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A NEW LOOK AT AN OLD COMPARISON THEOREM

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Abstract. We present an integral comparison theorem which guarantees the global existence of a solution of the generalized Riccati equation on the given interval $[a,b)$ when it is known that certain majorant Riccati equation has a global solution on $[a,b)$.

1. Introduction

The purpose of this note is to revisit an old and apparently forgotten comparison theorem due to Stafford and Heidel [7] which relates the existence of global solutions of a pair of scalar Riccati equations

$$r' + r^2 + q_1(t) = 0 \tag{1.1}$$

and

$$s' + s^2 + q_2(t) = 0\,, \tag{1.2}$$

where q_1 and q_2 are nonnegative continuous functions on $[a,b)$, $0 < a < b \le \infty$, and show that it can be extended and generalized to differential equations of the form

$$r' + \alpha p(t)|r|^{1+\frac{1}{\alpha}} + q_1(t) = 0 \tag{1.3}$$

and

$$s' + \alpha p(t)|s|^{1+\frac{1}{\alpha}} + q_2(t) = 0\,, \tag{1.4}$$

where $\alpha > 0$ is constant, p is a continuous function on $[a,b)$ with $p(t) > 0$ for $t \ge a$ and q_1 and q_2 are as before.

Generalized Riccati equations (1.3) and (1.4) arise, for example, in the study of nonoscillatory solutions of nonlinear planar differential systems of the form

$$x' = p(t)\varphi_{1/\alpha}(y)\,, \quad y' + q(t)\varphi_\alpha(x) = 0\,, \tag{1.5}$$

where $\varphi_\alpha(\xi)$ denotes the signed power function defined by $\varphi_\alpha(\xi) = |\xi|^\alpha \operatorname{sgn} \xi$ for $\xi \ne 0$ and $\varphi_\alpha(0) = 0$ (see [5] and [6]).

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Stafford-Heidel's result says that if

$$\int_a^t \tau^2 q_2(\tau) d\tau \le \int_a^t \tau^2 q_1(\tau)\, d\tau\,, \quad t \ge a\,, \tag{1.6}$$

and (1.1) has a solution r which exists on the whole interval $[a, b)$ and satisfies $ar(a) < 1$, then equation (1.2) also has a solution on $[a, b)$. This comparison result can be applied in the theory of linear differential equations of the second order to get criteria for disconjugacy and nonoscillation of such equations and complements in some sense the well-known Hille-Wintner and Levin's comparison theorems.

While the above result concerning the "classical" Riccati equations (1.1) and (1.2) has been improved and refined in various directions by several authors including Travis [8] and Erbe [2], we are not aware of any attempts to generalize it to nonlinear differential equations (1.3) and (1.4) which play an important role in the qualitative theory of nonlinear systems of the form (1.5) and the so called "half-linear" differential equations of the second order. For an excellent survey of the results on half-linear equations obtained through generalized Riccati equations we refer the reader to the monograph [1].

Our approach is different from the method used in [7] and relies on the Schauder-Tychonoff fixed point theorem for locally convex spaces applied to a suitable integral operator associated with the modified version of the Riccati equation under study.

2. Main result

Define $P(t) = \int_0^t p(\tau) d\tau$, $t \ge a$, and use the substitutions $u = P(t)^\alpha r$ and $v = P(t)^\alpha s$ to transform equations (1.3) and (1.4) into

$$P(t)u' = \alpha p(t)u - \alpha p(t)|u|^{1+\frac{1}{\alpha}} - P(t)^{\alpha+1} q_1(t) \tag{2.1}$$

and

$$P(t)v' = \alpha p(t)v - \alpha p(t)|v|^{1+\frac{1}{\alpha}} - P(t)^{\alpha+1} q_2(t)\,, \tag{2.2}$$

respectively. Integration of (2.1) from a to t yields

$$\begin{aligned} P(t)u(t) = P(a)u(a) + \int_a^t \left[(\alpha+1)u(\tau) - \alpha |u(\tau)|^{1+\frac{1}{\alpha}}\right] p(\tau)\, d\tau \\ - \int_a^t P(\tau)^{\alpha+1} q_1(\tau)\, d\tau\,. \end{aligned} \tag{2.3}$$

Analogous integral equation can be obtained also for equation (2.2).

We note that if (1.3) has a solution on $[a, b)$ such that $P(a)^\alpha r(a) \le 1$, then $u(t) \le 1$ for all $t \ge a$. Indeed, if $u(t_1) \ge 1$ for some $t_1 \ge a$, then from (2.1) it follows that $u'(t_1) \le 0$.

We are now ready to state and prove our main result.

Theorem 2.1. *If* (1.3) *has a solution* r *on* $[a,b)$ *such that* $P(a)^\alpha r(a) < 1$ *and*

$$\int_a^t P(\tau)^{\alpha+1} q_2(\tau)\, d\tau \le \int_a^t P(\tau)^{\alpha+1} q_1(\tau)\, d\tau\,, \quad t \ge a\,, \tag{2.4}$$

then (1.4) *has a solution* s *on* $[a,b)$ *such that* $r(t) \le s(t) \le P(t)^{-\alpha}$ *for* $t \ge a$.

Proof. Suppose that (1.3) has a solution r on $[a,b)$ satisfying $P(a)^\alpha r(a) < 1$. Then the function $u(t) = P(t)^\alpha r(t)$ solves integral equation (2.3) and, moreover, $u(t) \le 1$ for all $t \ge a$. Define

$$\mathcal{V} = \{v \in C[a,b) : u(t) \le v(t) \le 1,\ t \ge a\}$$

where $C[a,b)$ is understood to be a locally convex space of continuous functions on $[a,b)$ with the topology of uniform convergence on any compact subintervals of $[a,b)$. Then $\mathcal{V}$ is a closed convex subset of the space $C[a,b)$. Choose v_0 such that $u(a) \le v_0 \le 1$ and consider the integral operator $\mathcal{F}\colon \mathcal{V} \to C[a,b)$ defined by

$$P(t)(\mathcal{F}v)(t) = P(a)v_0 + \int_a^t \big[(\alpha+1)v(\tau) - \alpha|v(\tau)|^{1+\frac{1}{\alpha}}\big]p(\tau)\, d\tau - \int_a^t P(\tau)^{\alpha+1} q_2(\tau)\, d\tau\,.$$

If $v \in \mathcal{V}$, then since the function $G(\xi) := (\alpha+1)\xi - \alpha|\xi|^{1+\frac{1}{\alpha}}$ is increasing for $\xi \le 1$ and the inequality (2.4) holds, we obtain

$$\begin{aligned} P(t)(\mathcal{F}v)(t) \ge P(a)u(a) &+ \int_a^t \big[(\alpha+1)u(\tau) - \alpha|u(\tau)|^{1+\frac{1}{\alpha}}\big]p(\tau)\, d\tau \\ &- \int_a^t P(\tau)^{\alpha+1} q_1(\tau)\, d\tau = P(t)u(t) \end{aligned}$$

and

$$\begin{aligned} P(t)(\mathcal{F}v)(t) &\le P(a)v_0 + \int_a^t p(\tau)\, d\tau = P(a)v_0 + P(t) - P(a) \\ &\le P(a) + P(t) - P(a) = P(t) \end{aligned}$$

for all $t \in [a,b)$. This shows that $\mathcal{F}$ maps $\mathcal{V}$ into itself. It can be shown routinely that the operator $\mathcal{F}$ is continuous since it is easily checked that if a sequence $\{v_n(t)\}$ in $\mathcal{V}$ converges to $v(t)$ as $n \to \infty$ uniformly on compact subintervals of $[a,b)$, then exactly so does the sequence $\{\mathcal{F}v_n(t)\}$ to $\mathcal{F}v(t)$. The set $\mathcal{F}(\mathcal{V})$ is relatively compact in the topology of $C[a,b)$ because it is locally uniformly bounded and locally equicontinuous on $[a,b)$.

Thus, by the Schauder-Tychonoff fixed point theorem, there exists an element $v \in \mathcal{V}$ which satisfies $v(t) = (\mathcal{F}v)(t)$, or, equivalently,

$$\begin{aligned} P(t)v(t) = P(a)v_0 &+ \int_a^t \big[(\alpha+1)v(\tau) - \alpha|v(\tau)|^{1+\frac{1}{\alpha}}\big]p(\tau)\, d\tau \\ &- \int_a^t P(\tau)^{\alpha+1} q_2(\tau)\, d\tau\,. \end{aligned} \tag{2.5}$$

Define $s(t) = P(t)^{-\alpha} v(t)$. Then s is a solution of the integral equation

$$P(t)^{\alpha+1} s(t) = P(a)^{\alpha+1} v_0 + \int_a^t \big[(\alpha+1) P(\tau)^\alpha s(\tau) - \alpha P(\tau)^{\alpha+1} |s(\tau)|^{1+\frac{1}{\alpha}}\big] p(\tau)\, d\tau - \int_a^t P(\tau)^{\alpha+1} q_2(\tau)\, d\tau\,. \tag{2.6}$$

Differentiating (2.6) and dividing by $P(t)^{\alpha+1}$ shows that s is the solution of Riccati equation (1.4). The proof is complete. □

Corollary 2.2. *If*

$$\int_a^t P(\tau)^{\alpha+1} q_2(\tau)\, d\tau \le \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} P(t,a) \quad \text{for all} \quad t \ge a\,, \tag{2.7}$$

where $P(t,a) = \int_a^t p(\tau)\, d\tau$, *then* (1.4) *has a solution on* $[a,b)$.

Proof. Let

$$q_1(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}}\,.$$

Then equation (1.3) has the exact solution

$$r(t) = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} P(t)^{-\alpha}$$

which clearly satisfies $P(a)^\alpha r(a) < 1$. The assertion now follows from Theorem 2.1 □

Remark 2.3. Integral condition (2.7) can be rewritten as

$$\frac{1}{P(t,a)} \int_a^t p(\tau) \Big(\frac{\alpha}{\alpha+1}\Big)^{-\alpha-1} \frac{P(\tau)^{\alpha+1} q_2(\tau)}{p(\tau)}\, d\tau \le 1$$

and interpreted as the fact that the weighted average (with weight $p(t)$) of the ratio of the coefficient $q_2(t)$ of equation (1.4) and the coefficient $\big(\frac{\alpha}{\alpha+1}\big)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}}$ of the globally solvable Riccati equation associated with the nonoscillatory Euler-type half-linear differential equation of the second order, does not exceed 1.

Example 2.4. Consider the equation (1.4) where

$$q_2(t) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{p(t)}{P(t)^{\alpha+1}} \big[1 - \sin\big(P(t,a)\big)\big]\,.$$

Then the integral condition (2.7) is clearly satisfied and so by Corollary 2.2 equation (1.4) has a global solution on $[a,b)$.

Example 2.5. Let $b = \infty$ and

$$q_2(t) = \frac{\gamma p(t)}{P(t)^{\alpha+1}} \big[1 - f(P(t,a))\big]\,,$$

where $f(t) \le 1$ is a continuous ω-periodic function with mean value zero, i.e. $\int_0^\omega f(\tau)\, d\tau = 0$. If $\gamma \le \big(\alpha/(\alpha+1)\big)^{\alpha+1}$, then the assertion of Corollary 2.1 holds.

Notice that if f is an ω-periodic function with mean value zero, then

$$\int_a^t p(\tau) f\big(P(\tau,a)\big)\,d\tau = \int_0^{P(t,a)} f(y)\,dy = g\big(P(t,a)\big) - g(0)$$

for some ω-periodic function g. Without loss of generality we may assume that $g(0) = \max\{g(\tau),\ \tau \in [0,\omega]\}$ so that $g\big(P(t,a)\big) - g(0) \le 0$.

Remark 2.6. Half-linear differential equations of the second order with periodic coefficients (or, more generally, coefficients having mean values) have recently been studied in [3] and [4]. Our Example 2.5 indicates that Corollary 2.2 offers new perspectives in this study.

A closer look at the proof of Theorem 2.1 shows that we can relax the nonnegativity condition imposed on q_2 and improve the theorem as follows.

Theorem 2.7. *If equation* (1.3) *has a solution r on $[a,b)$, $0 < a < b \le \infty$, and there exists a constant v_0 such that the inequality*

$$\left| P(a)v_0 - \int_a^t P(\tau)^{\alpha+1} q_2(\tau)\,d\tau \right| \le -P(a)^{\alpha+1} r(a) + \int_a^t P(\tau)^{\alpha+1} q_1(\tau)\,d\tau \tag{2.8}$$

holds on $[a,b)$, then equation (1.4) *has a solution s on $[a,b)$ such that $P(a)^\alpha s(a) = v_0$.*

The proof is similar to the proof of Theorem 2.1 and is omitted.

Corollary 2.8. *If* (1.3) *has a solution r on $[a,b)$ such that $P(a)^\alpha r(a) \le 1$ and*

$$\left| \int_a^t P(\tau)^{\alpha+1} q_2(\tau)\,d\tau \right| \le \int_a^t P(\tau)^{\alpha+1} q_1(\tau)\,d\tau \tag{2.9}$$

holds on $[a,b)$, then equation (1.4) *has a solution s on $[a,b)$ such that $r(t) \le s(t) \le P(t)^{-\alpha}$ for $t \ge a$.*

Proof. Choose v_0 so that $P(a)^\alpha r(a) \le v_0 \le 1$. The assertion then follows from Theorem 2.7. $\square$

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