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INVOLUTIVITY DEGREE OF A DISTRIBUTION AT SUPERDENSITY POINTS OF ITS TANGENCIES

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Abstract.

Let $\Phi_1, \dots, \Phi_{k+1}$ (with $k \geq 1$) be vector fields of class C^k in an open set $U \subset \mathbb{R}^{N+m}$, let $\mathcal{M}$ be a N-dimensional C^k submanifold of U and define

$$\mathcal{T} := \{z \in \mathcal{M} : \Phi_1(z), \dots, \Phi_{k+1}(z) \in T_z\mathcal{M}\}$$

where $T_z\mathcal{M}$ is the tangent space to $\mathcal{M}$ at z. Then we expect the following property, which is obvious in the special case when z_0 is an interior point (relative to $\mathcal{M}$) of $\mathcal{T}$:

If $z_0 \in \mathcal{M}$ is a $(N+k)$-density point (relative to $\mathcal{M}$) of $\mathcal{T}$ then all the iterated Lie brackets of order less or equal to k

$$\Phi_{i_1}(z_0),\ [\Phi_{i_1}, \Phi_{i_2}](z_0),\ [[\Phi_{i_1}, \Phi_{i_2}], \Phi_{i_3}](z_0),\ \dots \qquad (h, i_h \leq k+1)$$

belong to $T_{z_0}\mathcal{M}$.

Such a property has been proved in [9] for $k = 1$ and its proof in the case $k = 2$ is the main purpose of the present paper. The following corollary follows at once:

Let $\mathcal{D}$ be a C^2 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and $\mathcal{M}$ be a N-dimensional C^2 submanifold of U. Moreover let $z_0 \in \mathcal{M}$ be a $(N+2)$-density point of the tangency set $\{z \in \mathcal{M} \,|\, T_z\mathcal{M} = \mathcal{D}(z)\}$. Then $\mathcal{D}$ must be 2-involutive at z_0, i.e., for every family $\{X_j\}_{j=1}^N$ of class C^2 in a neighborhood $V \subset U$ of z_0 which generates $\mathcal{D}$ one has

$$X_{i_1}(z_0), [X_{i_1}, X_{i_2}](z_0), [[X_{i_1}, X_{i_2}], X_{i_3}](z_0) \in T_{z_0}\mathcal{M}$$

for all $1 \leq i_1, i_2, i_3 \leq N$.

1. Introduction

Let $\Phi_1, \dots, \Phi_{k+1}$ (with $k \geq 1$) be vector fields of class C^k in an open set $U \subset \mathbb{R}^{N+m}$ (with $N, m \geq 1$), let $\mathcal{M}$ be a N-dimensional C^k submanifold of U and define

$$\mathcal{T} := \{z \in \mathcal{M} : \Phi_1(z), \dots, \Phi_{k+1}(z) \in T_z\mathcal{M}\}$$

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where $T_z\mathcal{M}$ is the tangent space to $\mathcal{M}$ at z. One has the following obvious property: *If z_0 is an interior point (relative to $\mathcal{M}$) of $\mathcal{T}$, then all the iterated Lie brackets of order less or equal to k*

$$\Phi_{i_1}(z_0),\ [\Phi_{i_1},\Phi_{i_2}](z_0),\ [[\Phi_{i_1},\Phi_{i_2}],\Phi_{i_3}](z_0),\ \dots \qquad (h, i_h \leq k+1)$$

belong to $T_{z_0}\mathcal{M}$.

With reference to this property, we are interested in understanding whether it remains true when we substitute the hypothesis that z_0 is an internal point of $\mathcal{T}$ with the assumption that it is a point of sufficiently high density of $\mathcal{T}$. In this regard, for the convenience of the reader, we recall that $z_0 \in \mathcal{M}$ is said to be a $(N+h)$-density point of a set $\mathcal{E} \subset \mathcal{M}$ (relative to $\mathcal{M}$) if $h \geq 0$ and

$$\mathcal{H}^N(B_{\mathcal{M}}(z_0,r) \setminus \mathcal{E}) = o(r^{N+h}) \qquad (\text{as } r \to 0+)$$

where $B_{\mathcal{M}}(z_0,r) \subset \mathcal{M}$ is the metric ball of radius r centered at z_0, compare Section 2 below. According to this definition, roughly speaking, we can say that: the larger h, the higher the concentration of $\mathcal{E}$ at z_0 (and the more z_0 will resemble an interior point of $\mathcal{E}$). Observe that, in the special case when $m = 0$ and $\mathcal{M} = \mathbb{R}^N$, the point z_0 is a N-density point of $\mathcal{E}$ if and only if it is a point of Lebesgue density of $\mathcal{E}$, that is $\mathcal{L}^N(B_{\mathbb{R}^N}(z_0,r) \cap E)/\mathcal{L}^N(B_{\mathbb{R}^N}(z_0,r)) \to 1$ (as $r \to 0+$).

In [9] we have proved the following result answering "yes" to the question above, when $k = 1$.

Theorem 1.1 ([9])**.** *Given an open set $U \subset \mathbb{R}^{N+m}$, consider Φ_1, $\Phi_2 \in C^1(U,\mathbb{R}^{N+m})$ and a N-dimensional C^1 submanifold $\mathcal{M}$ of U. Moreover define*

$$\mathcal{T} := \left\{z \in \mathcal{M} : \Phi_1(z), \Phi_2(z) \in T_z\mathcal{M}\right\}$$

and assume that $z_0 \in \mathcal{M}$ is a $(N+1)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$). Then one has $\Phi_1(z_0), \Phi_2(z_0), [\Phi_1,\Phi_2](z_0) \in T_{z_0}\mathcal{M}$.

The main purpose of this work is to prove that also for $k = 2$ the answer to the previous question is affirmative. More precisely one has

Theorem 1.2. *Given an open set $U \subset \mathbb{R}^{N+m}$, consider Φ_1, Φ_2, $\Phi_3 \in C^2(U,\mathbb{R}^{N+m})$ and a N-dimensional C^2 submanifold $\mathcal{M}$ of U. Moreover define*

$$\mathcal{T} := \left\{z \in \mathcal{M} : \Phi_1(z), \Phi_2(z), \Phi_3(z) \in T_z\mathcal{M}\right\}$$

and assume that $z_0 \in \mathcal{M}$ is a $(N+2)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$). Then one has $\Phi_{i_1}(z_0), [\Phi_{i_1},\Phi_{i_2}](z_0), [[\Phi_{i_1},\Phi_{i_2}],\Phi_{i_3}](z_0) \in T_{z_0}\mathcal{M}$ for all $1 \leq i_1, i_2, i_3 \leq 3$.

Now, in order to better understand the continuation of this introduction, we will recall some definitions and some well-known facts. First of all, let N, m, k be positive integers and recall that a C^k distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ is a map $\mathcal{D}$ assigning a N-dimensional vector subspace $\mathcal{D}(z)$ of $\mathbb{R}^{N+m}$ to each point $z \in U$ and satisfying the following property: If $z \in U$ then there exist a neighborhood $V^{(z)} \subset U$ of z and a family $\{X_i^{(z)}\}_{i=1}^N \subset C^k(V^{(z)},\mathbb{R}^{N+m})$ which generates $\mathcal{D}$ in $V^{(z)}$, i.e., such that $\{X_1^{(z)}(z'),\dots,X_N^{(z)}(z')\}$ is a basis of $\mathcal{D}(z')$ for all $z' \in V^{(z)}$. The distribution $\mathcal{D}$ is said to be k-involutive at $z \in U$ if all the iterated Lie brackets of order less or equal to k

$$X_{i_1}^{(z)}(z),\ [X_{i_1}^{(z)}, X_{i_2}^{(z)}](z),\ [[X_{i_1}^{(z)}, X_{i_2}^{(z)}], X_{i_3}^{(z)}](z), \dots \quad (\text{with } i_h \le N \text{ and } h \le k+1)$$

belong to $\mathcal{D}(z)$. Such a definition does not depend on the choice of the family $\{X_i^{(z)}\}_{i=1}^N$, compare Proposition 6.1 below. In the special case when $k = 1$ we will omit the prefix, i.e., we will simply say "involutive" instead of "1-involutive"(that makes this definition consistent with the classical one, compare [12, Definition 2.11.5]).

Let $\mathcal{D}$ be a C^1 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and let $\mathcal{M}$ be a N-dimensional C^1 submanifold of U. Then, according to a celebrated theorem by Frobenius (see [12, Section 2.11]), the distribution $\mathcal{D}$ is involutive at every point of U if and only if the following integrability property is verified: *For all $z_0 \in U$ there exists a C^1 submanifold $\mathcal{M}$ of U such that $z_0 \in \mathcal{M}$ and the tangency set of $\mathcal{M}$ with respect to $\mathcal{D}$ coincides with $\mathcal{M}$, namely $\tau(\mathcal{M}, \mathcal{D}) := \{z \in \mathcal{M} : T_z\mathcal{M} = \mathcal{D}(z)\} = \mathcal{M}$.*

The size of the tangency with respect to a noninvolutive distribution has been the subject of recent investigations in the field of sub-Riemannian geometry. The following list collects some of the results produced by this research activity. They describe the "integrability degree" of noninvolutive C^1 distributions $\mathcal{D}$, mainly by providing upper bounds for $\dim_H(\tau(\mathcal{M}, \mathcal{D}))$ as $\mathcal{M}$ varies among all the N-dimensional C^2 submanifolds of U (where $\dim_H$ denotes the Hausdorff dimension).

(1) Let $H\mathbb{H}^k$ be the horizontal subbundle of the tangent bundle to the Heisenberg group $\mathbb{H}^k$, that is the distribution of rank $2k$ on $\mathbb{R}^{2k+1}$ generated by the vector fields

$$(x_1, \dots, x_{2k+1}) \mapsto \frac{\partial}{\partial x_i} + 2x_{k+i}\,\frac{\partial}{\partial x_{2k+1}} \qquad (i = 1, \dots, k)$$

$$(x_1, \dots, x_{2k+1}) \mapsto \frac{\partial}{\partial x_{k+i}} - 2x_i\,\frac{\partial}{\partial x_{2k+1}} \qquad (i = 1, \dots, k)\,.$$

This distribution is noninvolutive everywhere and one has

$$\dim_H\left(\tau(\mathcal{M}, H\mathbb{H}^k)\right) \le k \tag{1.1}$$

for every $(2k)$-dimensional C^2 submanifold $\mathcal{M}$ of $\mathbb{R}^{2k+1}$ (see [1, Theorem 1.2], [2, Example 6.5], [7, Corollary 4.1]).

(2) An explicit estimate of the number

$$\sup\{\dim_H(\tau(\mathcal{M}, \mathcal{D})) : \mathcal{M} \text{ is } C^2\text{-smooth}\}$$

is provided by [2, Theorem 1.3] in terms of the involutiveness degree of $\mathcal{D}$. An elementary proof, based on the implicit function theorem, can be found in [6]. In [2, Example 6.5], already mentioned above, this result is used to prove the inequality (1.1).

(3) If $\mathcal{D}$ is of class C^∞ and fulfils the Hörmander noninvolutiveness condition (see [2, Definition 4.1]), then one has

$$\sup\{\dim_H(\tau(\mathcal{M}, \mathcal{D})) : \mathcal{M} \text{ is } C^2\text{-smooth}\} \le N - 1$$

compare [2, Theorem 4.5]. The well-known result by Derridj [10, Theorem 1] follows immediately from this property.

(4) Roughly speaking, the C^1 smooth submanifolds $\mathcal{M}$ are expected to produce much larger tangencies (with respect to $\mathcal{D}$) than those produced by C^2 smooth submanifolds. In fact, even if there are no points at which $\mathcal{D}$ is involutive, it can well be that a C^1 smooth $\mathcal{M}$ exists such that $\mathcal{H}^N(\tau(\mathcal{M},\mathcal{D})) > 0$. According to [2, Proposition 8.2], this is true for a large class of distributions including $H\mathbb{H}^k$ and there are good reasons to believe that it is true in general (compare [2, Problem 8.3]).

(5) If $\mathcal{D}$ is a C^1 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and $\mathcal{M}$ is a N-dimensional C^1 submanifold of U, then $\mathcal{D}$ must be involutive at each point $z_0 \in \mathcal{M}$ which is a $(N+1)$-density point of $\tau(\mathcal{M},\mathcal{D})$ (relative to $\mathcal{M}$) [8, Corollary 5.1]. In other words: despite (4), if $\mathcal{D}$ is not involutive at a point $z_0 \in \mathcal{M}$ then there is no N-dimensional C^1 submanifold $\mathcal{M}$ of U such that $z_0 \in \mathcal{M}$ and z_0 is a $(N+1)$-density point (relative to $\mathcal{M}$) of $\tau(\mathcal{M},\mathcal{D})$.

As we observed in [9], the result mentioned in (5) follows at once from Theorem 1.1. Analogously, the following corollary follows immediately from Theorem 1.2.

Corollary 1.1. *If $\mathcal{D}$ is a C^2 distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and $\mathcal{M}$ is a N-dimensional C^2 submanifold of U, then $\mathcal{D}$ must be 2-involutive at each point $z_0 \in \mathcal{M}$ which is a $(N+2)$-density point of $\tau(\mathcal{M},\mathcal{D})$ (relative to $\mathcal{M}$). In other words: if $\mathcal{D}$ is not 2-involutive at a point $z_0 \in \mathcal{M}$ then there is no N-dimensional C^2 submanifold $\mathcal{M}$ of U such that $z_0 \in \mathcal{M}$ and z_0 is a $(N+2)$-density point (relative to $\mathcal{M}$) of $\tau(\mathcal{M},\mathcal{D})$.*

When we started working on Theorem 1.2, our belief that it could be valid was rather weak, while (by virtue of Theorem 1.1 and Theorem 1.2) we are now firmly convinced that the following conjecture is true and will be the subject of future work.

Conjecture 1.1. *Given an open set $U \subset \mathbb{R}^{N+m}$, consider $\Phi_1, \ldots, \Phi_{k+1} \in C^k(U, \mathbb{R}^{N+m})$ and a N-dimensional C^k submanifold $\mathcal{M}$ of U (with $k \geq 1$). Moreover define*

$$\mathcal{T} := \left\{ z \in \mathcal{M} : \Phi_1(z), \ldots, \Phi_{k+1}(z) \in T_z\mathcal{M} \right\}.$$

and assume that $z_0 \in \mathcal{M}$ is a $(N+k)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$). Then all the iterated Lie brackets of order less or equal to k

$$\Phi_{i_1}(z_0),\ [\Phi_{i_1}, \Phi_{i_2}](z_0),\ [[\Phi_{i_1}, \Phi_{i_2}], \Phi_{i_3}](z_0),\ \ldots \qquad (h, i_h \leq k+1)$$

belong to $T_{z_0}\mathcal{M}$.

We conclude by observing that, just as Corollary 1.1 followed at once from Theorem 1.2, this property follows immediately for all $k \geq 1$ such that Conjecture 1.1 holds:

> If $\mathcal{D}$ is a C^k distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$ and $\mathcal{M}$ is a N-dimensional C^k submanifold of U, then $\mathcal{D}$ must be k-involutive at each point $z_0 \in \mathcal{M}$ which is a $(N+k)$-density point

of $\tau(\mathcal{M}, \mathcal{D})$ (relative to $\mathcal{M}$). In other words: if $\mathcal{D}$ is not k-involutive at a point $z_0 \in \mathcal{M}$ then there is no N-dimensional C^k submanifold $\mathcal{M}$ of U such that $z_0 \in \mathcal{M}$ and z_0 is a $(N+k)$-density point (relative to $\mathcal{M}$) of $\tau(\mathcal{M}, \mathcal{D})$.

2. General notation and preliminaries

We will have to deal with maps from $\mathbb{R}^N$ to $\mathbb{R}^m$. The standard basis of $\mathbb{R}^{N+m}$ and the corresponding coordinates are denoted by $e_1, \ldots, e_{N+m}$ and $(x_1, \ldots, x_N, y_1, \ldots, y_m)$, respectively. We may also write $\mathbb{R}^N_x$ in place of $\mathbb{R}^N$ and $\mathbb{R}^m_y$ in place of $\mathbb{R}^m$. If U is an open subset of $\mathbb{R}^N_x \times \mathbb{R}^m_y$ and $G \in C^1(U, \mathbb{R}^k)$, then $D_x G$ and $D_y G$ denote the Jacobian matrix of G with respect to x and the Jacobian matrix of G with respect to y, respectively, that is

$$D_x G := \left(\frac{\partial G}{\partial x_1} \bigg| \cdots \bigg| \frac{\partial G}{\partial x_N} \right), \quad D_y G := \left(\frac{\partial G}{\partial y_1} \bigg| \cdots \bigg| \frac{\partial G}{\partial y_m} \right).$$

In general, the Jacobian matrix of any C^1 vector field F is denoted by DF. The Hessian matrix of any C^2 function f is denoted by $D^2 f$, while $D^2_{ij} f$ stands for the (i,j)-entry of $D^2 f$. The h^{th}-order derivative of a C^h function of one variable g is indicated with $g^{(h)}$. For simplicity, we define

$$D_1 := \frac{\partial}{\partial x_1}, \ldots, D_N := \frac{\partial}{\partial x_N},\, D_{N+1} := \frac{\partial}{\partial y_1}, \ldots, D_{N+m} := \frac{\partial}{\partial y_m}\,.$$

For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, define

$$|\alpha| := \alpha_1 + \cdots + \alpha_N, \qquad D_\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

The Euclidean norms involved throughout this paper are all denoted by $\|\cdot\|$. The constants depending only on $p, q, \ldots$ are indicated by $C(p, q, \ldots)$. Let U be an open subset of $\mathbb{R}^{N+m}$. If $k \geq 1$ and

$$H = (H_1, \ldots, H_{N+m}),\, K = (K_1, \ldots, K_{N+m}) \in C^k(U, \mathbb{R}^{N+m})$$

then we recall that the Lie bracket product of H, K is the vector field

$$[H, K] = ([H,K]_1, \ldots, [H,K]_{N+m}) \in C^{k-1}(U, \mathbb{R}^{N+m})$$

where

$$[H,K]_j := \sum_{i=1}^{N+m} (H_i D_i K_j - K_i D_i H_j), \qquad j = 1, \ldots, N+m \tag{2.1}$$

compare [12, Remark 2.4.5]. Recall that the Lie bracket product is anti-symmetric, bilinear and verifies the following identity

$$[fH, gK] = f(H \cdot Dg)K - g(K \cdot Df)H + fg[H,K] \qquad (f, g \in C^k(U)) \tag{2.2}$$

compare [4, Chapter 1, Theorem 4.2]. If $k \geq 1$ and $X := \{X_1, \ldots, X_p\} \subset C^k(U, \mathbb{R}^{N+m})$, then we state the following inductive definition of h^{th}-order iterated

Lie brackets of the vector fields X_i, with $0 \le h \le k$ and $1 \le i_1, \ldots, i_{h+1} \le p$:

$$\Lambda^X_{(i_1,\ldots,i_{h+1})} := \begin{cases} X_{i_1} & \text{if} \quad h = 0 \\ \left[\Lambda^X_{(i_1,\ldots,i_h)}, X_{i_{h+1}}\right] & \text{if} \quad 1 \le h \le k \end{cases}$$

e.g. $\Lambda^X_{(1)} = X_1$, $\Lambda^X_{(1,2)} = [X_1, X_2]$, $\Lambda^X_{(1,2,1)} = [[X_1, X_2], X_1]$ (provided $k \ge 2$). Observe that

$$\Lambda^X_{(i_1,\ldots,i_{h+1})} \in C^{k-h}(U, \mathbb{R}^{N+m}). \tag{2.3}$$

Let $\mathcal{L}^X_h(z)$ be the vector space spanned by the family of the h^{th}-order iterated Lie brackets (of the vector fields X_i) at $z \in U$, namely

$$\mathcal{L}^X_h(z) := \operatorname{span}\left\{\Lambda^X_{(i_1,\ldots,i_{h+1})}(z) \,:\, 1 \le i_1, \ldots, i_{h+1} \le p\right\}$$

for all $z \in U$ and $0 \le h \le k$.

Let $\mathcal{M}$ be a N-dimensional C^1 submanifold of $\mathbb{R}^{N+m}$ and let d denote the distance defined on each connected component of $\mathcal{M}$ by taking the infimum over the joining paths (compare [3, Section 1.6]). Then for $z_0 \in \mathcal{M}$ and $r > 0$ we define

$$B_{\mathcal{M}}(z_0, r) := \{z \in \mathcal{M}^{(z_0)} \,|\, d(z, z_0) < r\}$$

where $M^{(z_0)}$ is the connected component of $\mathcal{M}$ containing z_0. Recall that for r small enough $\exp_{z_0}$ maps $B_{T_{z_0}\mathcal{M}}(0, r)$ diffeomorphically onto a neighborhood of z_0 and one has

$$\exp_{x_0}\left(B_{T_{z_0}\mathcal{M}}(0, r)\right) = B_{\mathcal{M}}(z_0, r)$$

compare [3, Theorem 1.6 and Corollary 1.1]. In the special case when $m = 0$ and $\mathcal{M} = \mathbb{R}^N$ the distance d reduces to the usual Euclidean distance and we denote $B_{\mathbb{R}^N}(z_0, r)$ simply by $B_r(z_0)$.

The Lebesgue outer measure on $\mathbb{R}^N$ and the N-dimensional Hausdorff measure on $\mathbb{R}^{N+m}$ will be denoted by $\mathcal{L}^N$ and $\mathcal{H}^N$, respectively.

A point $x \in \mathbb{R}^N$ is said to be a $(N+k)$-density point of $E \subset \mathbb{R}^N$ (where $k \in [0, +\infty)$) if

$$\mathcal{L}^N(B_r(x) \setminus E) = o(r^{N+k}) \qquad (\text{as} \quad r \to 0+)\,.$$

The set of all $(N+k)$-density points of E is denoted by $E^{(N+k)}$. Analogously, if $\mathcal{M}$ is a N-dimensional C^1 submanifold of $\mathbb{R}^{N+m}$ and $z_0 \in \mathcal{M}$, then we say that z_0 is a $(N+k)$-density point of $\mathcal{E} \subset \mathcal{M}$ (relative to $\mathcal{M}$) if

$$\mathcal{H}^N(B_{\mathcal{M}}(z_0, r) \setminus \mathcal{E}) = o(r^{N+k}) \qquad (\text{as} \quad r \to 0+).$$

The set of all $(N+k)$-density points of $\mathcal{E}$ (relative to $\mathcal{M}$) is denoted by $\mathcal{E}^{(N+k)}$. Observe that

$$\mathcal{E}^{(N+k)} \subset \mathcal{E}^{(N+h)}$$

for all $h \in [0, k]$. In particular, if k is a positive integer, one has

$$\mathcal{E}^{(N+k)} \subset \mathcal{E}^{(N+k-1)} \subset \cdots \subset \mathcal{E}^{(N)}\,. \tag{2.4}$$

By [11, 3.2.46] and the area formula [11, Theorem 3.2.3] one can prove that C^1 embeddings preserve density-degree, namely the following property holds [8, Proposition 3.3].

Proposition 2.1. *Let $\mathcal{M}$ be a N-dimensional C^1 submanifold of $\mathbb{R}^{N+m}$, let Ω be an open subset of $\mathbb{R}^N$ and let $F : \Omega \to \mathbb{R}^{N+m}$ be an injective immersion of class C^1 such that $F(\Omega) \subset \mathcal{M}$. Moreover let E be a subset of Ω and let $x_0 \in \Omega$. Then (for $k \geq 0$) one has*

$$\mathcal{L}^N(B_r(x_0) \setminus E) = o(r^{N+k}) \qquad (as \quad r \to 0+)$$

if and only if

$$\mathcal{H}^N(B_{\mathcal{M}}(F(x_0), r) \setminus F(E)) = o(r^{N+k}) \qquad (as \quad r \to 0+)\,.$$

In particular, $x_0 \in E^{(N+k)}$ if and only if $F(x_0) \in F(E)^{(N+k)}$.

We conclude this section with a remark which will be very useful below.

Remark 2.1. Let $H = (H_1, \dots, H_{N+m})$ be a vector field of class C^1 in an open set $U \subset \mathbb{R}^N_x \times \mathbb{R}^m_y$. Moreover let Ω be an open subset of $\mathbb{R}^N_x$ and $f = (f_1, \dots, f_m) \in C^1(\Omega, \mathbb{R}^m_y)$. Denote by Γ the graph of f, that is $\Gamma := F(\Omega)$ where

$$F\colon \Omega \to \mathbb{R}^N_x \times \mathbb{R}^m_y\,, \qquad F(x) := \big(x, f(x)\big)$$

and assume that $\Gamma \subset U$. Given $x \in \Omega$, obviously one has that $H(F(x)) \in T_{F(x)}\Gamma$ if and only if $H(F(x)) \in \mathrm{Im}(DF)$. Recalling that

$$DF = \begin{pmatrix} I \\ Df \end{pmatrix}$$

we get at once the following property: $H(F(x)) \in T_{F(x)}\Gamma$ if and only if

$$H_\#\big(F(x)\big) = Df(x) H_*\big(F(x)\big) \tag{2.5}$$

where we have defined

$$H_* := (H_1, \dots, H_N)\,, \qquad H_\# := (H_{N+1}, \dots, H_{N+m})\,.$$

Moreover, if $K = (K_1, \dots, K_{N+m})$ is another vector field of class C^1 in U and if one has $H(F(x)), K(F(x)) \in T_{F(x)}\Gamma$ for a certain $x \in \Omega$, then

$$DH\big(F(x)\big)K\big(F(x)\big) = D(H \circ F)(x) K_*\big(F(x)\big) \tag{2.6}$$

compare [9, Lemma 4.1].

3. Some localization properties at a superdensity point

Consider a function $g \in C^k(\mathbb{R})$, with $k \geq 1$, such that

$$0 \leq g \leq 1, \quad g|_{(-\infty,0]} \equiv 1, \quad g|_{[1,+\infty)} \equiv 0$$

and, for $\rho \in (0,1)$, define

$$\psi_\rho(x) := g\Big(\frac{\|x\| - \rho}{1 - \rho}\Big)\,, \qquad x \in \mathbb{R}^N\,.$$

Observe that

$$\psi_\rho|_{\overline{B_\rho(0)}} \equiv 1\,, \qquad \psi_\rho|_{\mathbb{R}^N \setminus B_1(0)} \equiv 0\,.$$

Proposition 3.1. *For all $\alpha \in \mathbb{N}^N \setminus \{0\}$ one has*

$$D_\alpha \psi_\rho(x) = \sum_{h=1}^{|\alpha|} (1-\rho)^{-h} g^{(h)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) \sum_{\{\beta_1,\dots,\beta_h\}\in\mathcal{P}_h(\alpha)} D_{\beta_1}\|x\| \cdots D_{\beta_h}\|x\|$$

where

$$\mathcal{P}_h(\alpha) := \Big\{ \{\beta_1,\dots,\beta_h\} \,:\, \beta_i \in \mathbb{N}^N \setminus \{0\},\ \sum_{i=1}^h \beta_i = |\alpha| \Big\}.$$

Proof. The statement is obvious if $|\alpha| = 1$. Then let k be a positive integer and assume that the identity holds whenever $|\alpha| \le k$. We have to prove that it continues to be true for any $\alpha \in \mathbb{N}^N$ such that $|\alpha| = k+1$. To this aim, without loss of generality, we can suppose that $\alpha_1 \ge 1$. If define

$$\varepsilon_1 := (1,0,\dots,0),\ \varepsilon_2 := (0,1,\dots,0),\ \dots,\ \varepsilon_N := (0,\dots,0,N)$$

then one has $|\alpha - \varepsilon_1| = k$, hence (by assumption)

$$D_{\alpha-\varepsilon_1}\psi_\rho(x) = \sum_{h=1}^{k} (1-\rho)^{-h} g^{(h)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) \sum_{\{\beta_1,\dots,\beta_h\}\in\mathcal{P}_h(\alpha-\varepsilon_1)} D_{\beta_1}\|x\| \cdots D_{\beta_h}\|x\|.$$

Thus

$$\begin{aligned}
D_\alpha\psi_\rho(x) &= D_{\varepsilon_1}\left(D_{\alpha-\varepsilon_1}\psi_\rho\right)(x) \\
&= \sum_{h=1}^{k} (1-\rho)^{-h-1} g^{(h+1)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) D_{\varepsilon_1}\|x\| \sum_{\{\beta_1,\dots,\beta_h\}\in\mathcal{P}_h(\alpha-\varepsilon_1)} D_{\beta_1}\|x\| \cdots D_{\beta_h}\|x\| \\
&\quad + \sum_{h=1}^{k} (1-\rho)^{-h} g^{(h)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) \sum_{\{\beta_1,\dots,\beta_h\}\in\mathcal{P}_h(\alpha-\varepsilon_1)} \big(D_{\beta_1+\varepsilon_1}\|x\| D_{\beta_2}\|x\| \cdots D_{\beta_h}\|x\| \\
&\quad + D_{\beta_1}\|x\| D_{\beta_2+\varepsilon_1}\|x\| \cdots D_{\beta_h}\|x\| + \cdots + D_{\beta_1}\|x\| \cdots D_{\beta_{h-1}}\|x\| D_{\beta_h+\varepsilon_1}\|x\|\big) \\
&= (1-\rho)^{-1} g^{(1)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) D_\alpha\|x\| + \sum_{h=2}^{k} (1-\rho)^{-h} g^{(h)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) \\
&\quad \times \Bigg[\sum_{\{\beta_1,\dots,\beta_{h-1}\}\in\mathcal{P}_{h-1}(\alpha-\varepsilon_1)} D_{\varepsilon_1}\|x\| D_{\beta_1}\|x\| \cdots D_{\beta_{h-1}}\|x\| \\
&\quad + \sum_{\{\beta_1,\dots,\beta_h\}\in\mathcal{P}_h(\alpha-\varepsilon_1)} \big(D_{\beta_1+\varepsilon_1}\|x\| D_{\beta_2}\|x\| \cdots D_{\beta_h}\|x\| \\
&\quad + D_{\beta_1}\|x\| D_{\beta_2+\varepsilon_1}\|x\| \cdots D_{\beta_h}\|x\| + \cdots + D_{\beta_1}\|x\| \cdots D_{\beta_{h-1}}\|x\| D_{\beta_h+\varepsilon_1}\|x\|\big)\Bigg] \\
&\quad + (1-\rho)^{-k-1} g^{(k+1)}\Big(\frac{\|x\|-\rho}{1-\rho}\Big) D_{\varepsilon_1}\|x\| D_{\varepsilon_1}^{\alpha_1-1}\|x\| D_{\varepsilon_2}^{\alpha_2}\|x\| \cdots D_{\varepsilon_N}^{\alpha_N}\|x\|
\end{aligned}$$

hence the conclusion follows immediately. $\square$

Remark 3.1. By a completely standard argument (e.g. by induction) one can easily prove that for all $\alpha \in \mathbb{N}^N$ one has

$$D_\alpha \|x\| = \frac{p_\alpha(x)}{\|x\|^{2|\alpha|-1}} \qquad (x \neq 0)$$

where p_α is a homogeneous polynomial of degree $|\alpha|$ whose coefficients depend only on α. It follows that

$$\max_{x \in \overline{B_1(0)} \setminus B_\rho(0)} \big| D_\alpha \|x\| \big| \leq \frac{C(\alpha)}{\rho^{2|\alpha|-1}} .$$

Corollary 3.1. *Let $x_0 \in \mathbb{R}^N$, $r > 0$, $\rho \in (1/2, 1)$ and define*

$$\varphi_{\rho,r}(x) := \psi_\rho\Big(\frac{x - x_0}{r}\Big) , \qquad x \in \mathbb{R}^N .$$

Then, for all $\alpha \in \mathbb{N}^N$, one has

$$\|D_\alpha \varphi_{\rho,r}\|_\infty \leq \frac{C(\alpha)}{(1-\rho)^{|\alpha|} r^{|\alpha|}} .$$

Proof. The statement is obvious for $|\alpha| = 0$, so we can assume $|\alpha| \geq 1$. Observe that

$$D_\alpha \varphi_{\rho,r}(x) = r^{-|\alpha|} (D_\alpha \psi_\rho)\Big(\frac{x - x_0}{r}\Big) , \qquad x \in \mathbb{R}^N .$$

Hence, by Proposition 3.1 and Remark 3.1, we get

$$\begin{aligned}
\|D_\alpha \varphi_{\rho,r}\|_\infty &= r^{-|\alpha|} \|D_\alpha \psi_\rho\|_\infty \\
&= r^{-|\alpha|} \max_{x \in \overline{B_1(0)} \setminus B_\rho(0)} |D_\alpha \psi_\rho(x)| \\
&\leq r^{-|\alpha|} \sum_{h=1}^{|\alpha|} \frac{\|g^{(h)}\|_\infty}{(1-\rho)^h} \sum_{\{\beta_1, \ldots, \beta_h\} \in \mathcal{P}_h(\alpha)} \frac{C(\beta_1) \ldots C(\beta_h)}{\rho^{2|\beta_1|-1} \ldots \rho^{2|\beta_h|-1}} \\
&\leq \frac{C(\alpha)}{r^{|\alpha|}} \sum_{h=1}^{|\alpha|} \frac{1}{(1-\rho)^h \rho^{2|\alpha|-h}} .
\end{aligned}$$

The conclusion follows by observing that if $1/2 < \rho < 1$, then one has

$$(1-\rho)^h \rho^{2|\alpha|-h} \geq \frac{(1-\rho)^{|\alpha|}}{2^{2|\alpha|}} \qquad (\text{for} \quad h = 1, \ldots, |\alpha|) .$$

□

Proposition 3.2. *Let Θ be a continuous function defined in a neighborhood of $x_0 \in \mathbb{R}^N$. Assume that for all $\rho \in (1/2, 1)$ one has*

$$\int_{B_r(x_0)} \Theta(x) \varphi_{\rho,r}(x)\, dx = o(r^N) \tag{3.1}$$

as $r \to 0+$. Then $\Theta(x_0) = 0$.

Proof. As in [9, Proposition 3.1]. □

Proposition 3.3. *Let E be a measurable subset of $\mathbb{R}^N$ and $x_0 \in E^{(N+k)}$, with $k \geq 1$. Moreover let Θ and Λ be a couple of continuous real valued functions defined in a neighborhood of x_0 such that $\Theta|_{E\cap B_r(x_0)} = \Lambda|_{E\cap B_r(x_0)}$ (for r small enough). If $\rho \in (1/2, 1)$, then one has*

$$\int_{B_r(x_0)} \Theta D_\alpha \varphi_{\rho,r}\, dx = \int_{B_r(x_0)} \Lambda D_\alpha \varphi_{\rho,r}\, dx + o(r^N) \qquad (as \quad r \to 0+)$$

for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$.

Proof. Let $\alpha \in \mathbb{N}^n$ be such that $|\alpha| \leq k$ and observe that (since the integral is linear) it will be enough to prove the statement for $\Lambda \equiv 0$. Then

$$\begin{aligned}\Big|\int_{B_r(x_0)} \Theta D_\alpha \varphi_{\rho,r}\, dx\Big| &= \Big|\int_{B_r(x_0)\setminus E} \Theta D_\alpha \varphi_{\rho,r}\, dx\Big| \\ &\leq \Big(\sup_{B_r(x_0)} |\Theta|\Big) \|D_\alpha \varphi_{\rho,r}\|_\infty \mathcal{L}^N(B_r(x_0)\setminus E)\,.\end{aligned}$$

We conclude by Corollary 3.1 and recalling that $x_0 \in E^{(N+k)}$. □

Definition 3.1. Let Θ and Λ be a couple of real valued functions, each one defined and summable in a neighborhood of $x_0 \in \mathbb{R}^N$, such that

$$\int_{B_r(x_0)} \Theta(x)\varphi_{\rho,r}(x)\, dx = \int_{B_r(x_0)} \Lambda(x)\varphi_{\rho,r}(x)\, dx + o(r^N) \qquad (\text{as} \quad r \to 0+)$$

for all $\rho \in (1/2, 1)$. Then we write $\Theta \overset{x_0}{\sim} \Lambda$ and say that Θ and Λ are equivalent at x_0.

Remark 3.2. It is trivial to verify that $\overset{x_0}{\sim}$ is actually an equivalence relation on the family of real valued functions defined and summable in a neighborhood of x_0.

Proposition 3.4. *Let Θ and Λ be a couple of real valued functions, each one defined and continuous in a neighborhood of $x_0 \in \mathbb{R}^N$, such that $\Theta \overset{x_0}{\sim} \Lambda$. Moreover, let g be a function of class C^1 in a neighborhood of x_0. Then one has $g\Theta \overset{x_0}{\sim} g\Lambda$.*

Proof. For r sufficiently small and $\rho \in (0, 1)$, one has

$$\begin{aligned}&\Big|\int_{B_r(x_0)} g(x)\Theta(x)\varphi_{\rho,r}(x)\, dx - \int_{B_r(x_0)} g(x)\Lambda(x)\varphi_{\rho,r}(x)\, dx\Big| \\ &= \Big|\int_{B_r(x_0)} [g(x)-g(x_0)]\Theta(x)\varphi_{\rho,r}(x)\, dx - \int_{B_r(x_0)} [g(x)-g(x_0)]\Lambda(x)\varphi_{\rho,r}(x)\, dx \\ &\quad + g(x_0)\Big(\int_{B_r(x_0)} \Theta(x)\varphi_{\rho,r}(x)\, dx - \int_{B_r(x_0)} \Lambda(x)\varphi_{\rho,r}(x)\, dx\Big)\Big| \\ &\leq C(N) \sup_{B_r(x_0)} |g-g(x_0)| \Big(\sup_{B_r(x_0)} |\Theta| + \sup_{B_r(x_0)} |\Lambda|\Big) r^N + |g(x_0)|\, o(r^N)\,.\end{aligned}$$

Moreover, for all $x \in B_r(x_0)$, one has

$$g(x) - g(x_0) = \int_0^1 Dg(x_0 + t(x-x_0))\cdot(x-x_0)\, dt$$

hence

$$\sup_{B_r(x_0)} |g - g(x_0)| \leq \Big(\sup_{B_r(x_0)} \|Dg\| \Big) r \,.$$

It follows that

$$\int_{B_r(x_0)} g(x)\Theta(x)\varphi_{\rho,r}(x)\,dx - \int_{B_r(x_0)} g(x)\Lambda(x)\varphi_{\rho,r}(x)\,dx = o(r^N) \quad (\text{as} \quad r \to 0+)\,.$$

$\square$

From Proposition 3.3 and the integration by parts formula it follows at once the following result.

Theorem 3.1. *Let E be a measurable subset of $\mathbb{R}^N$ and $x_0 \in E^{(N+k)}$, with $k \geq 1$. Moreover let Θ and Λ be a couple of real valued functions of class C^k in a neighborhood of x_0 such that $\Theta|_{E\cap B_r(x_0)} = \Lambda|_{E\cap B_r(x_0)}$ (for r small enough). Then one has*

$$D_\alpha\Theta \overset{x_0}{\sim} D_\alpha\Lambda \quad (\textit{hence } D_\alpha\Theta(x_0) = D_\alpha\Lambda(x_0),\ \textit{by Proposition 3.2})$$

for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$.

4. The proof of Theorem 1.2 (main result)

This section is devoted to the proof Theorem 1.2. It is an example of how Theorem 3.1 can serve to extend to $(N+k)$-density points a property which is known to hold at interior points. Actually we will prove the following result which is trivially equivalent to Theorem 1.2, by Theorem 1.1 and (2.4) (with $k = 2$ and $\mathcal{E} = \mathcal{T}$). We state it by using a subscript-free notation that will produce shorter formulas.

Theorem 4.1. *Let H, K, L be three vector fields of class C^2 in an open set $U \subset \mathbb{R}^{N+m}$. Moreover let $\mathcal{M}$ be a N-dimensional C^2 submanifold of U and define*

$$\mathcal{T} := \big\{ z \in \mathcal{M} : H(z), K(z), L(z) \in T_z\mathcal{M} \big\}\,.$$

If $z_0 \in \mathcal{M}$ is a $(N+2)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$) then $[[H,K],L](z_0) \in T_{z_0}\mathcal{M}$.

Proof. Since $\mathcal{M}$ is locally the graph of a C^2 function, we can assume that there exist an open set $\Omega \subset \mathbb{R}^N_x$ and $f = (f_1, \dots, f_m) \in C^2(\Omega, \mathbb{R}^m_y)$ such that

$$z_0 \in \Gamma := \{(x, f(x)) \,:\, x \in \Omega\} \subset \mathcal{M}\,.$$

Define $F \in C^2(\Omega, \mathbb{R}^{N+m})$ by

$$F(x) := \big(x, f(x)\big)\,, \qquad x \in \Omega\,.$$

Moreover let

$$x_0 := F^{-1}(z_0)\,, \qquad T := F^{-1}(\mathcal{T})$$

and observe that

$$x_0 \in \Omega \cap T^{(N+2)} \quad (\text{hence also } x_0 \in \Omega \cap T^{(N+1)})$$

by Proposition 2.1. The following notation will be useful: if A and B are functions defined in Ω such that $A|_T = B|_T$, then we write $A \overset{T}{=} B$.

If for all $h = 1, \dots, m$ define $\mathcal{D}_h \in C(\Omega)$ as

$$\mathcal{D}_h(x) := \big(Df(x)[[H,K],L]_*(F(x)) - [[H,K],L]_\#(F(x))\big) \cdot e_{N+h}$$

then, by Remark 2.1, we have to prove that

$$\mathcal{D}_h(x_0) = 0 \qquad (h = 1, \dots, m)\,. \tag{4.1}$$

From now on the argument is a very long and technical computation, divided into steps, whose hardest details are collected in the next section.
Step 1. First of all, observe that

$$\begin{aligned}
[[H,K],L]_j &= \sum_{i=1}^{N+m} ([H,K]_i D_i L_j - L_i D_i [H,K]_j) \\
&= \sum_{i,l=1}^{N+m} \big(H_l D_l K_i D_i L_j - K_l D_l H_i D_i L_j - L_i D_i (H_l D_l K_j - K_l D_l H_j)\big) \\
&= [(DK)H] \cdot DL_j - [(DH)K] \cdot DL_j - [(D^2 K_j)H] \cdot L + \\
&\quad - [(DH)L] \cdot DK_j + [(D^2 H_j)K] \cdot L + [(DK)L] \cdot DH_j
\end{aligned}$$

by (2.1). Hence we get (compare Section 5.1)

$$\mathcal{D}_h \overset{x_0}{\sim} \mathcal{G}_h(H,K,L) - \mathcal{G}_h(K,H,L) \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{G}_h(H,K,L) := {} & [D(K\circ F)(H_*\circ F)]\cdot\Big(\sum_{p=1}^{N}[(DL_p)\circ F]D_p f_h - [(DL_{N+h})\circ F]\Big) \\
& + [D(K\circ F)(L_*\circ F)]\cdot\Big(\sum_{p=1}^{N}[(DH_p)\circ F]D_p f_h - [(DH_{N+h})\circ F]\Big) \\
& - \sum_{p=1}^{N} D_p f_h \big([(D^2 K_p)\circ F](H\circ F)\big)\cdot(L\circ F) \\
& + \big([(D^2 K_{N+h})\circ F](H\circ F)\big)\cdot(L\circ F)\,.
\end{aligned} \tag{4.3}$$

Thus we are reduced to prove that

$$\mathcal{G}_h(H,K,L)(x_0) = \mathcal{G}_h(K,H,L)(x_0) \qquad (h = 1, \dots, m)\,.$$

Step 2. For $l = 1, \dots, N+m$ the following identity holds (compare Section 5.2)

$$\begin{aligned}
\big([(D^2 K_l)\circ F](H\circ F)\big)\cdot(L\circ F) \overset{x_0}{\sim} {} & \mathcal{A}_l(H,K,L) \\
& - [D(H\circ F)(L_*\circ F)]\cdot[(DK_l)\circ F]
\end{aligned} \tag{4.4}$$

where

$$\mathcal{A}_l(H,K,L) := D[D(K_l\circ F)\cdot(H_*\circ F)]\cdot(L_*\circ F).$$

Step 3. From (4.3) and (4.4), we obtain

$$
\begin{aligned}
\mathcal{G}_h(H,K,L) \overset{x_0}{\sim} & [D(K\circ F)(H_*\circ F)]\cdot\Big(\sum_{p=1}^{N}[(DL_p)\circ F]D_pf_h - [(DL_{N+h})\circ F]\Big) \\
& + \sum_{p=1}^{N} D_pf_h[D(K\circ F)(L_*\circ F)]\cdot[(DH_p)\circ F] + \\
& - \sum_{p=1}^{N} D_pf_h\, \mathcal{A}_p(H,K,L) + \sum_{p=1}^{N} D_pf_h[D(H\circ F)(L_*\circ F)]\cdot[(DK_p)\circ F] \\
& + \mathcal{A}_{N+h}(H,K,L) - [D(H\circ F)(L_*\circ F)]\cdot[(DK_{N+h})\circ F] \\
& - [D(K\circ F)(L_*\circ F)]\cdot[(DH_{N+h})\circ F]
\end{aligned}
$$

that is

$$
\begin{aligned}
\mathcal{G}_h(H,K,L) \overset{x_0}{\sim} & \mathcal{B}_h(H,K,L) + \mathcal{C}_h(H,K,L) \\
& + \mathcal{S}_h^{(1)}(H,K,L) + \mathcal{S}_h^{(2)}(H,K,L)
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\mathcal{B}_h(H,K,L) &:= \mathcal{A}_{N+h}(H,K,L) - \sum_{p=1}^{N} D_pf_h\mathcal{A}_p(H,K,L) \\
\mathcal{C}_h(H,K,L) &:= [D(K\circ F)(H_*\circ F)]\cdot\Big(\sum_{p=1}^{N}[(DL_p)\circ F]D_pf_h - [(DL_{N+h})\circ F]\Big) \\
\mathcal{S}_h^{(1)}(H,K,L) &:= -[D(H\circ F)(L_*\circ F)]\cdot[(DK_{N+h})\circ F] \\
&\quad - [D(K\circ F)(L_*\circ F)]\cdot[(DH_{N+h})\circ F] \\
\mathcal{S}_h^{(2)}(H,K,L) &:= \sum_{p=1}^{N} D_pf_h[D(K\circ F)(L_*\circ F)]\cdot[(DH_p)\circ F] \\
&\quad + \sum_{p=1}^{N} D_pf_h[D(H\circ F)(L_*\circ F)]\cdot[(DK_p)\circ F]\,.
\end{aligned}
$$

Observe that $\mathcal{S}_h^{(1)}(H,K,L)$ and $\mathcal{S}_h^{(2)}(H,K,L)$ are symmetric with respect to the couple (H,K), that is

$$
\mathcal{S}_h^{(1)}(H,K,L) = \mathcal{S}_h^{(1)}(K,H,L), \qquad \mathcal{S}_h^{(2)}(H,K,L) = \mathcal{S}_h^{(2)}(K,H,L)\,.
$$

Step 4. One has (compare Section 5.3)

$$
\mathcal{C}_h(H,K,L) \overset{x_0}{\sim} \mathcal{F}_h(H,K,L) + \mathcal{S}_h^{(3)}(H,K,L)
\tag{4.6}
$$

where

$$\mathcal{F}_h(H,K,L) := -\sum_{i=1}^{N}[D(K_i\circ F)\cdot(H_*\circ F)][D_i(Df_h)\cdot(L_*\circ F)]$$

$$\mathcal{S}_h^{(3)}(H,K,L) := \sum_{q=1}^{m}[D^2 f_q(K_*\circ F)]\cdot(H_*\circ F) \times \big([(D_{N+q}L_*)\circ F]\cdot Df_h - [(D_{N+q}L_{N+h})\circ F]\big)\,.$$

Observe that $\mathcal{S}_h^{(3)}(H,K,L)$ is symmetric with respect to the couple (H,K).
Step 5. One has (compare Section 5.4)

$$\begin{aligned}\mathcal{B}_h(H,K,L)+\mathcal{F}_h(H,K,L) \overset{x_0}{\sim} &\operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big)\\ &- \operatorname{div}\Big(\sum_{i=1}^{N} D_i f_h[D(K_i\circ F)\cdot(H_*\circ F)](L_*\circ F)\Big)\\ &+ \mathcal{S}_h^{(4)}(H,K,L)\end{aligned} \tag{4.7}$$

where

$$\mathcal{S}_h^{(4)}(H,K,L) := -[D^2 f_h(K_*\circ F)]\cdot(H_*\circ F)\operatorname{div}(L_*\circ F)$$

which is symmetric with respect to (H,K). From (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned}\mathcal{G}_h(H,K,L) \overset{x_0}{\sim} &\operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big)\\ &- \operatorname{div}\Big(\sum_{i=1}^{N} D_i f_h[D(K_i\circ F)\cdot(H_*\circ F)](L_*\circ F)\Big)\\ &+ \mathcal{S}_h(H,K,L)\end{aligned} \tag{4.8}$$

where

$$\mathcal{S}_h(H,K,L) := \sum_{l=1}^{4}\mathcal{S}_h^{(l)}(H,K,L)\,.$$

Step 6. One has (compare Section 5.5)

$$\begin{aligned}&\int_{B_r(x_0)}\varphi_{\rho,r}\operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big)\,dx\\ &\quad-\int_{B_r(x_0)}\varphi_{\rho,r}\operatorname{div}\Big(\sum_{i=1}^{N} D_i f_h[D(K_i\circ F)\cdot(H_*\circ F)](L_*\circ F)\Big)\,dx\\ &= \sigma_h(H,K,L)+o(r^N)\end{aligned} \tag{4.9}$$

as $r\to 0+$, where

$$\sigma_h(H,K,L) := -\int_{B_r(x_0)}[D^2 f_h(K_*\circ F)]\cdot(H_*\circ F)[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx\,.$$

Step 7 (the conclusion). One has

$$\int_{B_r(x_0)} \varphi_{\rho,r}\, \mathcal{G}_h(H,K,L)\, dx = \int_{B_r(x_0)} \varphi_{\rho,r}\, \mathcal{S}_h(H,K,L)\, dx + \sigma_h(H,K,L) + o(r^N) \tag{4.10}$$

as $r \to 0+$, by (4.8) and (4.9). Since the right hand side of (4.10) is symmetric with respect to the couple (H,K), we obtain

$$\mathcal{G}_h(H,K,L) \overset{x_0}{\sim} \mathcal{G}_h(K,H,L)$$

that is

$$\mathcal{D}_h \overset{x_0}{\sim} 0$$

by (4.2). Finally, the identity (4.1) follows from Proposition 3.2. □

Remark 4.1. Actually in [9] we have not proved Theorem 1.1, but the following "step-1" analogous of Theorem 4.1, which is however trivially equivalent to Theorem 1.1.

Theorem 4.2. *let H, K be two vector fields of class C^1 in an open set $U \subset \mathbb{R}^{N+m}$. Moreover let $\mathcal{M}$ be a N-dimensional C^1 submanifold of U and define*

$$\mathcal{T} := \left\{ z \in \mathcal{M} : H(z), K(z) \in T_z\mathcal{M} \right\}.$$

If $z_0 \in \mathcal{M}$ is a $(N+1)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$) then $[H,K](z_0) \in T_{z_0}\mathcal{M}$.

It is now quite clear that Theorem 4.1 and Theorem 4.2 strongly support the following conjecture, which is equivalent to Conjecture 1.1, by (2.4) (with $\mathcal{E} = \mathcal{T}$).

Conjecture 4.1. *Consider an open set $U \subset \mathbb{R}^{N+m}$, a N-dimensional C^k submanifold $\mathcal{M}$ of U, a family $\Phi := \{\Phi_1, \ldots, \Phi_{k+1}\} \subset C^k(U, \mathbb{R}^{N+m})$ (with $k \geq 1$) and define*

$$\mathcal{T} := \left\{ z \in \mathcal{M} : \Phi_1(z), \ldots, \Phi_{k+1}(z) \in T_z\mathcal{M} \right\}.$$

If $z_0 \in \mathcal{M}$ is a $(N+k)$-density point of $\mathcal{T}$ (relative to $\mathcal{M}$) then $\Lambda^{\Phi}_{(1,\ldots,k+1)}(z_0) \subset T_{z_0}\mathcal{M}$.

5. Collection of the computational details needed to prove Theorem 4.1

5.1. **Proof of** (4.2)**.** One has

$$\begin{aligned}
\mathcal{D}_h &= \sum_{p=1}^{N} D_p f_h [[H,K],L]_p \circ F - [[H,K],L]_{N+h} \circ F \\
&= \sum_{p=1}^{N} \big([(DK)\circ F](H\circ F) \big) \cdot [(DL_p)\circ F] D_p f_h \\
&\quad - \sum_{p=1}^{N} \big([(DH)\circ F](K\circ F) \big) \cdot [(DL_p)\circ F] D_p f_h
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{p=1}^{N}\big([(DH)\circ F](L\circ F)\big)\cdot[(DK_p)\circ F]D_p f_h \\
&+\sum_{p=1}^{N}\big([(DK)\circ F](L\circ F)\big)\cdot[(DH_p)\circ F]D_p f_h \\
&-\sum_{p=1}^{N} D_p f_h\big([(D^2K_p)\circ F](H\circ F)\big)\cdot(L\circ F)+\sum_{p=1}^{N} D_p f_h\big([(D^2H_p)\circ F](K\circ F)\big)\cdot(L\circ F) \\
&-\big([(DK)\circ F](H\circ F)\big)\cdot[(DL_{N+h})\circ F]+\big([(DH)\circ F](K\circ F)\big)\cdot[(DL_{N+h})\circ F] \\
&+\big([(DH)\circ F](L\circ F)\big)\cdot[(DK_{N+h})\circ F]-\big([(DK)\circ F](L\circ F)\big)\cdot[(DH_{N+h})\circ F] \\
&+\big([(D^2K_{N+h})\circ F](H\circ F)\big)\cdot(L\circ F)-\big([(D^2H_{N+h})\circ F](K\circ F)\big)\cdot(L\circ F)
\end{aligned}
$$

where

$$
\begin{aligned}
&[(DK)\circ F](H\circ F)\overset{T}{=}D(K\circ F)(H_*\circ F), \qquad [(DH)\circ F](K\circ F)\overset{T}{=}D(H\circ F)(K_*\circ F), \\
&[(DK)\circ F](L\circ F)\overset{T}{=}D(K\circ F)(L_*\circ F), \qquad [(DH)\circ F](L\circ F)\overset{T}{=}D(H\circ F)(L_*\circ F)
\end{aligned}
$$

by (2.6). Hence

$$
\mathcal{D}_h \overset{x_0}{\sim} \mathcal{G}_h(H,K,L)-\mathcal{G}_h(K,H,L)
$$

by Theorem 3.1.

5.2. **Proof of** (4.4)**.** By (2.6), one has

$$
[(DK_l)\circ F]\cdot(H\circ F)\overset{T}{=}[(DK_l\circ F)]\cdot(H_*\circ F)
$$

that is

$$
\sum_{j=1}^{N+m}[(D_jK_l)\circ F](H_j\circ F)\overset{T}{=}\sum_{i=1}^{N} D_i(K_l\circ F)(H_i\circ F)\,.
$$

By applying Theorem 3.1 with $k=1$, we get (for all $p=1,\dots,N$)

$$
D_p\Big(\sum_{j=1}^{N+m}[(D_jK_l)\circ F](H_j\circ F)\Big)\overset{x_0}{\sim} D_p\Big(\sum_{i=1}^{N} D_i(K_l\circ F)(H_i\circ F)\Big)
$$

namely

$$
\begin{aligned}
&\sum_{j=1}^{N+m}\Big([(D^2_{pj}K_l)\circ F]+\sum_{q=1}^{m}[(D^2_{j,N+q}K_l)\circ F]D_p f_q\Big)(H_j\circ F) \\
&\qquad+\sum_{j=1}^{N+m}[(D_jK_l)\circ F]\Big([(D_pH_j)\circ F]+\sum_{q=1}^{m}[(D_{N+q}H_j)\circ F]D_p f_q\Big) \\
&\overset{x_0}{\sim}\sum_{i=1}^{N} D^2_{pi}(K_l\circ F)(H_i\circ F)+\sum_{i=1}^{N} D_i(K_l\circ F)D_p(H_i\circ F)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\big([(D^2K_l)\circ F](H\circ F)\big)\cdot(L\circ F) &= \sum_{i,j=1}^{N+m}[(D^2_{ij}K_l)\circ F](H_j\circ F)(L_i\circ F)\\
&= \sum_{p=1}^{N}\sum_{j=1}^{N+m}[(D^2_{pj}K_l)\circ F](H_j\circ F)(L_p\circ F)\\
&\quad + \sum_{q=1}^{m}\sum_{j=1}^{N+m}[(D^2_{N+q,j}K_l)\circ F](H_j\circ F)(L_{N+q}\circ F)\\
&\overset{x_0}{\sim} \sum_{q=1}^{m}\sum_{j=1}^{N+m}[(D^2_{N+q,j}K_l)\circ F](H_j\circ F)(L_{N+q}\circ F)\\
&\quad + \sum_{i,p=1}^{N} D^2_{ip}(K_l\circ F)(H_i\circ F)(L_p\circ F) + \sum_{i,p=1}^{N} D_i(K_l\circ F)D_p(H_i\circ F)(L_p\circ F)\\
&\quad - \sum_{j=1}^{N+m}\sum_{p=1}^{N}[(D_jK_l)\circ F][(D_pH_j)\circ F](L_p\circ F)\\
&\quad - \sum_{j=1}^{N+m}\sum_{p=1}^{N}\sum_{q=1}^{m}[(D_jK_l)\circ F][(D_{N+q}H_j)\circ F](L_p\circ F)D_pf_q\\
&\quad - \sum_{j=1}^{N+m}\sum_{p=1}^{N}\sum_{q=1}^{m}[(D^2_{N+q,j}K_l)\circ F](H_j\circ F)(L_p\circ F)D_pf_q\\
&= \sum_{p=1}^{N} D_p[D(K_l\circ F)\cdot(H_*\circ F)](L_p\circ F) - \sum_{i,p=1}^{N} D_i(K_l\circ F)D_p(H_i\circ F)(L_p\circ F)\\
&\quad + \sum_{j=1}^{N+m}\sum_{q=1}^{m}[(D^2_{N+q,j}K_l)\circ F](H_j\circ F)\underbrace{\left[(L_{N+q}\circ F) - \sum_{p=1}^{N}(L_p\circ F)D_pf_q\right]}_{\overset{T}{=}0 \text{ by (2.5)}}\\
&\quad - \sum_{j=1}^{N+m}\sum_{q=1}^{m}[(D_jK_l)\circ F][(D_{N+q}H_j)\circ F]\underbrace{\sum_{p=1}^{N}(L_p\circ F)D_pf_q}_{\overset{T}{=}L_{N+q}\circ F \text{ by (2.5)}}\\
&\quad - \sum_{j=1}^{N+m}\sum_{p=1}^{N}[(D_jK_l)\circ F][(D_pH_j)\circ F](L_p\circ F)\\
&\quad + \sum_{i,p=1}^{N} D_i(K_l\circ F)D_p(H_i\circ F)(L_p\circ F)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{x_0}{\sim} \sum_{p=1}^{N} D_p[D(K_l \circ F)\cdot(H_* \circ F)](L_p \circ F) \\
&\quad - \sum_{j=1}^{N+m} [(D_j K_l)\circ F] \underbrace{[(DH_j)\circ F]\cdot(L\circ F)}_{\overset{T}{=} D(H_j \circ F)\cdot(L_* \circ F) \text{ by (2.6)}} \\
&\overset{x_0}{\sim} \mathcal{A}_l(H,K,L) - [D(H\circ F)(L_*\circ F)]\cdot[(DK_l)\circ F]\,.
\end{aligned}
$$

5.3. **Proof of** (4.6). Recall that

$$L_{N+h}\circ F \overset{T}{=} Df_h\cdot(L_*\circ F)$$

by (2.5). Then, from Theorem 3.1, we obtain (for $i = 1, \dots, N$)

$$
\begin{aligned}
D_i(L_{N+h}\circ F) &\overset{x_0}{\sim} D_i[Df_h\cdot(L_*\circ F)] \\
&= D_i(Df_h)\cdot(L_*\circ F) + Df_h\cdot D_i(L_*\circ F) \\
&= D_i(Df_h)\cdot(L_*\circ F) \\
&\quad + [(D_i L_*)\circ F]\cdot Df_h + \sum_{q=1}^{m} D_i f_q [(D_{N+q} L_*)\circ F]\cdot Df_h.
\end{aligned} \tag{5.1}
$$

Analogously (for $i = 1, \dots, N$ and $q = 1, \dots, m$)

$$D_i(K_{N+q}\circ F) \overset{x_0}{\sim} D_i(Df_q)\cdot(K_*\circ F) + Df_q\cdot D_i(K_*\circ F). \tag{5.2}$$

From (5.1) it follows that (for $i = 1, \dots, N$)

$$
\begin{aligned}
[(D_i L_*)\circ F]\cdot Df_h &\overset{x_0}{\sim} D_i(L_{N+h}\circ F) \\
&\quad - D_i(Df_h)\cdot(L_*\circ F) - \sum_{q=1}^{m} D_i f_q [(D_{N+q} L_*)\circ F]\cdot Df_h \\
&= [(D_i L_{N+h})\circ F] + \sum_{q=1}^{m} [(D_{N+q} L_{N+h})\circ F] D_i f_q \\
&\quad - D_i(Df_h)\cdot(L_*\circ F) - \sum_{q=1}^{m} D_i f_q [(D_{N+q} L_*)\circ F]\cdot Df_h\,.
\end{aligned} \tag{5.3}
$$

By (5.2), (5.3) and Proposition 3.4, we get

$$\begin{aligned}
\mathcal{C}_h(H,K,L) &= \sum_{i=1}^{N} D(K_i\circ F)\cdot(H_*\circ F)\big([(D_iL_*)\circ F]\cdot Df_h - [(D_iL_{N+h})\circ F]\big)\\
&\quad+\sum_{q=1}^{m} D(K_{N+q}\circ F)\cdot(H_*\circ F)\big([(D_{N+q}L_*)\circ F]\cdot Df_h - [(D_{N+q}L_{N+h})\circ F]\big)\\
&\overset{x_0}{\sim} \sum_{i=1}^{N} D(K_i\circ F)\cdot(H_*\circ F)\Big(\sum_{q=1}^{m}[(D_{N+q}L_{N+h})\circ F]D_if_q\\
&\quad - D_i(Df_h)\cdot(L_*\circ F) - \sum_{q=1}^{m} D_if_q[(D_{N+q}L_*)\circ F]\cdot Df_h\Big)\\
&\quad+\sum_{q=1}^{m}\sum_{i=1}^{N} D_i(Df_q)\cdot(K_*\circ F)(H_i\circ F)\big([(D_{N+q}L_*)\circ F]\cdot Df_h - [(D_{N+q}L_{N+h})\circ F]\big)\\
&\quad+\sum_{q=1}^{m}\sum_{p=1}^{N} Df_q\cdot D_p(K_*\circ F)(H_p\circ F)\big([(D_{N+q}L_*)\circ F]\cdot Df_h - [(D_{N+q}L_{N+h})\circ F]\big)\\
&= \mathcal{F}_h(H,K,L) + \mathcal{S}_h^{(3)}(H,K,L)
\end{aligned}$$

5.4. **Proof of** (4.7)**.** One has

$$\begin{aligned}
\mathcal{B}_h(H,K,L) &= D[D(K_{N+h}\circ F)\cdot(H_*\circ F)]\cdot(L_*\circ F)\\
&\quad-\sum_{p=1}^{N} D_pf_h D[D(K_p\circ F)\cdot(H_*\circ F)]\cdot(L_*\circ F)\\
&= \operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big)\\
&\quad-[D(K_{N+h}\circ F)\cdot(H_*\circ F)]\operatorname{div}(L_*\circ F)\\
&\quad-\sum_{p=1}^{N} D_pf_h \operatorname{div}\big([D(K_p\circ F)\cdot(H_*\circ F)](L_*\circ F)\big)\\
&\quad+\sum_{p=1}^{N} D_pf_h[D(K_p\circ F)\cdot(H_*\circ F)]\operatorname{div}(L_*\circ F)
\end{aligned}$$

where

$$\begin{aligned}
[D(K_{N+h}\circ F)\cdot(H_*\circ F)]\operatorname{div}(L_*\circ F) &\overset{x_0}{\sim} D\Big(\sum_{p=1}^{N} D_pf_h(K_p\circ F)\Big)\cdot(H_*\circ F)\operatorname{div}(L_*\circ F)\\
&= \sum_{p=1}^{N} D_pf_h[D(K_p\circ F)\cdot(H_*\circ F)]\operatorname{div}(L_*\circ F)\\
&\quad-\mathcal{S}_h^{(4)}(H,K,L)
\end{aligned}$$

by (2.5), Theorem 3.1 and Proposition 3.4. Hence

$$\begin{aligned}
\mathcal{B}_h(H,K,L) &\overset{x_0}{\sim} \operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big) \\
&\quad - \sum_{p=1}^{N} D_p f_h \operatorname{div}\big([D(K_p\circ F)\cdot(H_*\circ F)](L_*\circ F)\big) + \mathcal{S}_h^{(4)}(H,K,L) \\
&= \operatorname{div}\big([D(K_{N+h}\circ F)\cdot(H_*\circ F)](L_*\circ F)\big) \\
&\quad - \operatorname{div}\Big(\sum_{p=1}^{N} D_p f_h [D(K_p\circ F)\cdot(H_*\circ F)](L_*\circ F)\Big) \\
&\quad + \underbrace{\sum_{p=1}^{N} [D(K_p\circ F)\cdot(H_*\circ F)][D(D_p f_h)\cdot(L_*\circ F)]}_{=-\mathcal{F}_h(H,K,L)} + \mathcal{S}_h^{(4)}(H,K,L)\,.
\end{aligned}$$

5.5. **Proof of** (4.9)**.** Integrating by parts and using Proposition 3.3, one obtains

$$\begin{aligned}
\sigma_h(H,K,L) &+ \int_{B_r(x_0)} \varphi_{\rho,r} \operatorname{div}\Big(\sum_{i=1}^{N} D_i f_h [D(K_i\circ F)\cdot(H_*\circ F)](L_*\circ F)\Big)\,dx \\
&= -\int_{B_r(x_0)} [D^2 f_h (K_*\circ F)]\cdot(H_*\circ F)[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&\quad - \int_{B_r(x_0)} \sum_{i=1}^{N} D_i f_h [D(K_i\circ F)\cdot(H_*\circ F)][(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&= -\int_{B_r(x_0)} \big(D[Df_h\cdot(K_*\circ F)]\cdot(H_*\circ F)\big)[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&= -\int_{B_r(x_0)} \big(\operatorname{div}[Df_h\cdot(K_*\circ F)(H_*\circ F)]\big)[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&\quad + \int_{B_r(x_0)} [Df_h\cdot(K_*\circ F)\operatorname{div}(H_*\circ F)](L_*\circ F)\cdot D\varphi_{\rho,r}\,dx
\end{aligned}$$

where, by Proposition 3.3, one has:

$$\begin{aligned}
&-\int_{B_r(x_0)} \big(\operatorname{div}[Df_h\cdot(K_*\circ F)(H_*\circ F)]\big)[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&= \int_{B_r(x_0)} Df_h\cdot(K_*\circ F)(H_*\circ F)\cdot D[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx \\
&= \sum_{i=1}^{N} \int_{B_r(x_0)} Df_h\cdot(K_*\circ F)(H_*\circ F)\cdot D[(L_i\circ F) D_i\varphi_{\rho,r}]\,dx
\end{aligned}$$

$$= \sum_{i=1}^{N} \int_{B_r(x_0)} \underbrace{Df_h\cdot(K_*\circ F)[(H_*\circ F)\cdot D(L_i\circ F)]}_{\overset{\mathcal{T}}{=}(K_{N+h}\circ F)[(H_*\circ F)\cdot D(L_i\circ F)] \text{ by } (2.5)} D_i\varphi_{\rho,r}\,dx$$

$$+ \sum_{i=1}^{N} \int_{B_r(x_0)} \underbrace{(L_i\circ F)\,Df_h\cdot(K_*\circ F)(H_*\circ F)}_{\overset{\mathcal{T}}{=}(L_i\circ F)(K_{N+h}\circ F)(H_*\circ F) \text{ by } (2.5)} \cdot DD_i\varphi_{\rho,r}\,dx$$

$$= \sum_{i=1}^{N} \int_{B_r(x_0)} (K_{N+h}\circ F)[(H_*\circ F)\cdot D(L_i\circ F)]D_i\varphi_{\rho,r}\,dx$$

$$+ \sum_{i=1}^{N} \int_{B_r(x_0)} (L_i\circ F)(K_{N+h}\circ F)(H_*\circ F)\cdot DD_i\varphi_{\rho,r}\,dx + o(r^N)$$

$$= \int_{B_r(x_0)} (K_{N+h}\circ F)(H_*\circ F)\cdot D[(L_*\circ F)\cdot D\varphi_{\rho,r}]\,dx + o(r^N)$$

and

$$\int_{B_r(x_0)} \underbrace{[Df_h\cdot(K_*\circ F)\,\mathrm{div}(H_*\circ F)](L_*\circ F)}_{\overset{\mathcal{T}}{=}[(K_{N+h}\circ F)\mathrm{div}(H_*\circ F)](L_*\circ F) \text{ by } (2.5)} \cdot D\varphi_{\rho,r}\,dx$$

$$= \int_{B_r(x_0)} [(K_{N+h}\circ F)\,\mathrm{div}(H_*\circ F)](L_*\circ F)\cdot D\varphi_{\rho,r}\,dx + o(r^N)$$

as $r \to 0+$. Hence

$$\sigma_h(H,K,L) + \int_{B_r(x_0)} \varphi_{\rho,r}\mathrm{div}\left(\sum_{i=1}^{N} D_i f_h[D(K_i\circ F)\cdot(H_*\circ F)](L_*\circ F)\right)dx$$

$$= \int_{B_r(x_0)} (K_{N+h}\circ F)\,\mathrm{div}[(L_*\circ F)\cdot D\varphi_{\rho,r}\,(H_*\circ F)]\,dx + o(r^N)$$

$$= -\int_{B_r(x_0)} (L_*\circ F)\cdot D\varphi_{\rho,r}\,[(H_*\circ F)\cdot D(K_{N+h}\circ F)]\,dx + o(r^N)$$

$$= \int_{B_r(x_0)} \varphi_{\rho,r}\mathrm{div}[(H_*\circ F)\cdot D(K_{N+h}\circ F)\,(L_*\circ F)]\,dx + o(r^N)$$

as $r \to 0+$.

6. Appendix on k-involutive distributions

For the convenience of the reader we begin this section by recalling the definition of k-involutive distribution given in the Introduction. Here, as throughout the section, the letters k, N, m are used to denote three positive integers.

Definition 6.1. Let $\mathcal{D}$ be a C^k distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$. Then $\mathcal{D}$ is said to be k-involutive at $z_0 \in U$ if there exist a neighborhood V of z_0, with $V \subset U$, and a family $X := \{X_j\}_{j=1}^{N} \subset C^k(V, \mathbb{R}^{N+m})$ satisfying the following conditions:

(i) $\mathcal{L}_0^X(z) = \operatorname{span}\{X_j(z) : 1 \le j \le N\} = \mathcal{D}(z)$ for all $z \in V$;

(ii) $\mathcal{L}_h^X(z_0) \subset \mathcal{D}(z_0)$ for all $h = 0, \dots, k$.

When $k = 1$ we will simply say "involutive" instead of "1-involutive".

The following result shows that Definition 6.1 does not depend on the choice of the family $\{X_j\}_{j=1}^N$.

Proposition 6.1. *Given an open set $V \subset \mathbb{R}^{N+m}$, consider a C^k distribution $\mathcal{D}$ of rank N on V and two families $\{X_j\}_{j=1}^N, \{Y_j\}_{j=1}^N \subset C^k(V, \mathbb{R}^{N+m})$ satisfying*

$$\operatorname{span}\{X_j(z) : 1 \le j \le N\} = \operatorname{span}\{Y_j(z) : 1 \le j \le N\} = \mathcal{D}(z)$$

for all $z \in V$. Then, for all $z_0 \in V$ and $h \in \{1, \dots, k\}$, one has

$$\bigcup_{q=0}^{h} \mathcal{L}_q^X(z_0) = \bigcup_{q=0}^{h} \mathcal{L}_q^Y(z_0)$$

with $X := \{X_j\}_{j=1}^N$ and $Y := \{Y_j\}_{j=1}^N$.

Before proving it, we provide an example showing that, under the assumptions of Proposition 6.1, for $h \ge 1$ the equality $\mathcal{L}_h^X(z_0) = \mathcal{L}_h^Y(z_0)$ does not necessarily take place. Indeed, it may even happen that the inclusions $\mathcal{L}_h^X(z_0) \subset \mathcal{L}_h^Y(z_0)$ and $\mathcal{L}_h^X(z_0) \supset \mathcal{L}_h^Y(z_0)$ are both false.

Example 6.1 ($N = 2$, $m = 1$)**.** Let U be any open set in $\mathbb{R}^3$ not intersecting the plane $x_1 = 0$ and consider the vector fields X_1, X_2, Y_1, Y_2 on U defined by

$$X_1(z) := (1, 0, 2x_2), \quad X_2(z) := (0, 1, -2x_1)$$

and

$$Y_1(z) := X_1(z) = (1, 0, 2x_2), \quad Y_2(z) := x_1 X_2(z) = (0, x_1, -2x_1^2)$$

for all $z = (x_1, x_2, y_1) \in U$. Then

$$\operatorname{span}\{X_1(z), X_2(z)\} = \operatorname{span}\{Y_1(z), Y_2(z)\}$$

for all $z \in U$. Moreover, from (2.1) we easily obtain

$$[X_1, X_2](z) = (0, 0, -4), \quad [Y_1, Y_2](z) = (0, 1, -6x_1)$$

for all $z = (x_1, x_2, y_1) \in U$, hence (if $X := \{X_1, X_2\}$ and $Y := \{Y_1, Y_2\}$)

$$\mathcal{L}_1^X(z) \not\subset \mathcal{L}_1^Y(z), \quad \mathcal{L}_1^X(z) \not\supset \mathcal{L}_1^Y(z)$$

for all $z \in U$.

Proposition 6.1 is an immediate consequence of

Lemma 6.1. *Let $\mathcal{D}$ be a C^k distribution of rank N on an open set $V \subset \mathbb{R}^{N+m}$ and consider $\{X_j\}_{j=1}^N, \{Y_j\}_{j=1}^N \subset C^k(V, \mathbb{R}^{N+m})$ satisfying*

$$\operatorname{span}\{X_j(z) : 1 \le j \le N\} = \operatorname{span}\{Y_j(z) : 1 \le j \le N\} = \mathcal{D}(z)$$

for all $z \in V$. Then for all integers $h, i_1, \dots, i_{h+1}$ such that

$$0 \le h \le k, \qquad 1 \le i_i, \dots, i_{h+1} \le N$$

there exists a family of functions $\alpha^{(i_1,\ldots,i_{h+1})}_{(j_1,\ldots,j_{q+1})}$, *with* $0 \le q \le h$ *and* $0 \le j_i, \ldots, j_{q+1} \le N$, *such that*

$$\alpha^{(i_1,\ldots,i_{h+1})}_{(j_1,\ldots,j_{q+1})} \in C^{k-h+q}(V) \tag{6.1}$$

and

$$\Lambda^Y_{(i_1,\ldots,i_{h+1})} = \sum_{q=0}^{h} \sum_{j_1,\ldots,j_{q+1}=1}^{N} \alpha^{(i_1,\ldots,i_{h+1})}_{(j_1,\ldots,j_{q+1})} \Lambda^X_{(j_1,\ldots,j_{q+1})} \tag{6.2}$$

where $X := \{X_j\}_{j=1}^N$ *and* $Y := \{Y_j\}_{j=1}^N$.

Proof. First of all, let $\{a^{(i)}_p : 1 \le i, p \le N\} \subset C^k(V)$ be such that

$$Y_i = \sum_{p=1}^{N} a^{(i)}_p X_p \qquad (i = 1, \ldots, N).$$

We proceed by induction on h. If $h = 0$, one has obviously

$$\Lambda^Y_{(i_1)} = Y_{i_1} = \sum_{p=1}^{N} a^{(i_1)}_p X_p = \sum_{j_1=1}^{N} a^{(i_1)}_{j_1} \Lambda^X_{(j_1)}$$

that is (6.2) with $\alpha^{(i_1)}_{(j_1)} := a^{(i_1)}_{j_1} \in C^k(V)$. Then assume the statement is true for $h \le d$ and prove that it holds for $h = d+1$ (where $d \le k-1$). Indeed, setting for simplicity

$$I := (i_1, \ldots, i_{d+1}), \quad \mathcal{I}_r := \{1, \ldots, N\}^r,$$

one has, by (2.2):

$$\begin{aligned}
\Lambda^Y_{(i_1,\ldots,i_{d+2})} &= [\Lambda^Y_I, Y_{i_{d+2}}] = \sum_{q=0}^{d} \sum_{J \in \mathcal{I}_{q+1}} \sum_{p=1}^{N} [\alpha^I_J \Lambda^X_J, a^{(i_{d+2})}_p X_p] \\
&= \sum_{p=1}^{N} \sum_{q=0}^{d} \sum_{J \in \mathcal{I}_{q+1}} \alpha^I_J (\Lambda^X_J \cdot D a^{(i_{d+2})}_p) X_p - \sum_{q=0}^{d} \sum_{J \in \mathcal{I}_{q+1}} \sum_{p=1}^{N} a^{(i_{d+2})}_p (X_p \cdot D\alpha^I_J) \Lambda^X_J \\
&+ \sum_{q=0}^{d} \sum_{J \in \mathcal{I}_{q+1}} \sum_{p=1}^{N} \alpha^I_J a^{(i_{d+2})}_p [\Lambda^X_J, X_p]
\end{aligned}$$

that is

$$\begin{aligned}
\Lambda^Y_{(i_1,\ldots,i_{d+2})} &= \sum_{j_1=1}^{N} \sum_{q=0}^{d} \sum_{L \in \mathcal{I}_{q+1}} \alpha^I_L (\Lambda^X_L \cdot D a^{(i_{d+2})}_{j_1}) \Lambda^X_{(j_1)} \\
&- \sum_{q=0}^{d} \sum_{J \in \mathcal{I}_{q+1}} \sum_{p=1}^{N} a^{(i_{d+2})}_p (X_p \cdot D\alpha^I_J) \Lambda^X_J \\
&+ \sum_{q=1}^{d} \sum_{J \in \mathcal{I}_q} \sum_{j_{q+1}=1}^{N} \alpha^I_J a^{(i_{d+2})}_{j_{q+1}} \Lambda^X_{J \times \{j_{q+1}\}} + \sum_{J \in \mathcal{I}_{d+1}} \sum_{j_{d+2}=1}^{N} \alpha^I_J a^{(i_{d+2})}_{j_{d+2}} \Lambda^X_{J \times \{j_{d+2}\}}.
\end{aligned}$$

Hence

$$\alpha^{(i_1,\dots,i_{d+2})}_{(j_1)} = \sum_{q=0}^{d} \sum_{L\in\mathcal{I}_{q+1}} \alpha^I_L(\Lambda^X_L \cdot Da^{(i_{d+2})}_{j_1}) - \sum_{p=1}^{N} a^{(i_{d+2})}_p (X_p \cdot D\alpha^I_{(j_1)})\,,$$

$$\alpha^{(i_1,\dots,i_{d+2})}_{(j_1,\dots,j_{d+2})} = \alpha^I_{(j_1,\dots,j_{d+1})} a^{(i_{d+2})}_{j_{d+2}}$$

and, for $q = 1,\dots,d$

$$\alpha^{(i_1,\dots,i_{d+2})}_{j_1,\dots,j_{q+1}} = \alpha^I_{(j_1,\dots,j_q)} a^{(i_{d+2})}_{j_{q+1}} - \sum_{p=1}^{N} a^{(i_{d+2})}_p (X_p \cdot D\alpha^I_{(j_1,\dots,j_{q+1})})\,.$$

Now verifying the regularity of the coefficients is a trivial exercise based on (2.3) and (6.1). For example, for $q = 0$:

$$\alpha^{(i_1,\dots,i_{d+2})}_{(j_1)} = \sum_{r=0}^{d} \sum_{L\in\mathcal{I}_{r+1}} \underbrace{\alpha^I_L}_{\in C^{k-d+r}} (\underbrace{\Lambda^X_L}_{\in C^{k-r}} \cdot \underbrace{Da^{(i_{d+2})}_{j_1}}_{\in C^{k-1}}) - \sum_{p=1}^{N} \underbrace{a^{(i_{d+2})}_p}_{\in C^k} (\underbrace{X_p}_{\in C^k} \cdot \underbrace{D\alpha^I_{(j_1)}}_{\in C^{k-d-1}})$$

hence $\alpha^{(i_1,\dots,i_{d+2})}_{(j_1)} \in C^{k-(d+1)}(V) = C^{k-(d+1)+q}(V)$. Analogously one verifies the other cases. □

Proposition 6.2. *Let $\mathcal{D}$ be a C^k distribution of rank N on an open set $U \subset \mathbb{R}^{N+m}$. Then $\mathcal{D}$ is k-involutive everywhere if and only if it is involutive everywhere.*

Proof. If $\mathcal{D}$ is k-involutive everywhere, then it is obviously involutive everywhere too. Vice versa, assume that $\mathcal{D}$ is involutive everywhere and consider an arbitrary $z_0 \in U$. We have to prove that $\mathcal{D}$ is k-involutive at z_0. Indeed, from the Frobenius theorem [12, Theorem 2.11.9] it follows that there is a C^k submanifold $\mathcal{M}$ of U such that $z_0 \in \mathcal{M}$ and $\tau(\mathcal{M},\mathcal{D}) = \mathcal{M}$. If $X := \{X_j\}_{j=1}^N \subset C^k(V,\mathbb{R}^{N+m})$ generates $\mathcal{D}$ in a neighborhood $V \subset U$ of z_0, then we obtain $\mathcal{L}^X_h(z_0) \subset \mathcal{D}(z_0)$ for all $h = 0,\dots,k$. □

Example 6.2 ($N = 2$, $m = 1$)**.** Let $\mathcal{D}$ be the distribution of rank 2 on $\mathbb{R}^3$ defined as

$$\mathcal{D}(z) := \mathrm{span}\{H(z), K(z)\}, \quad z = (x_1, x_2, y_1) \in \mathbb{R}^3$$

where

$$H(z) := (1,0,0), \quad K(z) := (0,1,x_1^2)\,.$$

By a short computation based on (2.1), we find

$$[H,K](z) = (0,0,2x_1), \quad [[H,K],H](z) = (0,0,-2), \quad [[H,K],K](z) = 0$$

for all $z = (x_1,x_2,y_1) \in \mathbb{R}^3$. In particular, setting $X := \{H,K\}$, one has

$$\mathcal{L}^X_0(0) = \mathcal{D}(0) = \mathrm{span}\{H(0),K(0)\} = \mathrm{span}\{(1,0,0),(0,1,0)\} = \mathbb{R}^2 \times \{0\}$$

$$\mathcal{L}^X_1(0) = \mathrm{span}\{[H,K](0)\} = \mathrm{span}\{(0,0,0)\} \subset \mathcal{D}(0)$$

and

$$\mathcal{L}^X_2(0) = \mathrm{span}\{[[H,K],H](0), [[H,K],K](0)\} = \mathrm{span}\{(0,0,-2)\}$$

hence

$$\mathcal{L}^X_2(0) \not\subset \mathcal{D}(0)\,.$$

Then:

- The distribution $\mathcal{D}$ is 1-involutive (that is involutive) at 0 but it is not 2-involutive at 0;
- Moreover $\mathcal{D}$ cannot be involutive everywhere (or at every point in a neighborhood of 0), by Proposition 6.2;
- From Corollary 1.1 it follows that there is no 2-dimensional C^2 submanifold $\mathcal{M}$ of $\mathbb{R}^3$ such that $0 \in \mathcal{M}$ and 0 is a 4-density point (relative to $\mathcal{M}$) of $\tau(\mathcal{M}, \mathcal{D})$.

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