Choose $x \in Q - P$. Then there exist $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in F$. Hence $x \vee x' \in F \subseteq P$. Since $x \notin P$, we must have $x' \in P \subset Q$. Since $x \in Q$, we get $0 = x \wedge x' \in Q$, which is a contradiction to that Q is proper. Therefore P is a maximal F-filter of L. □

If $F = D$, then it clear that the F-quasicomplemented lattice will become quasicomplemented. Then the following corollary is a direct consequence of the above theorem.

Corollary 2.12. *In a quasicomplimented lattice, every prime D-filter is a maximal D-filter.*

Theorem 2.13. *Let F be a filter and P be a proper F-filter of a distributive lattice L. Then P is a prime F-filter of L if and only if $L-P$ is a prime ideal such that $(L-P)\cap F=\emptyset$.*

Proof. Assume that P is a prime F-filter of L. Since $P\neq L$, we get $L-P$ is non-empty. It can be easily shown that $L-P$ is an ideal of L. Suppose $(L-P)\cap F\neq\emptyset$. Choose $a\in(L-P)\cap F$. Hence $a\in F\subseteq P$, which is a contradiction to that $a\in L-P$. Therefore $(L-P)\cap F=\emptyset$.

Conversely, suppose that $L-P$ is a prime ideal of L such that $(L-P)\cap F=\emptyset$. It can be routinely verified that P is a filter of L. Let $x,y\in L$ be such that $x\notin P$ and $y\notin P$. Then $x\in L-P$ and $y\in L-P$. Since $L-P$ is an ideal, we get $x\vee y\in L-P$. Hence $x\vee y\notin P$, which means that P is a prime filter of L. Since $(L-P)\cap F=\emptyset$, we get $F\subseteq L-(L-P)=P$. Thus P is a prime F-filter of L. □

Corollary 2.14. *Let P be a proper D-filter of a distributive lattice L. Then P is a prime D-filter of L if and only if $L-P$ is a prime ideal such that $(L-P)\cap D=\emptyset$.*

Theorem 2.15. *Let F be a filter and G be an F-filter of a distributive lattice L. Suppose I is a non-empty subset of L, which is closed under the operation $\vee$ such that $G\cap I=\emptyset$. Then there exists a prime F-filter P of L such that $G\subseteq P$ and $P\cap I=\emptyset$.*

Proof. Let G be an F-filter of L and I a non-empty subset of L, which is closed under the operation $\vee$ such that $G\cap I=\emptyset$. Consider $\mathcal{P}=\{H\mid H$ is an F-filter of L, $G\subseteq H$ and $H\cap I=\emptyset\}$. Clearly $G\in\mathcal{P}$. Let $\{H_\alpha\}_{\alpha\in\Delta}$ be a chain of elements in $\mathcal{P}$. Since each H_α is an F-filter of L, we get $\bigcup\limits_{\alpha\in\Delta} H_\alpha$ is an F-filter of L. Since $G\subseteq H_\alpha$ for each $\alpha\in\Delta$, we get $G\subseteq\bigcup\limits_{\alpha\in\Delta} H_\alpha$. Since $H_\alpha\cap I=\emptyset$ for each $\alpha\in\Delta$, we get $\Big(\bigcup\limits_{\alpha\in\Delta} H_\alpha\Big)\cap I=\emptyset$. Hence $\bigcup\limits_{\alpha\in\Delta} H_\alpha$ is an upper bound for the chain $\{H_\alpha\}_{\alpha\in\Delta}$. By the Zorn's lemma, $\mathcal{P}$ has a maximal element say P. That is, P is an F-filter of L such that $G\subseteq P$ and $P\cap I=\emptyset$. Let $a,b\in L$ be such that $a\notin P$ and $b\notin P$. Then $P\vee\langle a\rangle_F$ and $P\vee\langle b\rangle_F$ are F-filter of L such that $P\subset P\vee\langle a\rangle_F$ and $P\subset P\vee\langle b\rangle_F$. Since P is maximal in $\mathcal{P}$ and $G\subset P\vee\langle a\rangle_F$, we get $(P\vee\langle a\rangle_F)\cap I\neq\emptyset$. Similarly, we obtain that $(P\vee\langle b\rangle_F)\cap I\neq\emptyset$. Choose $x\in(P\vee\langle a\rangle_F)\cap I$ and $y\in(P\vee\langle b\rangle_F)\cap I$. Then

$$\begin{aligned} x\vee y &\in \{P\vee\langle a\rangle_F)\}\cap\{P\vee\langle b\rangle_F\}\\ &= P\vee\{\langle a\rangle_F\cap\langle b\rangle_F\}\\ &= P\vee\langle a\vee b\rangle_F\,. \end{aligned}$$

If $a\vee b\in P$, then $\langle a\vee b\rangle_F\subseteq P$. Hence $x\vee y\in P$. Since $x,y\in I$, we get $x\vee y\in I$. Thus $x\vee y\in P\cap I$, which is a contradiction. Hence $a\vee b\notin P$. Thus P is a prime F-filter of L. □

Corollary 2.16. *Let F be a filter and G be an F-filter of a distributive lattice L. Suppose $a \in L$ such that $a \notin G$. Then there exists a prime F-filter P of L such that $G \subseteq P$ and $a \notin P$.*

In case of $F = D$, we have the following two special cases:

Corollary 2.17. *Let G be a D-filter of a distributive lattice L and I be a non-empty subset of L, which is closed under the operation $\vee$ such that $G \cap I = \emptyset$. Then there exists a prime D-filter P of L such that $G \subseteq P$ and $P \cap I = \emptyset$.*

Corollary 2.18. *Let G be a D-filter of a distributive lattice L and $a \in L$ be such that $a \notin G$. Then there exists a prime D-filter P of L such that $G \subseteq P$ and $a \notin P$.*

Theorem 2.19. *Let F be a filter of a distributive lattice L. Then the following assertions are equivalent:*

(1) *Every proper F-filter of L is prime;*
(2) *$\mathcal{F}_F(L)$ is a totally ordered set;*
(3) *$\mathcal{PF}_F(L)$ is a totally ordered set.*

Proof. (1) $\Rightarrow$ (2): Assume that every proper F-filter of L is a prime F-filter of L. Clearly $\mathcal{F}_F(L)$ is a partially ordered set with respect to the set inclusion $\subseteq$. Let G_1 and G_2 be two proper F-filters of L. By the assumption, we get that $G_1 \cap G_2$ is also a proper F-filter of L. Hence $G_1 \cap G_2$ is a prime F-filter of L. Since $G_1 \cap G_2 \subseteq G_1 \cap G_2$, we get $G_1 \subseteq G_1 \cap G_2 \subseteq G$ or $G_2 \subseteq G_1 \cap G_2 \subseteq G$. Therefore $\mathcal{F}_F(L)$ is a totally ordered set.
(2) $\Rightarrow$ (3): It is clear.
(3) $\Rightarrow$ (1): Assume that $\mathcal{PF}_F(L)$ is a totally ordered set. Let G be a proper F-filter of L. Choose $a, b \in L$ be such that $\langle a\rangle_F \cap \langle b\rangle_F \subseteq G$. Since $\langle a\rangle_F$ and $\langle b\rangle_F$ are F-filters of L, we get either $\langle a\rangle_F \subseteq \langle b\rangle_F$ or $\langle b\rangle_F \subseteq \langle a\rangle_F$. Hence $a \in \langle a\rangle_F = \langle a\rangle_F \cap \langle b\rangle_F \subseteq G$ or $b \in \langle b\rangle_F = \langle a\rangle_F \cap \langle b\rangle_F \subseteq G$. Therefore G is a prime F-filter of L. ☐

Corollary 2.20. *Following assertions are equivalent in a distributive lattice L:*

(1) *Every proper D-filter of L is prime;*
(2) *$\mathcal{F}_D(L)$ is a totally ordered set;*
(3) *$\mathcal{PF}_D(L)$ is a totally ordered set.*

3. Generalized minimal prime D-filters

In this section, the notion of generalized minimal prime D-filters is introduced as minimal prime F-filters of distributive lattices. A necessary and sufficient condition is derived for every prime D-filter to become a minimal prime D-filter. A set of equivalent conditions is established for any two distinct minimal prime D-filters to become co-maximal.

Definition 3.1. Let F be a filter of a distributive lattice L. For any $\emptyset \neq A \subseteq L$, define $(A, F) = \{x \in L \mid a \vee x \in F, \text{ for all } a \in A\}$.

For any distributive lattice L, it can be observed that $(L, F) = F$ and $(F, F) = L$. It can also be observed that $F \subseteq (A, F)$ for any subset A of a distributive lattice L. For $A = \{a\}$, we denote $(\{a\}, F)$ by (a, F). In case of $F = D$, we denote (A, D) by A°. For any $a \in L$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$.

Proposition 3.2. *Let F be a filter of a distributive lattice L. For any non-empty subset A of L, the set (A, F) is an F-filter of L.*

Proof. Clearly $F \subseteq (A, F)$. Let $x, y \in (A, F)$. For any $a \in A$, we have $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \in F$. Hence $x \wedge y \in (A, F)$. Let $x \in (A, F)$ and $y \in L$ with $x \leq y$. Then $a \vee x \leq a \vee y$ for all $a \in A$. Since $a \vee x \in F$ and F is a filter, we get $a \vee y \in F$. Hence $y \in (A, F)$ for all $a \in A$. Thus (A, F) is a filter of L. Since $F \subseteq (A, F)$, it concludes that (A, F) is an F-filter of L. □

Corollary 3.3. *For any non-empty subset A of a distributive lattice L, A° is a D-filter of L.*

Lemma 3.4. *Let F be a filter of a distributive lattice L. For any two non-empty subsets A and B of L, we have*

(1) $A \subseteq B$ *implies* $(B, F) \subseteq (A, F)$,
(2) $A \subseteq ((A, F), F)$,
(3) $(((A, F), F), F) = (A, F)$,
(4) $(A, F) = L$ *if and only if* $A \subseteq F$.

Proof. (1) Assume that $A \subseteq B$. Let $x \in (B, F)$. Then $a \vee x \in F$ for all $a \in B$. Since $A \subseteq B$, we get $a \vee x \in F$ for all $a \in A$. Hence $x \in (A, F)$. Therefore $(B, F) \subseteq (A, F)$.
(2) Let $a \in A$. Then for every $x \in (A, F)$, we get $a \vee x \in F$. Hence $a \vee x \in F$ for all $x \in (A, F)$. Thus $a \in ((A, F), F)$. Therefore $A \subseteq ((A, F), F)$.
(3) From (2), we have $A \subseteq ((A, F), F)$. By (1), we get $(((A, F), F), F) \subseteq (A, F)$. Again by (2), we have $(A, F) \subseteq (((A, F), F), F)$. Therefore $(((A, F), F), F) = (A, F)$.
(4) Suppose $(A, F) = L$. Then $0 \in (A, F)$. Now $a = a \vee 0 \in F$ for all $a \in A$. Hence $a \in F$ for all $a \in A$. Therefore $A \subseteq F$. Conversely, assume that $A \subseteq F$. Let $x \in L$. Since F is a filter, we get $a \vee x \in F$ for all $a \in A \subseteq F$. Hence $x \in (A, F)$. Therefore $(A, F) = L$. □

In the case of filters of distributive lattices, we have the following result.

Proposition 3.5. *For any filters F, G_1 and G_2 of a distributive lattice L, the following conditions hold:*

(1) $(G, F) \cap ((G, F), F) = F$,
(2) $(G_1 \vee G_2, F) = (G_1, F) \cap (G_2, F)$,
(3) $((G_1 \cap G_2, F), F) = ((G_1, F), F) \cap ((G_2, F), F)$.

Proof. (1) Clearly $F \subseteq (A, F) \cap ((A, F), F)$. Conversely, let $x \in (A, F) \cap ((A, F), F)$. Then $x = x \vee x \in F$. Hence $(A, F) \cap ((A, F), F) \subseteq F$. Therefore $(A, F) \cap ((A, F), F) = F$.

(2) Clearly $(G_1 \vee G_2, F) \subseteq (G_1, F) \cap (G_2, F)$. Conversely, let $x \in (G_1, F) \cap (G_2, F)$. Let $c \in G_1 \vee G_2$. Then $c = i \wedge j$ for some $i \in G_1$ and $j \in G_2$. Now $x \vee c = x \vee (i \wedge j) = (x \vee i) \wedge (x \vee j) \in F$. Hence $x \in (G_1 \vee G_2, F)$. Therefore $(G_1, F) \cap (G_2, F) \subseteq (G_1 \vee G_2, F)$.
(3) Clearly $((G_1 \cap G_2, F), F) \subseteq ((G_1, F), F) \cap ((G_2, F), F)$. Conversely, let $x \in ((G_1, F), F) \cap ((G_2, F), F)$, $y \in (G_1 \cap G_2, F)$, $f \in G_1$ and $g \in G_2$. Clearly $f \vee g \in G_1 \cap G_2$. Now

$$\begin{aligned}
y \in (G_1 \cap G_2, F) &\Rightarrow y \vee (f \vee g) \in F && \text{since } f \vee g \in G_1 \cap G_2 \\
&\Rightarrow (y \vee f) \vee g \in F && \text{for all } g \in G_2 \\
&\Rightarrow y \vee f \in (G_2, F) && \\
&\Rightarrow x \vee (y \vee f) \in F && \text{since } x \in ((G_2, F), F) \\
&\Rightarrow (x \vee y) \vee f \in F && \text{for all } f \in G_1 \\
&\Rightarrow x \vee y \in (G_1, F) && \\
&\Rightarrow x \vee y \in (G_1, F) \cap ((G_1, F), F) = F && \text{since } x \in ((G_1, F), F) \\
&\Rightarrow x \in (y, F) && \text{for all } y \in (G_1 \cap G_2, F) \\
&\Rightarrow x \in ((G_1 \cap G_2, F), F)\,.
\end{aligned}$$

Hence $((G_1, F), F) \cap ((G_2, F), F) \subseteq ((G_1 \cap G_2, F), F)$. Thus $((G_1 \cap G_2, F), F) = ((G_1, F), F) \cap ((G_2, F), F)$. □

The following corollaries are direct consequences of the above results.

Corollary 3.6. *Let F be a filter of a distributive lattice L and $a, b \in L$. Then we have*

(1) $([a), F) = (a, F)$,
(2) $a \leq b$ *implies* $(a, F) \subseteq (b, F)$,
(3) $(a \wedge b, F) = (a, F) \cap (b, F)$,
(4) $((a \vee b, F), F) = ((a, F), F) \cap ((b, F), F)$,
(5) $(a, F) = L$ *if and only if* $a \in F$.

Corollary 3.7. *Let L be a distributive lattice and $a, b \in L$. Then we have*

(1) $([a))^\circ = (a)^\circ$,
(2) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$,
(3) $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,
(4) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,
(5) $(a)^\circ = L$ *if and only if* a *is dense.*

Lemma 3.8. *Let F be a filter and P be a prime F-filter of a distributive lattice L. Then $x \notin P$ implies $(x, F) \subseteq P$ for any $x \in L$.*

In the following definition, the notion of minimal prime F-filters is introduced in distributive lattices.

Definition 3.9. Let F be an arbitrary filter of a distributive lattice L. Suppose G is an F-filter and P a prime F-filter of L such that $G \subseteq P$. Then P is called a *minimal prime F-filter belonging to G* if there exists no prime F-filter Q such that $G \subseteq Q \subset P$. A minimal prime F-filter belonging to the filter F is simply called *minimal prime F-filter* of L.

In case of $F = D$, we call P the *minimal prime D-filter belonging to G*. A minimal prime D-filter belonging to the filter D is simply called *minimal prime D-filter.*

Proposition 3.10. *Let F be an arbitrary filter of a distributive lattice L. Suppose G is an F-filter and P a prime F-filter of L such that $G \subseteq P$. Then P is a minimal prime F-filter belonging to G if and only if $L - P$ is a maximal ideal with respect to the property that $(L - P) \cap G = \emptyset$.*

Proof. Let P be a prime F-filter of L such that $G \subseteq P$. Assume that P is a minimal prime F-filter belonging to G. Clearly $L - P$ is an ideal such that $(L - P) \cap G = \emptyset$. Suppose Q is another ideal such that $Q \cap G = \emptyset$ and $L - P \subseteq Q$. Hence $G \subseteq L - Q \subseteq P$. By the minimality of P, we get $L - Q = P$. Hence $L - P$ is a maximal ideal with respect to the property $(L - P) \cap G = \emptyset$.

Conversely, let $L-P$ be a maximal ideal with respect to the property $(L-P) \cap G = \emptyset$. Suppose Q is a prime F-filter of L such that $F \subseteq G \subseteq Q \subseteq P$. Clearly $L - Q$ is an ideal such that $L - P \subseteq L - Q$ and $(L - Q) \cap G = \emptyset$, which contradicts the maximality of $L - P$. Hence P is the minimal prime F-filter belonging to G. $\square$

Corollary 3.11. *Let F be a D-filter and P a prime D-filter of a distributive lattice L such that $F \subseteq P$. Then P is a minimal prime D-filter belonging to F if and only if $L - P$ is a maximal ideal with respect to the property that $(L - P) \cap F = \emptyset$.*

Theorem 3.12. *Let F be an arbitrary filter of a distributive lattice L. Suppose G is an F-filter and P a prime F-filter of L such that $G \subseteq P$. Then P is a minimal prime F-filter belonging to G if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in G$.*

Proof. Let G be an F-filter and P a prime F-filter of L such that $G \subseteq P$. Assume that P is a minimal prime F-filter belonging to G. Then $L - P$ is a maximal ideal with respect to the property that $(L - P) \cap G = \emptyset$. Let $x \in P$. Then clearly $L - P \subset (L - P) \vee (x]$. By the maximality of $L - P$, we get $\{(L - P) \vee (x]\} \cap G \neq \emptyset$. Choose $a \in \{(L - P) \vee (x]\} \cap G$. Then we get $a = y \vee x$ for some $y \in L - P$ and $a \in G$. Hence $y \vee x = a \in G$ where $y \notin P$.

Conversely, assume that the condition holds. Suppose P is not a minimal prime F-filter belonging to G. Then there exists a prime F-filter Q of L such that $F \subseteq G \subseteq Q \subset P$. Choose $x \in P - Q$. Then, by the assumed condition, there exists $y \notin P$ such that $x \vee y \in G \subseteq Q$. Since $x \notin Q$, it yields that $y \in Q \subset P$, which is a contradiction. Therefore P is a minimal prime F-filter belonging to G. $\square$

Replacing the filter F by the D-filter D, the following are direct consequences.

Corollary 3.13. *Let G be a D-filter and P a prime D-filter of a distributive lattice L such that $G \subseteq P$. Then P is a minimal prime D-filter belonging to G if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in G$.*

Corollary 3.14. *A prime D-filter P of a distributive lattice L is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$.*

Definition 3.15. Let F be a filter of a distributive lattice L. For any prime F-filter P of L, define the set $O_F(P)$ as $O_F(P) = \{x \in L \mid x \in (a, F) \text{ for some } a \notin P\}$.

For a prime D-filter P, we denote $O_D(P)$ by $O(P) = \{x \in L \mid x \in (a)^\circ$ for some $a \notin P\}$.

Lemma 3.16. *Let F be a filter of a distributive lattice L. For any prime F-filter P of L, $O_F(P)$ is an F-filter such that $O_F(P) \subseteq P$.*

Proof. Clearly $F \subseteq O_F(P)$. Let $x, y \in O_F(P)$. Then $x \in (a, F)$ and $y \in (b, F)$ for some $a \notin P$ and $b \notin P$. Hence $((a, F), F) \subseteq (x, F)$ and $((b, F), F) \subseteq (y, F)$. Thus $((a \vee b, F), F) = ((a, F), F) \cap ((b, F), F) \subseteq (x, F) \cap (y, F) = (x \wedge y, F)$. Hence $x \wedge y \in ((x \wedge y, F), F) \subseteq (((a \vee b, F), F), F) = (a \vee b, F)$. Since $a \vee b \notin P$, we get that $x \wedge y \in O_F(P)$. Again, let $x \in O_F(P)$ and $x \leq y$. Then $x \in (a, F)$ for some $a \notin P$. Since (a, F) is a filter, we get $y \in (a, F)$. Hence $y \in O_F(P)$. Therefore $O_F(P)$ is an F-filter of L. Let $x \in O_F(P)$. Then $x \in (a, F)$ for some $a \notin P$. Hence $x \vee a \in F \subseteq P$. Since P is prime, we get $x \in P$. Therefore $O_F(P) \subseteq P$. $\square$

Lemma 3.17. *Let F be a filter and P be a prime F-filter of a distributive lattice L. Then every minimal prime F-filter belonging to $O_F(P)$ is contained in P.*

Proof. Let Q be a minimal prime F-filter belonging to $O_F(P)$. Suppose $Q \nsubseteq P$. Choose $x \in Q - P$. Then there exists $y \notin Q$ such that $x \vee y \in O_F(P)$. Hence $x \vee y \in (a, F)$ for some $a \notin P$. Thus $y \vee (x \vee a) = (x \vee y) \vee a \in F \subseteq P$. Since $x \notin P, a \notin P$, and P is prime, we get $x \vee a \notin P$. Hence $y \in O_F(P) \subseteq Q$, which is a contradiction. Therefore $Q \subseteq P$. $\square$

In case of $F = D$, the above two results conclude that $O(P)$ is a D-filter such that $O(P) \subseteq P$ and every minimal prime D-filter belonging to $O(P)$ is contained in P.

Proposition 3.18. *Let F be a filter and P be a prime F-filter of a distributive lattice L. Then $O_F(P)$ is the intersection of all minimal prime F-filters contained in P.*

Proof. Let P be a prime F-filter of L. By Zorn's lemma, P contains a minimal prime F-filter. Let $\{S_\alpha\}_{\alpha \in \Delta}$ be the family of all minimal prime F-filters contained in P. Let $x \in O_F(P)$. Then there exists $a \notin P$ such that $x \in (a, F)$. Since each $S_\alpha \subseteq P$ and $a \notin P$, we get $a \notin S_\alpha$ for all $\alpha \in \Delta$. Since $x \vee a \in F \subseteq S_\alpha$ and $a \notin S_\alpha$ for all $\alpha \in \Delta$, we get $x \in S_\alpha$ for all $\alpha \in \Delta$. Hence $x \in \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Therefore $O_F(P) \subseteq \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Conversely, let $x \notin O_F(P)$. Consider $S = (L - P) \vee (x]$. Suppose $F \cap S \neq \emptyset$. Choose $a \in F \cap S$. Since $a \in S$, we get $a = t \vee x$ for some $t \in L - P$. Since $a \in F$, we get $t \vee x \in F$. Hence $x \in (t, F)$ where $t \notin P$. Thus $x \in O_F(P)$, which is a contradiction. Therefore $S \cap F = \emptyset$. Let M be the maximal ideal such that $S \subseteq M$ and $M \cap F = \emptyset$. Then $L - M$ is a minimal prime F-filter such that

$L - M \subseteq P$ and $x \notin L - M$ because of $x \in S \subseteq M$. Hence $x \notin \bigcap\limits_{\alpha \in \Delta} S_\alpha$. Therefore $\bigcap\limits_{\alpha \in \Delta} S_\alpha \subseteq O_F(P)$. ☐

The following corollaries are direct consequences of the above theorem.

Corollary 3.19. *Let P be a prime D-filter of a distributive lattice L. Then $O(P)$ is the intersection of all minimal prime D-filters contained in P.*

Corollary 3.20. *If P is a minimal prime D-filter of a distributive lattice L, then $O(P) = P$.*

Definition 3.21. Let F be an arbitrary filter of a distributive lattice L. Then two F-filters G_1 and G_2 of L are called *co-maximal* if $G_1 \vee G_2 = L$.

Theorem 3.22. *Let F be an arbitrary filter of a distributive lattice L. Then the following assertions are equivalent:*

(1) *every prime F-filter contains a unique minimal prime F-filter;*
(2) *for any prime F-filter P, $O_F(P)$ is prime;*
(3) *for any $a, b \in L$ with $a \vee b \in F, (a, F) \vee (b, F) = L$;*
(4) *for any $a, b \in L, (a, F) \vee (b, F) = (a \vee b, F)$;*
(5) *any two distinct minimal prime F-filters are co-maximal.*

Proof. (1) $\Rightarrow$ (2): Assume that every prime F-filter of L contains a unique minimal prime F-filter. Then by Proposition 3.18, we get that $O_F(P)$ is a prime F-filter.
(2) $\Rightarrow$ (3): Assume the condition (2). Let $a, b \in L$ be such that $a \vee b \in F$. Suppose $(a, F) \vee (b, F) \neq L$. Then there exists a maximal filter P such that $(a, F) \vee (b, F) \subseteq P$. Hence $(a, F) \subseteq P$ and $(b, F) \subseteq P$. Thus we get $a \notin O_F(P)$ and $b \notin O_F(P)$. Since $O_F(P)$ is prime, we get $a \vee b \notin O_F(P)$. Hence $F \nsubseteq O_F(P)$, which is a contradiction. Therefore $(a, F) \vee (b, F) = L$.
(3) $\Rightarrow$ (4): Assume that the condition (3) holds. Let $a, b \in L$. Clearly $(a, F) \vee (b, F) \subseteq (a \vee b, F)$. Conversely, let $x \in (a \vee b, F)$. Then we get $(x \vee a) \vee (x \vee b) = x \vee (a \vee b) \in F$. Hence by condition (3), we get $(x \vee a, F) \vee (x \vee b, F) = L$. Therefore $x \in (x \vee a, F) \vee (x \vee b, F)$. Hence $x = r \wedge s$ for some $r \in (x \vee a, F)$ and $s \in (x \vee b, F)$. Since $r \in (x \vee a, F)$, we get $r \vee x \in (a, F)$. Similarly, we get $s \vee x \in (b, F)$. Now

$$\begin{aligned} x &= x \vee x \\ &= x \vee (r \wedge s) \\ &= (x \vee r) \wedge (x \vee s) \in (a, F) \vee (b, F). \end{aligned}$$

Hence $(a \vee b, F) \subseteq (a, F) \vee (b, F)$. Therefore $(a, F) \vee (b, F) = (a \vee b, F)$ for all $a, b \in L$.
(4) $\Rightarrow$ (5): Assume the condition (4). Let P and Q be two distinct minimal prime F-filters of L. Choose $x \in P - Q$ and $y \in Q - P$. Since P and Q are minimal, there exist $x_0 \notin P$ and $y_0 \notin Q$ such that $x \vee x_0 \in F$ and $y \vee y_0 \in F$. Hence $x \vee x_0 \vee y \vee y_0 \in F$. Since $x_0 \notin P$ and $y \notin P$, we get $x_0 \vee y \notin P$ and hence we conclude $(x_0 \vee y, F) \subseteq P$. Similarly, we can obtain $(x \vee y_0, F) \subseteq Q$. By Corollary

3.6(5), we get

$$\begin{aligned} L &= (x \vee x_0 \vee y \vee y_0, F) \\ &= ((x \vee y_0) \vee (x_0 \vee y), F) \\ &= (x \vee y_0, F) \vee (x_0 \vee y, F) \qquad \text{by (4)} \\ &\subseteq P \vee Q \end{aligned}$$

which gives $P \vee Q = L$. Therefore P and Q are co-maximal.

(5) $\Rightarrow$ (1): Assume that any two distinct minimal prime F-filters of L are co-maximal. Let P be a prime F-filter of L. Suppose P contains two minimal prime F-filters, say Q_1 and Q_2. Then by condition (5), we get $L = Q_1 \vee Q_2 \subseteq P$, which is a contradiction. Therefore every prime F-filter contains a unique minimal prime F-filter. ☐

Corollary 3.23. *Following assertions are equivalent in a distributive lattice L:*

(1) *every prime D-filter contains a unique minimal prime D-filter;*
(2) *for any prime D-filter P, $O(P)$ is prime;*
(3) *for any $a, b \in L$ with $a \vee b \in D, (a)^{\circ} \vee (b)^{\circ} = L$;*
(4) *for any $a, b \in L, (a)^{\circ} \vee (b)^{\circ} = (a \vee b)^{\circ}$;*
(5) *any two distinct minimal prime D-filters are co-maximal.*

4. Congruences and prime D-filters

In this section, we first study a congruence in terms of ideals. Later, an equivalency is obtained between the minimal prime D-filters of a distributive lattice and its quotient algebra with respect to this congruence. The following proposition can be routinely verified.

Proposition 4.1. *Let I be an ideal of a distributive lattice L. For any $x, y \in L$, define a binary relation ψ_I on L by $(x, y) \in \psi_I$ if and only if $x \vee a = y \vee a$ for some $a \in I$. Then ψ_I is a congruence on L with I as a congruence class modulo ψ_I.*

For any distributive lattice L, it can be shown that the quotient algebra $L_{/\psi_I}$ is also a distributive lattice with respect to the following operations:

$$[x]_{\psi_I} \wedge [y]_{\psi_I} = [x \wedge y]_{\psi_I} \text{ and } [x]_{\psi_I} \vee [y]_{\psi_I} = [x \vee y]_{\psi_I}$$

where $[x]_{\psi_I}$ is the congruence class of x modulo ψ_I. It can be routinely verified that the mapping $\Psi \colon L \longrightarrow L_{/\psi_I}$ defined by $\Psi(x) = [x]_{\psi_I}$ is a homomorphism.

Lemma 4.2. *The following conditions hold in a distributive lattice L:*

(1) *If d is a dense element of L, then $[d]_{\psi_I}$ is a dense element of $L_{/\psi_I}$,*
(2) *If F is a D-filter of $L_{/\psi_I}$, then $\Psi^{-1}(F)$ is a D-filter of L,*
(3) *If P is a prime D-filter of $L_{/\psi_I}$, then $\Psi^{-1}(P)$ is a prime D-filter of L.*

Proof. (1). It is clear.

(2). Let F be a D-filter of $L_{/\psi_I}$. Clearly $\Psi^{-1}(F)$ is a filter of L. Let $x \in D \subseteq L$. Then by (1), we get that $[x]_{\psi_I}$ is a dense element of $L_{/\psi_I}$. Hence $\Psi(x) = [x]_{\psi_I} \in F$,

which implies $x \in \Psi^{-1}(F)$. Thus $D \subseteq \Psi^{-1}(F)$. Therefore $\Psi^{-1}(F)$ is a D-filter of L.
(3). By the nature of homomorphisms, it can be obtained easily. $\square$

Definition 4.3. Let I be an ideal of a distributive lattice L. For any filter F of L, define $\overline{F} = \{[x]_{\psi_I} \mid x \in F\}$.

By the nature of congruences of distributive lattices, it can be easily observed that $\overline{F}$ is a D-filter in $L_{/\psi_I}$ whenever F is a D-filter in L. In the following, we observe some properties of prime D-filters of a distributive lattice with respect to congruence classes modulo Ψ_I.

Proposition 4.4. *Let P be a prime D-filter and I an ideal of a distributive lattice L such that $P \cap I = \emptyset$. Then the following conditions hold:*

(1) *$x \in P$ if and only if $[x]_{\psi_I} \in \overline{P}$,*
(2) *$\overline{P} \cap \overline{I} = \emptyset$,*
(3) *If P is a prime D-filter of L, then $\overline{P}$ is a prime D-filter of $L_{/\psi_I}$.*

Proof. (1). Clearly $x \in P$ implies $[x]_{\psi_I} \in \overline{P}$. Conversely, let $[x]_{\psi_I} \in \overline{P}$. Then $[x]_{\psi_I} = [t]_{\psi_I}$ for some $t \in P$. Hence $(x, t) \in \psi_I$. Thus $x \vee a = t \vee a \in P$ for some $a \in I$. Since $P \cap I = \emptyset$, we get $a \notin P$. Since $x \vee a \in P$ and $a \notin P$, we must have $x \in P$.
(2). Suppose $\overline{P} \cap \overline{I} \neq \emptyset$. Then choose $[x]_{\psi_I} \in \overline{P} \cap \overline{I}$. Then by (1), we get $x \in P$ and $[x]_{\psi_I} \in \overline{I}$. Hence we can have the following:

$$\begin{aligned}
[x]_{\psi_I} \in \overline{I} &\Rightarrow [x]_{\psi_I} = [y]_{\psi_I} && \text{for some } y \in I \\
&\Rightarrow (x, y) \in \psi_I \\
&\Rightarrow x \vee a = y \vee a && \text{for some } a \in I \\
&\Rightarrow x \vee a \in I && \text{since } y \vee a \in I \\
&\Rightarrow x \vee a \in P \cap I && \text{since } x \in P
\end{aligned}$$

which is a contradiction to $P \cap I = \emptyset$. Therefore $\overline{P} \cap \overline{I} = \emptyset$.
(3). Since P is a filter of L, it is clear that $\overline{P}$ is a filter of $L_{/\psi_I}$. Let $[x]_{\psi_I} \in \overline{D}$. Then, we get $x \in D \subseteq P$. Hence $[x]_{\psi_I} \in \overline{P}$. Therefore $\overline{P}$ is a D-filter of $L_{/\psi_I}$. Since P is a proper filter of L, by (1), we get that $\overline{P}$ is a proper filter in $L_{/\psi_I}$. Let $[x]_{\psi_I}, [y]_{\psi_I} \in L_{/\psi_I}$. Then we have

$$\begin{aligned}
[x]_{\psi_I} \vee [y]_{\psi_I} \in \overline{P} &\Rightarrow [x \vee y]_{\psi_I} \in \overline{P} \\
&\Rightarrow x \vee y \in P && \text{from (1)} \\
&\Rightarrow x \in P && \text{or } y \in P \\
&\Rightarrow [x]_{\psi_I} \in \overline{P} && \text{or } [y]_{\psi_I} \in \overline{P}
\end{aligned}$$

Therefore $\overline{P}$ is a prime D-filter in $L_{/\psi_I}$. $\square$

Proposition 4.5. *Let I be an ideal of a distributive lattice L. Then the mapping $P \mapsto \overline{P}$ is an order isomorphism of the set of all prime D-filters of L disjoint from I onto the set of all prime D-filters of $L_{/\psi_I}$.*

Proof. Let P and Q be two prime D-filters of L such that $P \cap I = \emptyset$ and $Q \cap I = \emptyset$. Then by Proposition 4.4(1), we get that $P \subseteq Q \Leftrightarrow \overline{P} \subseteq \overline{Q}$. Let P be a prime D-filter of L such that $P \cap I = \emptyset$. Then by Proposition 4.4(3), we get that $\overline{P}$ is a prime D-filter of $L_{/\psi_I}$. Let R be a prime D-filter of $L_{/\psi_I}$. Consider $P = \{x \in L \mid [x]_{\psi_I} \in R\}$. Since R is a D-filter of $L_{/\psi_I}$, we get that P is a D-filter of L. Let $x, y \in L$ be such that $x \vee y \in P$. Then $[x]_{\psi_I} \vee [y]_{\psi_I} = [x \vee y]_{\psi_I} \in R$. Since R is prime, we get either $[x]_{\psi_I} \in R$ or $[y]_{\psi_I} \in R$. Hence either $x \in P$ or $y \in P$. Therefore P is a prime D-filter of L. Clearly $\overline{P} = R$. Suppose $P \cap I \neq \emptyset$. Choose $a \in P \cap I$. Then $[a]_{\psi_I} \in R$ and $a \in I$. Let $[y]_{\psi_I} \in L_{/\psi_I}$ be an arbitrary element. Now for any $a \in I$ and $y \in L$, we have

$$\begin{aligned}
a \vee y = a \vee y \vee a &\Rightarrow (y, y \vee a) \in \psi_I \\
&\Rightarrow [y]_{\psi_I} = [y \vee a]_{\psi_I} \\
&\Rightarrow [y]_{\psi_I} = [y]_{\psi_I} \vee [a]_{\psi_I} \in R \quad \text{since } R \text{ is a filter} \\
&\Rightarrow [y]_{\psi_I} \in R
\end{aligned}$$

Hence $L_{/\psi_I} \subseteq R$, which is a contradiction. Thus $P \cap I = \emptyset$. Therefore $P \mapsto \overline{P}$ is an order isomorphism from the set of all prime D-filters of L which are disjoint from I onto the set of all prime D-filters of $L_{/\psi_I}$. $\square$

The following corollary is a direct consequence of the above theorem.

Corollary 4.6. *Let L be a distributive lattice. Then the above map $P \mapsto \overline{P}$ induces a one-to-one correspondence between the set of all minimal prime D-filters of L which are disjoint from I and the set of all minimal prime D-filters of $L_{/\psi_I}$.*

Theorem 4.7. *Let I be an ideal of a distributive lattice L. Then any two distinct minimal prime D-filters of L are co-maximal if and only if any two distinct minimal prime D-filters of $L_{/\psi_I}$ are co-maximal.*

Proof. Assume that any two distinct minimal prime D-filters of L are co-maximal. Let P_1, P_2 be two distinct minimal prime D-filters of $L_{/\psi_I}$. Then by the above corollary, there exist two minimal prime D-filters Q_1 and Q_2 of L such that $Q_1 \cap I = \emptyset$ and $Q_2 \cap I = \emptyset$. Also $\overline{Q}_1 = P_1$ and $\overline{Q}_2 = P_2$. Since P_1 and P_2 are distinct, we get that Q_1 and Q_2 are distinct. By the assumption, we get that $Q_1 \vee Q_2 = L$. Hence for any $x \in L$, we can have

$$x = x_1 \wedge x_2 \qquad \text{where } x_1 \in Q_1 \text{ and } x_2 \in Q_2$$

Since $x_1 \in Q_1$, we get $[x_1]_{\psi_I} \in \overline{Q}_1 = P_1$. Similarly, we get $[x_2]_{\psi_I} \in \overline{Q}_2 = P_2$. Hence we get

$$[x]_{\psi_I} = [x_1 \wedge x_2]_{\psi_I} = [x_1]_{\psi_I} \wedge [x_2]_{\psi_I} \in P_1 \vee P_2.$$

Thus, for any $x \in L$, we obtained $[x]_{\psi_I} \in P_1 \vee P_2$. Hence $P_1 \vee P_2 = L_{/\psi_I}$.

Conversely, assume that any two distinct minimal prime D-filters of $L_{/\psi_I}$ are co-maximal. Let P be a prime D-filter of L. Suppose P contains two distinct minimal prime D-filters, say P_1 and P_2. Consider $S = L - P$. Then clearly S is an ideal of L such that $P_1 \cap S = \emptyset = P_2 \cap S$. Then by Corollary 4.6, we get that $\overline{P}_1$ and $\overline{P}_2$ are distinct minimal prime D-filters of $L_{/\psi_I}$ such that $\overline{P}_1, \overline{P}_2 \subseteq \overline{P}$. Thus $\overline{P}$ is containing two distinct minimal prime D-filters of $L_{/\psi_I}$, which is a contradiction.

Hence P contains a unique minimal prime D-filter. Therefore, by Corollary 3.23, any two distinct minimal prime D-filters of L are co-maximal. □

5. The space of prime D-filters

In this section, some topological properties of the space of all prime D-filters of a distributive lattice are studied. In the following proposition, we first observe the generalization of the Stone's theorem of prime ideals to the case of prime D-filters of distributive lattices:

Proposition 5.1. *Every D-filter of a distributive lattice is the intersection of all prime D-filters containing it.*

Proof. Let F be a D-filter of a distributive lattice L. Consider

$$F_0 = \bigcap\{P \mid P \text{ is a prime } D\text{-filter such that } F \subseteq P\}$$

Clearly $F \subseteq F_0$. Conversely, let $x \notin F$. Then there exists a prime filter P such that $F \subseteq P$ and $x \notin P$. As F a D-filter, P is also a D-filter. Hence $x \notin F_0$, which completes the proof. □

Corollary 5.2. *The intersection of all prime D-filters of a distributive lattice is equal to D.*

Corollary 5.3. *Let L be a distributive lattice and $x \in L$. If $x \notin D$, then there exists a prime D-filter P of L such that $x \notin P$.*

Let us denote the set of all prime D-filters of a lattice L by $\mathrm{Spec}_F^D(L)$. For any $x \in L$, define $\mathfrak{K}(x) = \{P \in \mathrm{Spec}_F^D(L) \mid x \notin P\}$. Then we have the following result whose proof is straightforward.

Lemma 5.4. *Let L be a distributive lattice and x, $y \in L$. Then the following conditions hold:*

(1) $\bigcup\limits_{x\in L} \mathfrak{K}(x) = \mathrm{Spec}_F^D(L)$,
(2) $\mathfrak{K}(x) \cap \mathfrak{K}(y) = \mathfrak{K}(x \vee y)$,
(3) $\mathfrak{K}(x) \cup \mathfrak{K}(x) = \mathfrak{K}(x \wedge y)$,
(4) $\mathfrak{K}(x) = \emptyset$ *if and only if* $x \in D$,
(5) $\mathfrak{K}(0) = \mathrm{Spec}_F^D(L)$.

From the above result, it can be easily observed that the collection $\{\mathfrak{K}(x) \mid x \in L\}$ forms a base for a topology on $\mathrm{Spec}_F^D(L)$. Under this topology, we have the following result.

Theorem 5.5. *The following conditions hold in a distributive lattice L:*

(1) *For any $x \in L, \mathfrak{K}(x)$ is compact in* $\mathrm{Spec}_F^D(L)$,
(2) *Let C be a compact open subset of* $\mathrm{Spec}_F^D(L)$*. Then $C = \mathfrak{K}(x)$ for some $x \in L$,*
(3) $\mathrm{Spec}_F^D(L)$ *is a T_0-space,*

(4) *The map $x \mapsto \mathfrak{K}(x)$ is an anti-homomorphism from L onto the lattice of all compact open*

subsets of $\mathrm{Spec}_F^D(L)$.

Proof. (1) Let $x \in L$. Let $A \subseteq L$ be such that $\mathfrak{K}(x) \subseteq \bigcup_{y \in A} \mathfrak{K}(y)$. Let F be a D-filter generated by the set A. Suppose $x \notin F$. Then there exists a prime D-filter P such that $F \subseteq P$ and $x \notin P$. Since $A \subseteq F \subseteq P$, we get $P \notin \mathfrak{K}(y)$ for all $y \in A$. Since $x \notin P$, we get $P \in \mathfrak{K}(x)$, which is a contradiction. Hence $x \in F$. So we can write $x = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ for some $a_1, a_2, \ldots, a_n \in A$ and $n \in \mathbb{N}$. Then, we get

$$\mathfrak{K}(x) = \mathfrak{K}(\bigwedge_{i=1}^{n} a_i) = \bigcup_{i=1}^{n} \mathfrak{K}(a_i)$$

which is a finite subcover for $\mathfrak{K}(x)$. Therefore $\mathfrak{K}(x)$ is compact.

(2) Let C be a compact open subset of $\mathrm{Spec}_F^D(L)$. Since C is open, we get $C = \bigcup_{a \in A} \mathfrak{K}(a)$ for some $A \subseteq L$. Since C is compact, there exist $a_1, a_2, \ldots, a_n \in A$ such that

$$C = \bigcup_{i=1}^{n} \mathfrak{K}(a_i) = \mathfrak{K}(\bigwedge_{i=1}^{n} a_i).$$

Therefore $C = \mathfrak{K}(x)$ for some $x \in L$.

(3) Let P and Q be two distinct prime D-filters of L. Without loss of generality, assume that $P \nsubseteq Q$. Choose $x \in L$ such that $x \in P$ and $x \notin Q$. Hence $P \notin \mathfrak{K}(x)$ and $Q \in \mathfrak{K}(x)$. Therefore $\mathrm{Spec}_F^D(L)$ is a T_0-space.

(4) It can be obtained from (1), (2) and by the above lemma. □

Proposition 5.6. *Following assertions are equivalent in a distributive lattice L:*

(1) $\mathrm{Spec}_F^D(L)$ *is a Hausdorff space;*

(2) *for each $P \in \mathrm{Spec}_F^D(L)$, P is the unique member of* $\mathrm{Spec}_F^D(L)$ *such that $O(P) \subseteq P$;*

(3) *every prime D-filter is maximal;*

(4) *every prime D-filter is minimal.*

Proof. (1) ⇒ (2): Assume that $\mathrm{Spec}_F^D(L)$ is a Hausdorff space. Let $P \in \mathrm{Spec}_F^D(L)$. Clearly $O(P) \subseteq P$. Suppose $Q \in \mathrm{Spec}_F^D(L)$ such that $Q \neq P$ and $O(P) \subseteq Q$. Since $\mathrm{Spec}_F^D(L)$ is Hausdorff, there exists $x, y \in L$ such that $P \in \mathfrak{K}(x)$, $Q \in \mathfrak{K}(y)$ and $\mathfrak{K}(x \vee y) = \mathfrak{K}(x) \cap \mathfrak{K}(y) = \emptyset$. Hence $x \notin P, y \notin Q$ and $x \vee y \in D$. Therefore $y \in O(P) \subseteq Q$, which is a contradiction to that $y \notin Q$. Hence $P = Q$. Therefore P is the unique member of $\mathrm{Spec}_D F(L)$ such that $O(P) \subseteq P$.

(2) ⇒ (3): Assume the condition (2). Let P be a prime D-filter of L. Suppose P is not minimal. Let Q be a prime D-filter in L such that $Q \subseteq P$. Hence $O(Q) \subseteq Q \subseteq P$, which is a contradiction to the assumption. Therefore P is a minimal prime D-filter of L.

(3) ⇒ (4): Since every maximal D-filter is prime, it is clear.

(4) ⇒ (1): Assume that every prime D-filter is minimal. Let P and Q be two distinct

elements of $\mathrm{Spec}_F^D(L)$. Hence $O(Q) \nsubseteq P$. Choose $x \in O(Q)$ such that $x \notin P$. Since $x \in O(Q)$, there exists $y \notin Q$ such that $x \in (y)^{\circ}$. Hence $x \vee y \in D$. Thus it yields, $P \in \mathfrak{K}(x)$, $Q \in \mathfrak{K}(y)$. Since $x \vee y \in D$, we get that $\mathfrak{K}(x) \cap \mathfrak{K}(y) = \mathfrak{K}(x \vee y) = \emptyset$. Therefore $\mathrm{Spec}_F^D(L)$ is Hausdorff. □

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