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A HALF-SPACE TYPE PROPERTY IN THE EUCLIDEAN SPHERE

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Dedicated to Manfredo P. do Carmo, in memory

Abstract. We study the notion of strong r-stability for the context of closed hypersurfaces Σ^n $(n \geq 3)$ with constant $(r+1)$-th mean curvature H_{r+1} immersed into the Euclidean sphere $\mathbb{S}^{n+1}$, where $r \in \{1, \dots, n-2\}$. In this setting, under a suitable restriction on the r-th mean curvature H_r, we establish that there are no r-strongly stable closed hypersurfaces immersed in a certain region of $\mathbb{S}^{n+1}$, a region that is determined by a totally umbilical sphere of $\mathbb{S}^{n+1}$. We also provide a rigidity result for such hypersurfaces.

1. Introduction and statements of the results

The notion of *stability* concerning closed hypersurfaces of constant mean curvature in Riemannian manifolds was first studied by Barbosa and do Carmo in [8], and Barbosa, do Carmo and Eschenburg in [9], where they proved that geodesic spheres are the only stable critical points in a simply connected space form of the area functional for volume-preserving variations. On the other hand, with respect to the notion of *strong stability* related to constant mean curvature closed hypersurfaces (that is, for all variations, not necessarily volume-preserving variations), it is well known that *there are no strongly stable closed hypersurfaces with constant mean curvature in the Euclidean sphere* $\mathbb{S}^{n+1}$ (for instance, see [3, Section 2]). Following the same direction, the author together with Aquino, de Lima and dos Santos obtained in [6] an extension of this result when the space form is either the Euclidean space $\mathbb{R}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$. More precisely, they proved that there does not exist a strongly stable closed hypersurface with constant mean curvature H immersed in either $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ $(n \geq 3)$ and such that its total umbilicity operator Φ satisfies the condition

$$|\Phi| \leq \frac{2\sqrt{n(n-1)}(H^2+c)}{(n-2)|H|},$$

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where $c = 0$ or $c = -1$ according to the space form be $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$, respectively. When $n = 2$ they also showed that there does not exist strongly stable closed surface with constant mean curvature immersed in either $\mathbb{R}^3$ or $\mathbb{H}^3$.

In [1], Alencar, do Carmo and Colares extended the results of [8] and [9] to the context of closed hypersurfaces with constant scalar curvature in a space form. More specifically, they showed that closed hypersurfaces with constant scalar curvature of a space form are the critical points of the so-called 1-area functional for volume-preserving variations and, for the case $\mathbb{S}^{n+1}$ and $\mathbb{R}^{n+1}$, they also proved that a closed hypersurface with constant scalar curvature is stable if and only if it is a geodesic sphere. More recently Alías, Brasil and Sousa [4] and Cheng [12] have studied the notion of strong stability of closed hypersurfaces with constant (normalized) scalar curvature R immersed into $\mathbb{S}^{n+1}$, where they obtained characterizations of the Clifford torus via some estimates of the first eigenvalue of stability when $R = 1$ and $R > 1$, respectively.

The natural generalization of mean and scalar curvatures for an n-dimensional hypersurface of space forms are the r-th mean curvatures H_r, for $r \in \{0, \dots, n\}$, where H_0 is identically equal to 1 by definition. In fact, H_1 is just the mean curvature H and H_2 defines a geometric quantity which is related to the scalar curvature.

In [7], Barbosa and Colares studied the notion of *r-stability* (see item (a) of Remark 2 to understand this concept) for closed hypersurfaces immersed with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{0, \dots, n-2\}$, in space forms. In this setting, they showed that such hypersurfaces in a simply connected space form are r-stable if and only if they are geodesic spheres. Moreover, in [14], the author and de Lima were able to establish another characterization result concerning r-stability through the analysis of the first eigenvalue of an operator naturally attached to the r-th mean curvature.

Motivated by all the work described above, a question appears naturally:

> *Are there closed hypersurfaces which are strongly r-stable with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1, \dots, n-2\}$?*

With the intention of addressing this issue and seeking a possible answer (affirmative or not), we can slightly change our question and propose the new question:

> *On what conditions is it possible to guarantee the existence (or nonexistence) of hypersupefaces with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1, \dots, n-2\}$, that are strongly r-stable?*

Our proposal here is to investigate the strong r-stability concerning closed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1, \dots, n-2\}$, immersed into the $(n+1)$-dimensional Euclidean sphere $\mathbb{S}^{n+1}$, with $n \geq 3$ (see Definition 1). For this, in Section 2 we recorded some main facts about the hypersurfaces immersed in $\mathbb{S}^{n+1}$ and in Section 3 we describe the variational problem that gives rise to the notion of strong r-stability. Next, initially we prove that geodesic spheres of $\mathbb{S}^{n+1}$ are strongly r-stable (see Proposition 2), which provides an affirmative answer to our first question. Afterwards, to achieve our goals, we make use of the Riemannian warped product $(0,\pi)\times_{\sin\tau}\mathbb{S}^n$, $\tau\in(0,\pi)$, which models a certain open region Ω^{n+1} of $\mathbb{S}^{n+1}$ (see equations (4.1), (4.2) and (4.3)) and, in Proposition 3, we calculate the differential operator L_r (associated with the variational problem that defines the notion of strong r-stability) acting on an support function ξ (see equation (4.9)) naturally attached to a hypersurface $\psi:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{1,\dots,n-2\}$, immersed in Ω^{n+1}. Then, under a suitable restriction on H_r and H_{r+1}, we use the formula of $L_r(\xi)$ to show that if a closed hypersurface $\psi:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{1,\dots,n-2\}$, in $\mathbb{S}^{n+1}$ is strongly r-stable, then it must be a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\frac{\pi}{4}$ (for a better understanding of this region, we recommend the reader to see Definition 2), which provides a partial converse of Proposition 2. More specifically, we have established the following rigidity result for strongly r-stable hypersurfaces in $\mathbb{S}^{n+1}$:

Theorem 1. *Let $\psi\colon\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ $(n\geq 3)$ be a strongly r-stable closed hypersurface with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{1,\dots,n-2\}$. If the r-th mean curvature H_r of $\psi:\Sigma^n\looparrowright\Omega^{n+1}$ obeys the condition*

$$H_{r+1}\geq H_r\geq 1\quad\text{on }\ \Sigma^n,\tag{1.1}$$

then $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\pi/4$.

The motivation to assume the hypothesis (1.1) in Theorem 1 is described in Remark 3, while the restrictions $r\neq\{0,n-1,n\}$ are explained in item (b) of Remark 2. As an immediate consequence of this result, we establish a result of nonexistence for strongly r-stable closed hypersurfaces immersed in $\mathbb{S}^{n+1}$, which can be understood as an answer to our second question.

Theorem 2. *There is no strongly r-stable closed hypersurface Σ^n $(n\geq 3)$ with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{1,\dots,r+2\}$, immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\pi/4$, with r-th mean curvature H_r satisfying the inequality $H_{r+1}\geq H_r\geq 1$ on Σ^n.*

From our results listed above we can conclude that the region of $\mathbb{S}^{n+1}$ that contains the set of closed hypersurfaces $\psi:\Sigma^n\looparrowright\mathbb{S}^{n+1}$ $(n\geq 3)$ with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{1,\dots,n-2\}$, which are strongly r-stable and whose r-th mean curvature H_r satisfies the condition (1.1), is small. It is in this configuration that our results can be understood as a half-space type property of strongly r-stable closed hypersurfaces in the Euclidean sphere $\mathbb{S}^{n+1}$ (cf. Remark 4).

Finally, in Corollary 1 and 2 we write Theorems 1 and 2 for the case of closed hypersurfaces immersed into $\mathbb{S}^{n+1}$ with constant (normalized) scalar curvature R. The proofs of the main results of this work is carried out in Section 4.

2. Background

Unless stated otherwise, all manifold considered on this work will be connected, while *closed* means compact without boundary. Let $\mathbb{S}^{n+1}$ be the $(n+1)$-dimensional

Euclidean sphere. We will consider immersions $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ of closed orientable hypersurfaces Σ^n in $\mathbb{S}^{n+1}$. In this setting, we denote by $d\Sigma$ the volume element with respect to the metric induced by ψ, $C^\infty(\Sigma^n)$ the ring of real functions of class C^∞ defined on Σ^n and by $\mathfrak{X}(\Sigma^n)$ the $C^\infty(\Sigma^n)$-module of vector fields of class C^∞ on Σ^n. Since Σ^n is orientable, one can choose a globally defined unit normal vector field N on Σ^n. Let

$$\begin{array}{rcccl} A & : & \mathfrak{X}(\Sigma^n) & \to & \mathfrak{X}(\Sigma^n) \\ & & Y & \mapsto & A(Y) = -\overline{\nabla}_Y N\,. \end{array} \tag{2.1}$$

denote the shape operator with respect to N, so that, at each $q \in \Sigma^n$, A restricts to a self-adjoint linear map $A_q \colon T_q\Sigma \to T_q\Sigma$.

According to the ideas established by Reilly [16], for $1 \le r \le n$, if we let $S_r(q)$ denote the r-th *elementary symmetric function* on the eigenvalues of A_q, we get n functions $S_r \in C^\infty(\Sigma^n)$ such that

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r\, t^{n-r}\,,$$

where $I \colon \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ is the identity operator and $S_0 = 1$ by definition. If $q \in \Sigma^n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_q\Sigma$ formed by eigenvectors of A_q, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n)\,, \tag{2.2}$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

For $1 \le r \le n$, one defines the *r-th mean curvature* H_r (also called *higher order mean curvature*) of $\psi \colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ by

$$\binom{n}{r} H_r = S_r = S_r(\lambda_1, \ldots, \lambda_n)\,. \tag{2.3}$$

In particular, for $r = 1$,

$$H_1 = \frac{1}{n}\sum_{k=1}^{n} \lambda_k = H$$

is the *mean curvature* of the hypersurface $\psi \colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, which is the main extrinsic curvature. When $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) *normalized scalar curvature* R of $\psi \colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$. More precisely, it follows from the Gauss equation of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ that

$$R = 1 + H_2. \tag{2.4}$$

We can also define (cf. [16, Section 1]), for $0 \le r \le n$, the so-called *r-th Newton transformation* $P_r \colon \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by setting $P_0 = I$ and, for $1 \le r \le n$, via the recurrence relation

$$P_r = S_r I - A P_{r-1}\,.$$

A trivial induction shows that

$$P_r = S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r,$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p(\Sigma^n)$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^\perp \subset T_p(\Sigma^n)$, it is easy to see that

$$\det(tI - A_i) = \sum_{j=0}^{n-1} (-1)^j S_j(A_i)\, t^{n-1-j},$$

where

$$S_j(A_i) = \sum_{\substack{1 \le j_1 < \cdots < j_m \le n \\ j_1, \ldots, j_m \neq i}} \lambda_{j_1} \cdots \lambda_{j_m}\,. \tag{2.5}$$

With the above notations, it is also immediate to check that

$$P_r(e_i) = S_r(A_i) e_i\,, \tag{2.6}$$

and hence (cf. [7, Lemma 2.1])

$$\begin{cases} \mathrm{tr}(P_r) = (n-r)S_r = b_r H_r\,; \\ \mathrm{tr}(AP_r) = (r+1)S_{r+1} = b_r H_{r+1}\,; \\ \mathrm{tr}(A^2 P_r) = S_1 S_{r+1} - (r+2)S_{r+2} = n\frac{b_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2}\,, \end{cases} \tag{2.7}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r \colon C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$, given by

$$L_r(f) = \mathrm{tr}\left(P_r \,\mathrm{Hess}\, f\right)\,. \tag{2.8}$$

We observed that $L_0 = \Delta$, the Laplacian operator on Σ^n, and $L_1 = \Box$, the Yau's square operator on Σ^n (cf. [13, Equation (1.7)]).

3. The variational problem

For a closed orientable hypersurface $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ as in the previous section, a *variation* of it is a smooth mapping $X \colon (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{RP}^{n+1}$ such that, for every $t \in (-\epsilon, \epsilon)$, the map

$$\begin{array}{rcccl} X_t : \Sigma^n & \looparrowright & \mathbb{S}^{n+1} & & \\ q & \mapsto & X_t(q) & = & X(t,q) \end{array} \tag{3.1}$$

is an immersion, with $X_0 = x$. In what follows, we let $d\Sigma_t$ denote the volume element of the metric induced on Σ^n by X_t, and N_t will stand for the unit normal vector field along X_t.

The *variational field* associated to the variation $X \colon (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{S}^{n+1}$ is $\frac{\partial X}{\partial t}|_{t=0} \in \mathfrak{X}(X((-\epsilon, \epsilon) \times \Sigma^n))$. Letting

$$f_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle, \tag{3.2}$$

we get

$$\frac{\partial X}{\partial t} = f_t N_t + \left(\frac{\partial X}{\partial t}\right)^{\top},$$

where $(\,\cdot\,)^{\top}$ stands for the tangential component.

The *balance of volume* of the variation $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is the functional

$$\begin{array}{rcl} \mathcal{V} : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{V}(t) \;=\; \displaystyle\int_{\Sigma^n\times[0,t]} X^*\,(dV)\,, \end{array}$$

and we say that $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a *volume-preserving* variation for $x\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ if $\mathcal{V}(t)=\mathcal{V}(0)=0$, for all $t\in(-\epsilon,\epsilon)$. Moreover, following [7], we define the *r-th area functional*

$$\begin{array}{rcl} \mathcal{A}_r : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{A}_r(t) \;=\; \displaystyle\int_{\Sigma^n} F_r\left(S_1(t),S_2(t),\ldots,S_r(t)\right)d\Sigma_t\,, \end{array}$$

where $S_r(t)=S_r(t,\cdot)$ is the r-th elementary symmetric fuunction of Σ^n via the immersion (3.1) and F_r is recursively defined by setting $F_0=1$, $F_1=S_1(t)$ and, for $2\le r\le n-1$,

$$F_r = S_r(t) + \frac{(n-r+1)}{r-1}\,F_{r-2}\,.$$

The following lemma is well known and can be found in [7].

Lemma 1. *Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a closed hypersurface. If $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a variation of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ then*

(a) $\frac{d}{dt}\mathcal{V}(t)=\int_{\Sigma^n} f_t d\Sigma_t$, *where f_t is the function defined in* (3.2). *In particular, $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a volume-preserving variation for $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ if and only if $\int_{\Sigma^n} f_t d\Sigma_t = 0$ for all $t\in(-\epsilon,\epsilon)$.*

(b) $\frac{d}{dt}\mathcal{A}_r(t) = -b_r\int_{\Sigma^n} H_{r+1}(t) f_t d\Sigma_t$, *where $b_r=(r+1)\binom{n}{r+1}$ and $H_{r+1}(t)=H_{r+1}(t,\cdot)$ is the $(r+1)$-th mean curvature of Σ^n via the immersion* (3.1).

Remark 1. From [9, Lemma 2.2], given a closed hypersurface $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, if $f\in C^\infty(\Sigma^n)$ is such that

$$\int_{\Sigma^n} f\,d\Sigma = 0\,, \tag{3.3}$$

then there exists a volume-preserving variation $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ for $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ whose variational field is just $\frac{\partial X}{\partial t}|_{t=0} = fN$.

In order to characterize hypersurfaces of $\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature, we will consider the variational problem of minimizing the r-th area functional $\mathcal{A}_r$ for all volume-preserving variations of the closed hypersurface $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$.

The *Jacobi functional* $\mathcal{J}_r$ associated to the problem is given by

$$\begin{array}{rcl} \mathcal{J}_r : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{J}_r(t) \;=\; \mathcal{A}_r(t)+\varrho\mathcal{V}(t)\,, \end{array}$$

where ϱ is a constant to be determined. As an immediate consequence of Lemma 1 we get

$$\frac{d}{dt}\mathcal{J}_r(t) = \int_{\Sigma^n} \{-b_r H_{r+1}(t) + \varrho\} f_t d\Sigma_t \,,$$

where f_t is the function defined in (3.2) and $b_r = (r+1)\binom{n}{r+1}$ and $H_{r+1}(t) = H_{r+1}(t,\cdot)$ is the $(r+1)$-th mean curvature of Σ^n via the immersion (3.1). In order to choose ϱ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}(\Sigma^n)} \int_{\Sigma^n} H_{r+1} d\Sigma$$

be a integral mean of the function H_{r+1} along the Σ^n. We call the attention to the fact that, in the case that H_{r+1} is constant, one has

$$\overline{\mathcal{H}} = H_{r+1} \,, \tag{3.4}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\varrho = b_r\overline{\mathcal{H}}$, we arrive at

$$\frac{d}{dt}\mathcal{J}_r(t) = b_r \int_{\Sigma^n} \left\{-H_{r+1}(t) + \overline{\mathcal{H}}\right\} f_t d\Sigma_t \,.$$

In particular,

$$\frac{d}{dt}\mathcal{J}_r(t)\Big|_{t=0} = b_r \int_{\Sigma^n} \left\{-H_{r+1} + \overline{\mathcal{H}}\right\} f_0 d\Sigma. \tag{3.5}$$

Now, following the same ideas of [8, Proposition 2.7], from (3.5), (3.4) and Remark 1 we can establish the following result, which characterizes all the critical points of the variational problem described above.

Proposition 1. *Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a closed hypersurface. The following statements are equivalent:*

(a) $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ *has constant* $(r+1)$*-th mean curvature functions* H_{r+1}*;*

(b) *we have* $\delta_f\mathcal{A}_r = \frac{d}{dt}\mathcal{A}_r(t)|_{t=0} = 0$ *for all volume-preserving variations of* $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$*;*

(c) *we have* $\delta_f\mathcal{J}_r = \frac{d}{dt}\mathcal{J}_r(t)|_{t=0} = 0$ *for all variations of* $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$.

Motivated by the ideas established in [4], [2] and [12], we exchanged our studying problem and now we wish to detect hypersurfaces $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ which minimize the Jacobi functional $\mathcal{J}_r$ for all variations of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$. Then, Proposition 1 shows that the critical points for this new variational problem coincide with those of the first variational problem, namely, are the closed hypersurfaces $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature H_{r+1}. Currently, geodesic spheres of $\mathbb{S}^{n+1}$ and Clifford hypersurfaces of $\mathbb{S}^{n+1}$ are examples for these critical points. So, for such a critical point, we need computing the second variation $\delta^2_f\mathcal{J}_r = \frac{d^2}{dt^2}\mathcal{J}_r(t)|_{t=0}$ of the Jacobi functional $\mathcal{J}_r$. This will motivate us to establish the following notion of stability.

Definition 1. Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ $(n \geq 3)$ be a closed hypersurface with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1,\dots,n-2\}$. We say that $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ is strongly r-stable if $\delta^2_f\mathcal{J}_r \geq 0$ for all $f \in C^\infty(\Sigma^n)$.

From [7, Proposition 4.4] we get that the sought formula for the second variation $\delta_f^2\mathcal{J}_r$ of $\mathcal{J}_r$ is given by

$$\delta_f^2\mathcal{J}_r = -(r+1)\int_{\Sigma^n} f\mathcal{L}(f)d\Sigma\,, \tag{3.6}$$

where

$$\mathcal{L} = L_r + \frac{nb_r}{r+1}HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r \tag{3.7}$$

is the *Jacobi differential operator* associated with our variational problem. Here, L_r is the differential operator defined in (2.8), H, H_r, H_{r+1} and H_{r+2} are the mean curvature, the r-th mean curvature, the $(r+1)$-th mean curvature and the $(r+2)$-th mean curvature of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, respectively, and $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r, r+1\}$.

Remark 2. Regarding our definition of strong stability, we note that:

(a) From a geometrical point of view, the notion of r-stability, namely, when $\delta_f^2\mathcal{A}_r \geq 0$ for all $f \in C^\infty(\Sigma^n)$ satisfying the condition (3.3), is more natural than the notion the strong r-stability. However, from an analytical point of view, the strong r-stability is more natural and easier to use. The analytical interest is due to its possible applications to Geometric Analysis such as: the approach of bifurcation techniques related to our variational problem, the study of evolution problems related to the differential operator of Jacobi $\mathcal{L}$, problems of eigenvalue of $\mathcal{L}$, the search for notions of parabolicity for $\mathcal{L}$, uniqueness (or multiliqueness) of solutions to problems of initial value involving $\mathcal{L}$, among others.

(b) In Definition 1, we put the restriction $r \neq 0$ due to the fact that there are no strongly stable constant mean curvature closed hypersurfaces in $\mathbb{S}^{n+1}$ (cf. [3, Section 2]), whereas the constraint $r \neq \{n+1, n\}$ is due to the explicit expression that admits $\delta_f^2\mathcal{J}_r$ (see equations (3.6) and (3.7)).

In [7, Proposition 5.1] was established that the geodesic spheres of $\mathbb{S}^{n+1}$ are r-stable. We note that the proof of this result can be used to affirm that the geodesic spheres of $\mathbb{S}^{n+1}$ are also strongly r-stable. Here, for completeness of content, we present a proof.

Proposition 2. *For any $r \in \{1, \ldots, n-2\}$, the geodesic spheres of $\mathbb{S}^{n+1}$ $(n \geq 3)$ are strongly r-stable.*

Proof. Let Σ^n be a geodesic sphere of $\mathbb{S}^{n+1}$ and let $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be its inclusion map into $\mathbb{S}^{n+1}$. Since Σ^n is totally umbilical then its principal curvatures are all equal to a certain constant λ. By choosing the normal vector we may assume that $\lambda \geq 0$. Thus, from (2.2), (2.3) and (2.5), respectively, we have for $r \in \{1, \ldots, n-2\}$ that

$$S_r = \binom{n}{r}\lambda^r = \text{constant}\,, \qquad H_r = \lambda^r = \text{constant}$$

and

$$S_r(A_i) = \binom{n-1}{r}\lambda^r = \text{constant}. \tag{3.8}$$

Next, if $e_1, \ldots, e_n$ are the principal directions of Σ^n, from (2.8), (2.6) and (3.8), we get

$$\begin{aligned} L_r(f) &= \sum_{i=1}^{n} \langle \mathrm{Hess}(f)(e_i), P_r(e_i) \rangle \\ &= \binom{n-1}{r}\lambda^r \sum_{i=1}^{n} \langle \mathrm{Hess}(f)(e_i), e_i \rangle = \binom{n-1}{r}\lambda^r \Delta f\,, \end{aligned}$$

for all $f \in C^\infty(\Sigma^n)$.

Then, from (3.6), (3.7) and (2.7), we obtain

$$\begin{aligned} \delta_f^2 \mathcal{J}_r &= -\int_{\Sigma^n} \Big\{ \binom{n-1}{r}\lambda^r \Delta f + b_r H_r f \\ &\quad + \Big(n\frac{b_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2}\Big) f \Big\} f d\Sigma \\ &= -\int_{\Sigma^n} \Big\{ \binom{n-1}{r}\lambda^r f \Delta f + (n-r)\binom{n}{r}\lambda^r f^2 \\ &\quad + \Big[n\binom{n}{r+1}\lambda^{r+2} - (n-r-1)\binom{n}{r+1}\lambda^{r+2} \Big] f^2 \Big\} d\Sigma \\ &= -\binom{n-1}{r}\lambda^r \int_{\Sigma^n} \{ f\Delta f + n f^2 + n\lambda^2 f^2 \}\, d\Sigma \\ &= \binom{n-1}{j}\lambda^r \int_{\Sigma^n} \{ -f\Delta f - n(1+\lambda^2) f^2 \}\, d\Sigma\,. \end{aligned} \tag{3.9}$$

Now, let η_1 be the first eigenvalue of the Laplacian Δ of $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, which admits the following min-max characterization (cf. [11, Section 1.5])

$$\eta_1 = \min \Big\{ -\int_{\Sigma^n} f \Delta f d\Sigma \Big/ \int_{\Sigma^n} f^2\, d\Sigma \ :\ f \in C^\infty(\Sigma^n)\ , f \neq 0 \Big\}. \tag{3.10}$$

Since $\lambda \geq 0$, from (3.9) and (3.10) we get

$$\delta_f^2\, \mathcal{J}_r \geq \binom{n-1}{r}\lambda^r \int_{\Sigma^n} \{ \eta_1 - n(1+\lambda^2) \} f^2 d\Sigma\,,$$

for all $f \in C^\infty(\Sigma^n)$. But, since $\iota(\Sigma^n)$ is isometric to an n-dimensional Euclidean sphere with constant sectional curvature equal to λ^2+1, we have that $\eta_1 = n(\lambda^2+1)$. Hence, for every $f \in C^\infty(\Sigma^n)$ we get

$$\delta_f^2 \mathcal{J}_r \geq \binom{n-1}{r}\lambda^r \int_{\Sigma^n} \{ \eta_1 - n(1+\lambda^2) \} f^2 d\Sigma = 0\,.$$

Therefore, according to Definition 1, $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ must be strongly r-stable. $\square$

4. Proof of the main results

In order to obtain a rigidity result concerning to strongly r-stable closed hypersurfaces immersed into $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$, we need to describe a Riemannian warped product that models a certain region of $\mathbb{S}^{n+1}$.

Let $\mathbf{P}$ be the *north pole* of $\mathbb{S}^{n+1}$ and $\mathbb{S}^n$ be the *equator* orthogonal to $\mathbf{P}$. From [15, Example 2], the open region

$$\Omega^{n+1} := \mathbb{S}^{n+1} \setminus \{\mathbf{P}, -\mathbf{P}\} \tag{4.1}$$

is isometric to the Riemannian warped product

$$(0,\pi) \times_{\sin\tau} \mathbb{S}^n, \quad \tau \in (0,\pi)\ . \tag{4.2}$$

At the moment, making $\mathbf{P} = (0,\ldots,0,1) \in \mathbb{S}^{n+1}$ and identifying the point $q = (q_1,\ldots,q_{n+1}) \in \mathbb{S}^n$ with $q = (q_1,\ldots,q_{n+1},0) \in \mathbb{S}^{n+1}$, we have that the correspondence

$$\begin{array}{rcccl} \Psi\ :\ (0,\pi)\times_{\sin\tau}\mathbb{S}^n & \rightarrow & \Omega^{n+1}\subset\mathbb{S}^{n+1} & & \\ (\tau,q) & \mapsto & \Psi(\tau,q) & = & (\cos\tau)\,q + (\sin\tau)\,\mathbf{P}, \end{array} \tag{4.3}$$

defines an isometry between (4.2) and (4.1). We denote by

$$\Phi : \Omega^{n+1} \subset \mathbb{S}^{n+1} \to (0,\pi) \times_{\sin\tau} \mathbb{S}^n \tag{4.4}$$

as being the inverse of Ψ.

If $d\tau^2$ and $d\sigma^2$ denote the metrics of $(0,\pi)$ and $\mathbb{S}^n$, respectively, then

$$\langle\,,\rangle = (\pi_I)^*\left(d\tau^2\right) + (\sin\tau)^2(\pi_{\mathbb{S}^n})^*\left(d\sigma^2\right)\ ,$$

is the tensor metric of the Riemannian warped product (4.2), where π_I and $\pi_{\mathbb{S}^n}$ denote the projections onto the $(0,\pi)$ and $\mathbb{S}^n$, respectively. In this context, the vector field

$$(\sin\tau)\,\frac{\partial}{\partial\tau} \in \mathfrak{X}\left((0,\pi)\times_{\sin\tau}\mathbb{S}^n\right)$$

is a *conformal* and *closed* one (in the sense that its dual 1-form is closed), with conformal factor $\cos\tau$. Moreover, from [15, Proposition 1], for each $\tau_0 \in (0,\pi)$, the *slice* $\{\tau_0\}\times\mathbb{S}^n$ of the *foliation*

$$(0,\pi) \ni \tau_0 \longmapsto \{\tau_0\}\times\mathbb{S}^n$$

is a n-dimensional geodesic sphere of $\mathbb{S}^{n+1}$, parallel to the equator $\mathbb{S}^n$, with shape operator (see (2.1)) A_{τ_0} given by

$$\begin{array}{rcccl} A_{\tau_0}\ :\ \mathfrak{X}(\{\tau_0\}\times\mathbb{S}^n) & \rightarrow & \mathfrak{X}(\{\tau_0\}\times\mathbb{S}^n) & & \\ Y & \mapsto & A_{\tau_0}(Y) & = & -\overline{\nabla}_Y(-\partial_\tau) = \dfrac{(\cos\tau_0)}{(\sin\tau_0)}\,Y \end{array} \tag{4.5}$$

with respect to the orientation given by $-\frac{\partial}{\partial\tau}$. Thus, from (2.2), (2.3) and (4.5), we get for $r\in\{0,\ldots,n\}$ that the r-th elementary symmetric function $\mathcal{S}_r$ and the r-th mean curvature $\mathcal{H}_r$ of each slice $\{\tau_0\}\times\mathbb{S}^n$ are

$$\mathcal{S}_r = \binom{n}{r}(\cot\tau_0)^r \quad \text{and} \quad \mathcal{H}_r = (\cot\tau_0)^r\,, \tag{4.6}$$

respectively. We note that $\mathcal{S}_r$ and $\mathcal{H}_r$ are constant on $\{\tau_0\}\times\mathbb{S}^n$.

In order to facilitate the understanding of certain regions in the Euclidean sphere, we have established the following notions.

Definition 2. Fixed $\tau_0 \in (0, \pi)$, the region

$$\Phi^{-1}\left((0, \tau_0) \times_{\sin\tau} \mathbb{S}^n \right) = \{ q \in \mathbb{S}^{n+1} \; : \; \Phi(q) \in (0, \tau_0) \times_{\sin\tau} \mathbb{S}^n \}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(0, \tau_0) \times_{\sin\tau} \mathbb{S}^n \subset (0, \pi) \times_{\sin\tau} \mathbb{S}^n$$

will be called of upper domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0. Similarly, the region

$$\Phi^{-1}\left((\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \right) = \{ q \in \mathbb{S}^{n+1} \; : \; \Phi(q) \in (\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \subset (0, \pi) \times_{\sin\tau} \mathbb{S}^n$$

will be called of lower domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0. In turn, the regions

$$\Phi^{-1}\left((0, \tau_0] \times_{\sin\tau} \mathbb{S}^n \right) = \{ q \in \mathbb{S}^{n+1} \; : \; \Phi(q) \in (0, \tau_0] \times_{\sin\tau} \mathbb{S}^n \}$$

and

$$\Phi^{-1}\left([\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \right) = \{ q \in \mathbb{S}^{n+1} \; : \; \Phi(q) \in [\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(0, \tau_0] \times_{\sin\tau} \mathbb{S}^n \subset (0, \pi) \times_{\sin\tau} \mathbb{S}^n$$

and

$$[\tau_0, \pi) \times_{\sin\tau} \mathbb{S}^n \subset (0, \pi) \times_{\sin\tau} \mathbb{S}^n,$$

respectively, will be called of closure of the upper domain and closure of the lower domain enclosed by the geodesic sphere of Ω^{n+1} of level τ_0, where Φ is the isometry given in (4.4).

For example, from Definition 2 we have that the upper domain enclosed by the geodesic sphere of Ω^{n+1} of level $\tau = \pi/2$ is the open upper hemisphere (minus the north pole $\mathbf{P}$) of $\mathbb{S}^{n+1}$, which is isometric to the Riemannian warped product

$$\left(0, \frac{\pi}{2}\right) \times_{\sin\tau} \mathbb{S}^n, \quad \tau \in (0, \pi/2)$$

According to the ideas established in [5, Section 5], we will consider that the orientable hypersurfaces $\psi \colon \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ for which their Gauss map N satisfies

$$-1 \leq \left\langle \Phi_*(N(q)), \frac{\partial}{\partial\tau} \right\rangle_{\Phi(\psi(q))} < 0$$

for all $q \in \Sigma^n$. In this setting, for such a hypersurface $\psi \colon \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ we define the *normal angle* θ as being the smooth function

$$\begin{array}{rcccl} \theta : \Sigma^n & \to & \left[0, \dfrac{\pi}{2}\right) & & \\ q & \mapsto & \theta(q) & = & \arccos\left(-\left\langle \Phi_*(N(q)), \dfrac{\partial}{\partial\tau} \right\rangle_{\Phi(\psi(q))} \right) . \end{array} \tag{4.7}$$

Thus, on Σ^n the normal angle θ verifies

$$0 < \cos\theta = -\Big\langle \Phi_*(N), \frac{\partial}{\partial\tau}\Big\rangle \leq 1\,. \tag{4.8}$$

Moreover, since the orientation of the slice $\{\tau_0\}\times\mathbb{S}^n$ is given by $-\frac{\partial}{\partial\tau}$, the normal angle θ of $\{\tau_0\}\times\mathbb{S}^n$ is such that $\cos\theta = 1$.

We need the following result, whose proof is a consequence of a suitable formula due to Barros and Sousa [10].

Proposition 3. *Let $\psi\colon \Sigma^n \looparrowright \Omega^{n+1}\subset\mathbb{S}^{n+1}$ $(n\geq 2)$ be an orientable hypersurface with constant $(r+1)$-th mean curvature H_{r+1}, $r\in\{0,\dots,n-2\}$. If*

$$\begin{array}{rcccl} \xi\,:\,\Sigma^n & \rightarrow & \mathbb{R} & & \\ q & \mapsto & \xi(q) & = & -\sin\tau\cos\theta(q), \end{array} \tag{4.9}$$

where θ is the normal angle of Σ^n defined in (4.7), then the formula of the differential operator L_r defined in (2.8) acting on ξ is given by

$$\begin{aligned} L_r(\xi) = &-\left(\frac{nb_r}{r+1}\,HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r\right)\xi \\ &- b_rH_r\sin\tau\cos\theta - b_rH_{r+1}\cos\tau. \end{aligned} \tag{4.10}$$

where H, H_r, H_{r+1} and H_{r+2} are the mean curvature, r-th mean curvature, $(r+1)$-th mean curvature and $(r+2)$-th mean curvature of $\psi:\Sigma^n\looparrowright\mathbb{S}^{n+1}$, respectively, and $b_k = (k+1)\binom{n}{k+1}$ for $k\in\{r,r+1\}$. Here, for simplicity we are adopting the abbreviated notations $H_j = H_j\circ\psi^{-1}\circ\Phi^{-1}$, $j\in\{1,r,r+1,r+2\}$, where Φ is the isometry described in (4.4).

Proof. From Theorem 2 of [10],

$$\begin{aligned} L_r(\xi) = &-\Big(\frac{nb_r}{r+1}\,HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r\Big)\xi \\ &-\, b_rH_r\,\Phi_*(N)(\cos\tau)(\cos\tau) - b_rH_{r+1}\cos\tau\,. \end{aligned} \tag{4.11}$$

Observing that

$$\overline{\nabla}\cos\tau = \Big\langle \overline{\nabla}\cos\tau, \frac{\partial}{\partial\tau}\Big\rangle\frac{\partial}{\partial\tau} = (\cos\tau)'\frac{\partial}{\partial\tau} \;=\; -\sinh\tau\frac{\partial}{\partial\tau}\,,$$

from (4.8) we have that

$$\begin{aligned} \Phi_*(N)(\cos\tau) &= \langle\overline{\nabla}\cos\tau, \Phi_*(N)\rangle \\ &= -\Big\langle\frac{\partial}{\partial\tau}, \Phi_*(N)\Big\rangle\sin\tau \;=\; \sin\tau\cos\theta. \end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.11) we obtain (4.10). □

Remark 3. For $1\leq r\leq n-1$, from (4.6) we can observe that the $(r+1)$-th mean curvature $\mathcal{H}_{r+1}$, of slice the $\{\tau_0\}\times\mathbb{S}^n$, with $\tau_0\in(0,\frac{\pi}{4})$, of the Riemannian warped product $(0,\pi)\times_{\sin\tau}\mathbb{S}^n$ verify the inequalities

$$\mathcal{H}^{r+1} = \mathcal{H}_{r+1} > \mathcal{H}_r > \cdots > \mathcal{H}_2 > \mathcal{H} > 1.$$

Taking into account this situation, we established in Theorem 1 a rigidity result for strongly r-stable closed hypersurfaces immersed into $\mathbb{S}^{n+1}$.

Proof of Theorem 1. Since the hypersurface

$$\Phi \circ \psi \colon \Sigma^n \looparrowright (0,\pi) \times_{\sin\tau} \mathbb{S}^n \tag{4.13}$$

is strongly r-stable, where Φ is the isometry described in (4.4), from (3.6) and (3.7) following Definition 1 we get

$$0 \le -\int_{\Phi(x(\Sigma^n))} \left\{ L_r(f) + \left(\frac{nb_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2} + b_r H_r \right) f \right\} f d\Phi(\Sigma)$$

for all $f \in C^\infty(\Sigma^n)$, where L_r is the differential operator defined in (2.8), $d\Phi(\Sigma)$ denotes the volume element of Σ^n induced by (4.13), $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r, r+1\}$ and, for simplicity, we use the notations $H_j = H_j \circ \psi^{-1} \circ \Phi^{-1}$, $j \in \{1, r, r+1, r+2\}$. In particular, considering the smooth function $\xi = -\sin\tau\cos\theta$ defined in (4.9), from Proposition 3 we obtain

$$\begin{aligned} 0 &\le b_r \int_{\Phi(\psi(\Sigma^n))} (-H_r \sin\tau\cos\theta - H_{r+1}\cos\tau) \sin\tau\cos\theta \, d\Phi(\Sigma) \\ &\le b_r \int_{\Phi(\psi(\Sigma^n))} (H_r \cos\theta - H_{r+1}) \cos\tau \sin\tau\cos\theta \, d\Phi(\Sigma) \\ &\le b_r \int_{\Phi(\psi(\Sigma^n))} (\cos\theta - 1)\, H_r \cos\tau\sin\tau\cos\theta \, d\Phi(\Sigma) \end{aligned} \tag{4.14}$$

where in the last inequality we use the condition (1.1). Now, since $H_r \ge 1$ on Σ^n, the normal angle θ of Σ^n verifies the inequalities established in(4.8), and $\cos\tau$ and $\sin\tau$ are positive values when $\tau \in (0, \pi/4]$, then from the (4.14) we obtain

$$0 \le b_r \int_{\Phi(\psi(\Sigma^n))} (\cos\theta - 1)\, H_r \cos\tau \sin\tau\theta \, d\Phi(\Sigma) \le 0\,.$$

Therefore, $\cos\theta = 1$ on Σ^n and, consequently, there is $\tau_0 \in (0, \pi/4]$ such that $\Phi(\psi(\Sigma^n)) = \{\tau_0\} \times \mathbb{S}^n$. □

With respect to the notion of strong stability related to closed hypersurfaces with constant mean curvature immersed into Euclidean sphere $\mathbb{S}^{n+1}$, it is well known that *there are no strongly stable closed hypersurfaces with constant mean curvature in* $\mathbb{S}^{n+1}$ (cf. [3, Section 2]). In the context of the higher order mean curvatures, from Theorem 1 we can establish a nonexistent result to strongly r-stable closed hypersurfaces immersed in $\mathbb{S}^{n+1}$ (see Theorem 2).

Proof of Theorem 2. Assuming that there is a strongly r-stable closed hypersurface $\psi \colon \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \ge 3)$ with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1, \dots, r+2\}$, immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$ and with r-th mean curvature H_r satisfying $H_{r+1} \ge H_r \ge 1$ on Σ^n, from Theorem 1 we get that $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, obtaining a contradiction. □

Remark 4. Consider all closed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ $(n \geq 3)$ with constant $(r+1)$-th mean curvature H_{r+1}, $r \in \{1, \ldots, n-2\}$, which are strongly r-stable and that satisfy the condition $H_{r+1} \geq H_r \geq 1$, where H_r is the r-th mean curvature of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, from Theorems 1 and 2 we can conclude that the region of the Euclidean sphere $\mathbb{S}^{n+1}$ that contains all these hypersurfaces is small when compared to the set of closed hypersurfaces of $\mathbb{S}^{n+1}$ that do not verify all these assumptions. It is in this context that our results can be understood as a half-space type property for this class of hypersurfaces of $\mathbb{S}^{n+1}$.

For the case $r = 1$, taking into account (2.4), we can exchange the second mean curvature H_2 for the normalized scalar curvature R in equation (3.5) and then rewrite our Definition 1 in terms of R. In this context, an immediate application of Theorem 1 and Theorem 2 gives the following results.

Corollary 1. *Let $\psi : \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 3)$ be a strongly 1-stable closed hypersurface with constant normalized scalar curvature R. If the mean curvature H of $\psi : \Sigma^n \looparrowright \Omega^{n+1}$ obeys the condition $R - 1 \geq H \geq 1$ on Σ^n, then $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$.*

Corollary 2. *There is no strongly 1-stable closed hypersurface Σ^n $(n \geq 3)$ with constant normalized scalar curvature R immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 3)$ of level $\tau_0 = \pi/4$, with mean curvature H satisfying the condition $R - 1 \geq H \geq 1$ on Σ^n.*

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