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AN UPPER BOUND OF A GENERALIZED UPPER HAMILTONIAN NUMBER OF A GRAPH

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Abstract. In this article we study graphs with ordering of vertices, we define a generalization called a pseudoordering, and for a graph H we define the H-Hamiltonian number of a graph G. We will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We will prove equivalent characteristics of an isomorphism of graphs G and H using H-Hamiltonian number of G. Furthermore, we will show that for a fixed number of vertices, each path has a maximal upper H-Hamiltonian number, which is a generalization of the same claim for upper Hamiltonian numbers and upper traceable numbers. Finally we will show that for every connected graph H only paths have maximal H-Hamiltonian number.

1. Introduction

In this article we study a part of graph theory based on an ordering of vertices. We define a generalization called a pseudoordering of a graph. We will show how to generalize a Hamiltonian number, for a graph H we define the H-Hamiltonian number of a graph G and we will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We get them by a special choice of graph H. Furthermore, we will study a maximalization of upper H-Hamiltonian number for a fixed number of vertices. We will show that, for a fixed number of vertices, each path has a maximal upper H-Hamiltonian number. From the definition it will be obvious that a lower bound of the H-Hamiltonian number is the number of edges $|E(H)|$ and the graph G has a minimal lower H-Hamiltonian number if and only if H is a subgraph of G. Now we can say that G having a maximal upper H-Hamiltonian number is dual to H being a subgraph of G. Furthermore, by above for every two finite graphs G and H such that G is connected satisfying $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, we get that $G \cong H$ if and only if the lower H-Hamiltonian number of G is $|E(H)|$.

In [2] it is proved that G has a maximal upper traceable number if and only if G is a path. The same is proved for Hamiltonian number. We will show that for

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H connected G has a maximal H-Hamiltonian number if and only if G is a path. This shows that this generalization of ordering of vertices is natural.

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In this article we will study a generalization of Hamiltonian spectra of undirected finite graphs. Recall that, a graph G is a pair

$$G = \bigl(V(G), E(G)\bigr)\,,$$

where $V(G)$ is a finite set of vertices of G and $E(G) \subseteq V(G) \times V(G)$, a symmetric Antireflexive relation, is a set of edges. We will denote an edge between v and u by $\{v, u\}$.

Recall that, an *ordering* on the graph G is a bijection

$$f\colon \{1, 2, \dots, |V(G)|\} \to V(G)\,,$$

we denote

$$s(f,G) = \sum_{i=1}^{|V(G)|} \rho_G\bigl(f(i), f(i+1)\bigr)\,,$$
$$\bar{s}(f,G) = \sum_{i=1}^{|V(G)|-1} \rho_G\bigl(f(i), f(i+1)\bigr)\,,$$

where $\rho_G(x, y)$ is the distance of x, y in the graph G and $f(|V(G)|+1) := f(1)$, for better notation. We will write only $s(f)$, $\bar{s}(f)$ if the graph is clear from context. Then

$$\{s(f,G) \mid f \ \text{ ordering on } \ G\}$$
$$\{\bar{s}(f,G) \mid f \ \text{ ordering on } \ G\}$$

are the *Hamiltonian spectrum* of the graph G and the *traceable spectrum* of the graph G, respectively.

We want to generalize the notion of an ordering of a graph.

Definition 1.1. Let G, H be graphs such that $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a bijection, then we call f a *pseudoordering* on the graph G (by H), denote

$$s_H(f,G) = \sum_{\{x,y\}\in E(H)} \rho_G\bigl(f(x), f(y)\bigr)\,,$$

where $\rho_G(x, y)$ is the distance of x, y in the graph G. We will call $s_H(f,G)$ the sum of the pseudoordering f. Then

$$\{s_H(f,G) \mid f \ \text{ pseudoordering on } \ G \text{ by } H\}$$

is the *H-Hamiltonian spectrum* of the graph G.

The minimum and the maximum of a Hamiltonian spectrum and of a traceable spectrum are called the (*lower*) *Hamiltonian number* and the *upper Hamiltonian number*, respectively. Furthermore, the (lower) traceable number and the upper traceable number of a graph G are denoted by

$$h(G) = \min\{s(f,G) \mid f \ \text{ ordering on } \ G\}\,,$$
$$h^+(G) = \max\{s(f,G) \mid f \ \text{ ordering on } \ G\}\,,$$
$$t(G) = \min\{\bar{s}(f,G) \mid f \ \text{ ordering on } \ G\}\,,$$
$$t^+(G) = \max\{\bar{s}(f,G) \mid f \ \text{ ordering on } \ G\}\,.$$

Now we define generalized versions.

Definition 1.2.

$$h_H(G) = \min\{s_H(f,G) \mid f \ \text{ pseudoordering on } \ G\},$$
$$h_H^+(G) = \max\{s_H(f,G) \mid f \ \text{ pseudoordering on } \ G\}\,.$$

We will call them the *lower H-Hamiltonian number* and the *upper H-Hamiltonian number* of a graph G, respectively.

Now take $H = C_{|V(G)|}$, where C_n is the cycle with n vertices. When we denote the vertices of $C_{|V(G)|}$ by $\{1,2,\dots,|V(G)|\}$ we can see that

$$s(f,G) = s_{C_{|V(G)|}}(f,G)\,.$$

Analogously for $H = P_{|V(G)|-1}$, where P_{n-1} is the path of length $n-1$, we get that

$$\bar{s}(f,G) = s_{P_{|V(G)|-1}}(f,G)\,.$$

Remark 1.3. The $C_{|V(G)|}$-Hamiltonian spectrum of a graph G is equal to the Hamiltonian spectrum of G for $|V(G)| \geq 3$, and the $P_{|V(G)|-1}$-Hamiltonian spectrum of G is equal to the traceable spectrum of G for $|V(G)| \geq 2$.

Lemma 1.4. *Let G be a connected finite graph and H be a graph such that $|V(G)| = |V(H)|$, then $h_H(G) = |E(H)|$ if and only if H is isomorphic to some subgraph of G.*

Proof. Let $f\colon V(H) \to V(G)$ be a pseudoordering satisfying $s(f,G) = |E(H)|$, then f is an injective graph homomorphism. The opposite implication is obvious. □

Lemma 1.5. *Let G be a connected finite graph and H be a graph such that $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, then $h_H(G) = |E(H)|$ if and only if H is isomorphic to the graph G.*

Proof. The graph H is isomorphic to a subgraph of G and furthermore $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, hence $H \cong G$. The opposite implication is obvious. □

2. Maximalization of the upper H-Hamiltonian number of a graph G

In this section we will prove that for every pair of connected graphs H, G and each pseudoordering f there exists a pseudoordering

$$g\colon V(H) \to \{1,2,\dots,|V(G)|\}$$

such that

$$s_H(f,G) \leq s_H(g, P_{|V(G)|-1})\,.$$

At first, let G be a tree. We will only work with graphs which have at least 2 vertices.

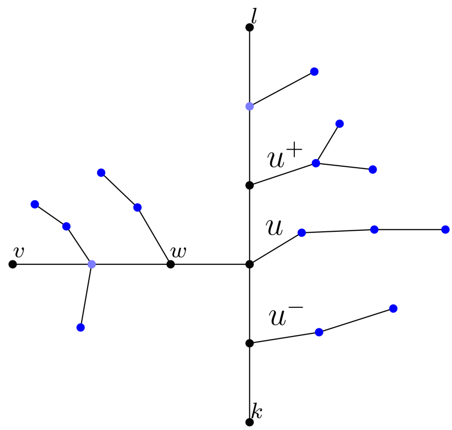
Definition 2.1. Let G and H be graphs such that G is connected, $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering. Furthermore, let $a, b \in V(G)$, we define $a \sim_{H,f} b$ if and only if $\{f^{-1}(a), f^{-1}(b)\} \in E(H)$.

Definition 2.2. Let G be a tree such that G is not a path. Denote three pairwise distinct leaves by l, k, $v \in V(G)$. Because G is not a path then G has at least 3 leaves, connect l, k with a path $l = x_1, x_2, \ldots, x_m = k$. Connect v, l with a path l $v = y_1, y_2, \ldots, y_s = l$ and take the minimum of a set

$$i_m = \min\{i \mid \exists j \in \{1, \ldots, m\}, y_i = x_j\}\,.$$

Take j_m such that $y_{i_m} = x_{j_m}$. Now we define $u = y_{i_m}$, $w = y_{i_m-1}$, $u^+ = x_{j_m-1}$, $u^- = x_{j_m+1}$.

Example.



Remark 2.3. $l \neq u \neq k$.

Definition 2.4. Define a set $K(v, G) \subseteq V(G)$ as a set of vertices $z \in V(G)$ such the path between z and l uses the edge $\{w, u\}$.

Remark 2.5. $K(v, G)$ is the connected component of $\big(V(G), E(G) \setminus \{w, u\}\big)$, G without edge $\{w, u\}$, which contains v.

Lemma 2.6. (i) *Paths between vertices from $K(v, G)$ don't use the edge $\{w, u\}$.*

(ii) *Paths between vertices from $V(G) \setminus K(v, G)$ don't use the edge $\{w, u\}$.*

(iii) *Paths joining a vertex from $V(G) \setminus K(v, G)$ to a vertex from $K(v, G)$ use the edge $\{w, u\}$.*

Proof. Because G is a tree, there is a unique path between each pair of vertices, then it is obvious by remark 2.5. $\square$

Definition 2.7. Define graphs

$$\bar{G} = \big(V(G), E(G) \setminus \{\{w,u\}\} \cup \{\{w,l\}\}\big)\,,$$
$$\tilde{G} = \big(V(G), E(G) \setminus \{\{w,u\}\} \cup \{\{w,k\}\}\big)\,.$$

Lemma 2.8. *$\bar{G}$ and $\tilde{G}$ are trees.*

Proof. At first we show connectivity, let a, $b \in V(G)$, connect them with a path. If both are in $K(v,G)$ or in $V(G) \setminus K(v,G)$, then by Lemma 2.6, the path in G uses only edges which are also in $\bar{G}, \tilde{G}$. Hence it is path also there.

Let $a \in K(v,G)$ and $b \in V(G) \setminus K(v,G)$. We can see $w \in K(v,G)$, by Lemma 2.6 a path between a and w, $a = a_1, a_2, \ldots, a_p = w$, doesn't use $\{w,u\}$ and all vertices of this path are in $K(v,G)$. If not, there is a path between vertices from $K(v,G)$ and $V(G) \setminus K(v,G)$ which doesn't use $\{w,u\}$, that is a contradiction with Lemma 2.6. Connect l and b with a path, $l = b_1, b_2, \ldots, b_q = b$. It doesn't use $\{w,u\}$ and all vertices are in $V(G) \setminus K(v,G)$. Then $a = a_1, a_2, \ldots, a_p = w,\ l = b_1, b_2, \ldots, b_q = b$ is a path between a, b in the graph $\bar{G}$, analogously for $\tilde{G}$.

Now we show that they don't contain a cycle, for contradiction suppose that $\bar{G}$ contains a cycle $C \subseteq \bar{G}$. If C doesn't use the edge $\{w,l\}$, then $C \subseteq G$, but G is a tree, this is a contradiction. If C uses $\{w,l\}$, then there exists a path in G between w, l, which doesn't use the edge $\{w,l\}$. Then there exists a path in G between w, l, which doesn't use the edge $\{w,u\}$, but $w \in K(v,G)$ and $l \in V(G) \setminus K(v,G)$, that is contradiction with Lemma 2.6. Analogously for $\tilde{G}$. □

We want to show that

$$s_H(G,f) \le s_H(\bar{G},f)$$

or

$$s_H(G,f) \le s_H(\tilde{G},f)\,.$$

Lemma 2.9.

$$a,b \in K(v,G), \qquad \text{then} \quad \rho_G(a,b) = \rho_{\bar{G}}(a,b) = \rho_{\tilde{G}}(a,b)\,,$$
$$a,b \in V(G) \setminus K(v,G), \quad \text{then} \quad \rho_G(a,b) = \rho_{\bar{G}}(a,b) = \rho_{\tilde{G}}(a,b)\,.$$

Proof. A path in G between a, b, by Lemma 2.6, doesn't use $\{u,w\}$, hence it is a path in $\bar{G}$ and $\tilde{G}$ too, then the distance of a, b is the same in G, $\bar{G}$ and $\tilde{G}$. □

Definition 2.10. Define subsets

$$F^+,\ F^-,\ F^0 \subseteq K(v,G) \times \big(V(G) \setminus K(v,G)\big)$$

such that $(a,b) \in F^+$ if a path between a, b uses the edge $\{u,u^+\}$. $(a,b) \in F^-$ if a path between a, b uses the edge $\{u,u^-\}$ and $(a,b) \in F^0$ if a path between a, b doesn't use neither $\{u,u^-\}$ nor $\{u,u^+\}$.

Lemma 2.11. *F^+, F^-, F^0 are pairwise disjoint and*

$$F^+ \cup F^- \cup F^0 = K(v,G) \times \big(V(G) \setminus K(v,G)\big)\,.$$

Proof. From the definition of F^+, F^-, F^0 we have F^- and F^0, F^+ and F^0 are disjoint. Let $(a,b) \in F^+ \cap F^-$, then the path between a, b uses edges $\{u, u^-\}, \{u, u^+\}$ and by lemma 2.6, it also uses the edge $\{w, u\}$. Hence it is a path which has a vertex of degree 3 and that is contradiction. $\square$

Lemma 2.12. *Let $x, \bar{x} \in K(v,G)$ and $y, \bar{y} \in V(G) \setminus K(v,G)$ such that $(x,y) \in F^+$ and $(\bar{x}, \bar{y}) \in F^-$. Then*

$$\rho_{\bar{G}}(x,y) + \rho_{\bar{G}}(\bar{x},\bar{y}) \geq \rho_G(x,y) + \rho_G(\bar{x},\bar{y})\,,$$

$$\rho_{\tilde{G}}(x,y) + \rho_{\tilde{G}}(\bar{x},\bar{y}) \geq \rho_G(x,y) + \rho_G(\bar{x},\bar{y})\,.$$

Moreover, both sides are equal, in the first inequality, if and only if $y = l$ and, in the second inequality, if and only if $\bar{y} = k$.

Proof. Let z denote the first common vertex of paths $Q\colon l = y_1, y_2, \dots, y_s = k$ and $P\colon y = x_1, x_2, \dots, x_m = x$. Consider

$$i_m = \min\{i \mid \exists j \in \{1, \dots, m\}, y_i = x_j\}$$

and therefore $z = y_{i_m}$, let T be the path from z to l, we will show that z is the only one common vertex of T and P, vertices from P split into the 4 subpaths, P_1 from y to z, P_2 from z to u, edge $\{u,w\}$ and P_3 from w to x. Vertices from P_1 are not in Q (except for z) from the definition of z. Vertices from P_2 are not in T (except for z) from the uniqueness of paths in trees and vertices from P_3 belong to $K(v,G)$ and every vertex of T belongs to $V(G) \setminus K(v,G)$. By composition of paths P_1, $T, \{l,w\}$, P_3, we get a path from y to x in the graph $\bar{G}$.

Let $\bar{P}$ denote the path from $\bar{y}$ to $\bar{x}$, analogously define $\bar{z}$ as the first common vertex of paths $\bar{P}$ and Q (first in the direction from $\bar{y}$ to $\bar{x}$). We split $\bar{P}$ into the subpaths $\bar{P}_1$ from $\bar{y}$ to $\bar{z}$, $\bar{P}_2$ from $\bar{z}$ to u, edge $\{u,w\}$ and $\bar{P}_3$ from u to $\bar{x}$. Let $\bar{T}$ be the path from u to l, analogously we get that u is the only one common vertex of $\bar{P}$ and $\bar{T}$. Hence $\bar{P}_1, \bar{P}_2, \bar{T}, \{l,w\}, \bar{P}_3$ is a path between $\bar{y}, \bar{x}$ in the graph $\bar{G}$.

And for paths from u to z and from u to $\bar{z}$, u is the only one common vertex, by uniqueness of path in trees.

Now we can calculate

$$\rho_G(x,y) = \rho_G(x,w) + 1 + \rho_G(u,z) + \rho_G(z,y)\,,$$

$$\rho_G(\bar{x},\bar{y}) = \rho_G(\bar{x},w) + 1 + \rho_G(u,\bar{z}) + \rho_G(\bar{z},\bar{y})\,,$$

$$\rho_{\bar{G}}(x,y) = \rho_G(x,w) + 1 + \rho_G(l,z) + \rho_G(z,y)\,,$$

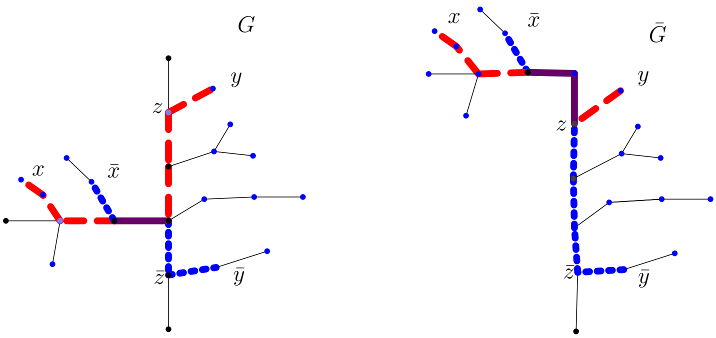
$$\rho_{\bar{G}}(\bar{x},\bar{y}) = \rho_G(\bar{x},w) + 1 + \rho_G(l,z) + \rho_G(z,u) + \rho_G(u,\bar{z}) + \rho_G(\bar{z},\bar{y})\,,$$

hence

$$\rho_{\bar{G}}(\bar{x},\bar{y}) + \rho_{\bar{G}}(x,y) = \rho_G(\bar{x},\bar{y}) + \rho_G(x,y) + 2\rho_G(l,z)\,.$$

Now we get our inequality and we see that both are equal if and only if $l = z$. But l is a leaf, hence z is a leaf, then $y = z = l$. For $\tilde{G}$ analogously. $\square$

Example. Paths between x, y and $\bar{x}$, $\bar{y}$ in graphs G and $\bar{G}$.



Lemma 2.13. *Let* $(x, y) \in F^0$ *then*

$$\rho_{\bar{G}}(x, y) > \rho_G(x, y)\,,$$
$$\rho_{\tilde{G}}(x, y) > \rho_G(x, y)\,.$$

Proof. Let P be a path from x to y and Q be a path from l to k in G, for P and Q, u is the only one common vertex because $(x, y) \in F^0$. Hence $x \to w - l \to u \to y$ is a path in $\bar{G}$, where paths of type $a \to b$ are subpaths of P and Q and $-$ denotes an edge. Now we can calculate the following

$$\rho_{\bar{G}}(x, y) = \rho_G(x, u) + 1 + \rho_G(l, u) + \rho_G(u, y) = \rho_G(x, y) + \rho_G(l, u)$$

and from $l \neq u$ we have inequality.

For $\tilde{G}$ analogously. $\square$

Lemma 2.14.

$$\rho_{\bar{G}}(x, y) > \rho_G(x, y) \quad \textit{for} \quad (x, y) \in F^-\,,$$
$$\rho_{\tilde{G}}(x, y) > \rho_G(x, y) \quad \textit{for} \quad (x, y) \in F^+\,.$$

Proof. We will prove the first inequality. As well as in lemma 2.12 denote z the first common vertex of paths from y to x and from k to l, formally we can define it as well as in lemma 2.12. Now we consider a path
$x \to w - l \to u \to z \to y$. Hence

$$\begin{aligned}\rho_{\bar{G}}(x, y) &= \rho_G(x, w) + 1 + \rho_G(l, u) + \rho_G(u, z) + \rho_G(z, y)\\ &= \rho_G(x, y) + \rho_G(l, u)\end{aligned}$$

and from $l \neq u$ we have inequality.

For second inequality analogously. $\square$

Definition 2.15. Let G be a tree and H be a graph such that

$$|V(G)| \;=\; |V(H)|$$

and

$$f\colon V(H) \to V(G)$$

is a pseudoordering, we define a set

$$L = \{(x,y) \in K(v,G) \times \big(V(G) \setminus K(v,G)\big) \mid x \sim_{H,f} y\},$$

where $K(v,G)$ is the set from Definition 2.4.

Lemma 2.16. *Let G be a tree and H be a graph such that, $|V(G)| = |V(H)|$ and*

$$f\colon V(H) \to V(G)$$

is a pseudoordering. Then

$$s_H(f,\bar{G}) \geq s_H(f,G)$$

or

$$s_H(f,\tilde{G}) \geq s_H(f,G)\,,$$

the first case occurs when

$$|L \cap F^+| \leq |L \cap F^-|\,,$$

the second case occurs when

$$|L \cap F^+| \geq |L \cap F^-|\,.$$

Proof. Denote $n^+ = |L \cap F^+|$, $n^- = |L \cap F^-|$, $m = |L \cap F^0|$,

$$\bar{m} = \frac{\Big|\big\{(x,y) \in \big(K(v,G)^2\big) \cup \Big(\big(V(G) \setminus K(v,G)\big)^2\Big) \mid x \sim_{H,f} y\big\}\Big|}{2}\,,$$

where square $K(v,G)^2$ means $K(v,G) \times K(v,G)$. $\bar{m}$ is number of edges $\{x,y\} \in E(H)$, which satisfy that $f(x)$ and $f(y)$ lie in the same component of

$$(V(G), E(G) \setminus \{w,u\})\,.$$

Let $n^+ \geq n^-$, the second case is analogous, we rearrange the sum $s_H(f,G)$ in this way

$$s_H(f,G) = \sum_{i=1}^{n^-} \big(\rho_G(x_i,y_i) + \rho_G(\bar{x}_i,\bar{y}_i)\big) + \sum_{i=n^-+1}^{n^+} \rho_G(x_i,y_i)$$

$$+ \sum_{i=1}^{m} \rho_G(a_i,b_i) + \sum_{i=1}^{\bar{m}} \rho_G(c_i,d_i)\,,$$

where

$$(x_i,y_i) \in F^+, \quad (\bar{x}_i,\bar{y}_i) \in F^-, \quad (a_i,b_i) \in F^0,$$

$$(c_i,d_i) \in \big\{(x,y) \in \big(K(v,G)^2\big) \cup \Big(\big(V(G) \setminus K(v,G)\big)^2\Big) \mid x \sim_{H,f} y\big\}\,.$$

Now, by Lemma 2.12

$$\rho_G(x_i,y_i) + \rho_G(\bar{x}_i,\bar{y}_i) \leq \rho_{\tilde{G}}(x_i,y_i) + \rho_{\tilde{G}}(\bar{x}_i,\bar{y}_i)\,,$$

by Lemma 2.14

$$\rho_G(x_i,y_i) \leq \rho_{\tilde{G}}(x_i,y_i)\,,$$

by Lemma 2.13
$$\rho_G(a_i,b_i) \le \rho_{\tilde{G}}(a_i,b_i)$$
and by Lemma 2.9
$$\rho_G(c_i,d_i) = \rho_{\tilde{G}}(c_i,d_i)\,.$$
Hence
$$\begin{aligned} s_H(f,G) &\le \sum_{i=1}^{n^-}(\rho_{\tilde{G}}(x_i,y_i)+\rho_{\tilde{G}}(\bar{x}_i,\bar{y}_i)) \\ &\quad+\sum_{i=n^-+1}^{n^+}\rho_{\tilde{G}}(x_i,y_i)+\sum_{i=1}^{m}\rho_{\tilde{G}}(a_i,b_i)+\sum_{i=1}^{\bar{m}}\rho_{\tilde{G}}(c_i,d_i) \\ &= s_H(f,\tilde{G})\,. \end{aligned}$$
□

Lemma 2.17. *Let G be a tree and H be a graph such that, $|V(G)| = |V(H)|$ and $f\colon V(H)\to V(G)$ is a pseudoordering. Then there exists a pseudoordering*
$$g\colon V(H)\to \{x_1,x_2,\dots,x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \textit{such that}$$
$$s_H(f,G) \le s_H(g,P_{|V(G)|-1})\,.$$

Proof. We denote
$$\alpha(G) = \sum_{\substack{v\in V(G)\\ \deg_G v\ge 3}} \deg_G v,$$
from the definition of u, l and k we know that $\deg_G u \ge 3$ and $\deg_G l = \deg_G k = 1$. From the construction of $\bar{G}$ and $\tilde{G}$ we have $\deg_{\bar{G}} u = \deg_{\tilde{G}} u \le \deg_G u$, $\deg_{\bar{G}} l = \deg_{\tilde{G}} k = 2$ and all other vertices have the same degree as before. Hence
$$\alpha(\bar{G}) < \alpha(G)\,,$$
$$\alpha(\tilde{G}) < \alpha(G)\,.$$
Let S be a tree, which is not a path, we choose any three pairwise distinct leaves in $V(S)$ and define S^* as one of graphs $\bar{S}, \tilde{S}$, which satisfy $s_H(f,S^*) \ge s_H(f,S)$. Denote $G_0 = G$ and for $i\ge 0$ denote $G_{i+1} = G_i^*$ if G_i is not a path, otherwise define $G_{i+1} = G_i$. For contradiction we assume that the tree G_i is not a path for every $i\in\mathbb{N}_0$. We know $\alpha(G_i)\in\mathbb{N}_0$ for every i and
$$\alpha(G_{i+1}) \le \alpha(G_i) - 1\,,$$
hence
$$\alpha(G_{\alpha(G_0)+1}) \le \alpha(G_0) - \alpha(G_0) - 1 = -1$$
and this is contradiction. Therefore there exists some j such that G_j is a path, from Lemma 2.16 we get
$$s_H(f,G_{i+1}) \ge s_H(f,G_i)$$

and hence

$$s_H(f, G_j) \geq s_H(f, G) .$$

□

Theorem 2.18. *Let G and H be graphs such that G is connected, $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering, then there exists a pseudordering*

$$g\colon V(H) \to \{x_1, x_2, \ldots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \textit{such that}$$

$$s_H(f, G) \leq s_H(g, P_{|V(G)|-1}) .$$

Proof. Let K be any spanning tree of G, x, $y \in V(G)$, we connect x and y with a path in graph K, this path is also a path in G. Hence

$$\rho_G(x, y) \leq \rho_K(x, y)$$

for every x, y, hence

$$s_H(f, G) \leq s_H(f, K),$$

by Lemma 2.17 there exists a pseudoordering

$$g\colon V(H) \to \{x_1, x_2, \ldots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \quad \text{such that}$$

$$s_H(f, G) \leq s_H(f, K) \leq s_H(g, P_{|V(G)|-1}) .$$

□

Corollary 2.19. *Let G and H be graphs such that G is connected, $|V(G)| = |V(H)|$, then*

$$h_H^+(G) \leq h_H^+(P_{|V(G)|-1}) .$$

3. Graphs with a maximal upper H-Hamiltonian number

In this section we will prove that if in Corollary 2.19 the graph H is connected, then in the inequality in Corollary 2.19 both sides are equal.

Remark 3.1. For easier writing, we will denote vertices of H the same as vertices of G, we will rename them in this way $v \in H \mapsto f(v)$. We can naturally see it as graph with two sets of edges.

In inequalities in Lemma 2.16 both sides are equal under specific conditions, if $L \cap F^0 \neq \emptyset$, then in Lemma 2.13 there is a strict inequality and then also the same happens in Theorem 2.18.

If $(L \setminus K(v, G) \times \{l\}) \cap F^+ \neq \emptyset$, then in Lemma 2.12 there is a strict inequality and then also the same happens in Theorem 2.18. Analogously if

$$(L \setminus K(v, G) \times \{k\}) \cap F^- \neq \emptyset .$$

Overall we get that the only nontrivial case is

$$L \subseteq K(v, G) \times \{k, l\} . \tag{1}$$

Remark 3.2. Remark 3.1 holds for every triple of distinct leaves k, l, v in G.

Lemma 3.3. *Let G be a tree, H connected graph such that $|V(G)| = |V(H)|$ and $f\colon V(H) \to V(G)$ is a pseudoordering, which satisfy*

$$s_H(f,G) = h^+_H(P_{|V(G)|-1})\,,$$

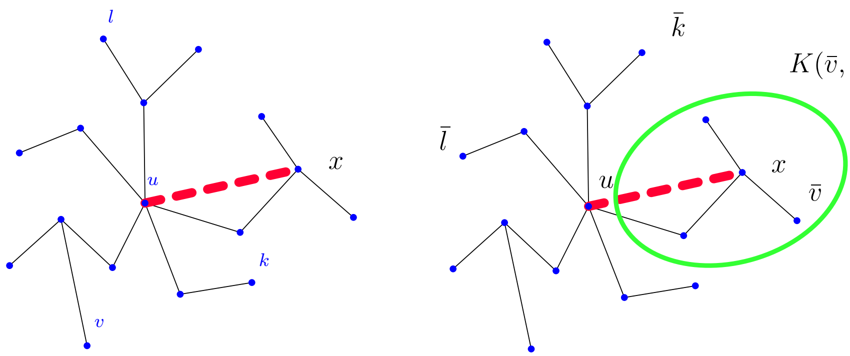
then G is path.

Proof. For contradiction suppose that G is not a path, then there exist three pairwise distinct leaves k, l, v, we denote in the same way as before, vertex u and set of vertices $K(v,G)$. Because graph H is connected there exists a vertex x such that $\{u,x\} \in E(H)$. Let $X \subseteq V(G)$ be a set of vertices of components of graph $G \setminus u$, containing x. $G \setminus u$ has, by definition of u, at least 3 components. Let now $\bar{v}$ be an arbitrary leaf (leaf in G) in X. Choose $\bar{k}$, $\bar{l}$ as arbitrary leaves in pairwise distinct components of $G \setminus u$ and different from X.

Now $(x,u) \in \bar{L}$, where $\bar{L}$ is alternative of L for $\bar{k}$, $\bar{l}$, $\bar{v}$ and by Remark 3.1 for $\bar{k}$, $\bar{l}$, $\bar{v}$ and by $k \neq u \neq l$ we get contradiction.

□

Example. We show the idea of the last proof in the following picture.



Remark 3.4. Let G be a graph with a maximal H-Hamiltonian number, then every spanning tree of G has a maximal H-Hamiltonian number, therefore every spanning tree is a path. We will show that the only graphs with this property are cycles and paths.

Lemma 3.5. *Let G be a connected graph such that $|V(G)| \geq 2$, then there is a vertex, which is not an articulation point.*

Proof. Consider a block-cut tree of G and a block B, which is a leaf of the block-cut tree or if this tree has only one vertex, then $B = G$. B is, by definition of a block, 2-connected. Because B is leaf we get that in B there is only one articulation and in B there are at least 2 vertices. Hence in B there is at least one vertex, which is not an articulation point. □

Lemma 3.6. *Let G be a finite connected graph such that $|V(G)| \geq 2$ and every spanning tree of G is a path, then G is a path or a cycle.*

Proof. We will prove it by induction with respect to the number of vertices. Let n be the number of vertices, for $n = 2$ and $n = 3$ it is obviously true. Let it be true for $n \geq 3$, let G be a graph with $n+1$ vertices such that every spanning tree of G is a path. Let $v \in V(G)$ be a vertex, which is not an articulation point, by lemma 3.5 it exists. We denote G' the subgraph induced by the set of vertices $V(G) \setminus \{v\}$. G' is connected, we will show that every spanning tree of G' is a path. Let there exist a spanning tree which is not a path, let $u \in V(G)$ be a vertex such that $\{v, u\} \in E(G)$. Now when we add this edge to the spanning tree, we get a spanning tree of G, which is not a path and it is a contradiction. By induction hypothesis G' is a path or a cycle, we denote $A = \{u \in V(G) | \{v, u\} \in E(G)\}$. For contradiction we assume G' is a cycle and let $u \in A$, in G' be an edge e such that u is not incident to e. Consider the subgraph B of G, $B = (V(G), E(G') \setminus e \cup \{v, u\})$, and this is a spanning tree of G which is not a path, contradiction.

Therefore G' is a path, let x, y be endpoints of this path, for contradiction we assume that there exists some another vertex $u \in A$. Hence G' together with $\{u, v\}$ form a spanning tree which is not a path. Hence $A \subseteq \{x, y\}$, because G is connected we get also $A \neq \emptyset$. Finally there are the two cases for G, if $|A| = 1$, then G will be a path and if $|A| = 2$, then G will be a cycle. □

Theorem 3.7. *Let G and H be connected finite graphs such that $|V(G)| = |V(H)|$, then*

$$h_H^+(G) \leq h_H^+(P_{|V(G)|-1}) \,,$$

moreover, both sides are equal if and only if G is a path.

Proof. The first part follows from Theorem 2.18, let G be a graph, f be a pseudoordering such that

$$s_H(f, G) = h_H^+(G) = h_H^+(P_{|V(G)|-1}) \,.$$

From the proof of Theorem 2.18 we know that every spanning tree also satisfies the equation above. Hence, by Lemma 3.3, every spanning tree of G is a path. By Lemma 3.6 G is a path or a cycle, for contradiction we assume, that it is a cycle. We denote $n = |V(G)|$, we will show that there are two vertices $v, u \in V(G)$ such that $v \sim_{H,f} u$ and $\rho_G(u, v) < \frac{n}{2}$.

Because G is cycle, $|V(H)| = n \geq 3$ and H is connected we see that there is a vertex of degree at least 2. Let v be a vertex such that $deg_H(v) \geq 2$, there exists at least two vertices u such that $v \sim_{H,f} u$. There exists at most one vertex such that $\rho_G(u, v) \geq \frac{n}{2}$, hence at least one of them satisfies $\rho_G(u, v) < \frac{n}{2}$.

Now we connect v and u with a shorter path in G. Let e be some edge on this path, we define a graph $\bar{G} = \big(V(G), E(G) \setminus e\big)$, it is a path, where every distance is greater or equal as in G. But $\rho_G(u, v) < \rho_{\bar{G}}(u, v)$ and then

$$s_H(f, \bar{G}) = s_H(f, \bar{G}) > h_H^+(P_{|V(G)|-1}) \,,$$

and this is contradiction with Theorem 2.18. □

4. Conclusion

When we use following equations which can be found for example in [2, Theorem 1.3] and [2, Corollary 2.2]

$$h^+(P_{|V(G)|-1}) = \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor , \quad t^+(P_{|V(G)|-1}) = \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1 \,.$$

This result is also calculated in [1] and when we use Theorem 3.7 for $H = P_{|V(G)|-1}$ and for $H = C_{|V(G)|}$ we get the following theorem.

Theorem 4.1 ([2])**.**

$$h^+(G) \le \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor , \quad t^+(G) \le \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1 \,.$$

Moreover, both sides are equal if and only if G is a path.

First part is [2, Corollary 2.2] and second part is [2, Theorem 4.2]. Now we can see, that Theorem 3.7 is generalization of Theorem 4.1 which is from article [2].

References

[1] Dzúrik, M., *Metrické vlastnosti grafů*, bachelor thesis (2018).

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