ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 35–47

THE RIBES-ZALESSKII PROPERTY OF SOME ONE RELATOR GROUPS

Gilbert Mantika, Narcisse Temate-Tangang, and Daniel Tieudjo

Abstract. The profinite topology on any abstract group G, is one such that the fundamental system of neighborhoods of the identity is given by all its subgroups of finite index. We say that a group G has the Ribes-Zalesskii property of rank k, or is RZ_k with k a natural number, if any product $H_1 H_2 \cdots H_k$ of finitely generated subgroups $H_1, H_2, \cdots, H_k$ is closed in the profinite topology on G. And a group is said to have the Ribes-Zalesskii property or is RZ if it is RZ_k for any natural number k. In this paper we characterize groups which are RZ_2. Consequently, we obtain condition under which a free product with amalgamation of two RZ_2 groups is RZ_2. After observing that the Baumslag-Solitar groups $BS(m,n)$ are RZ_2 and clearly RZ if $m = n$, we establish some suitable properties on the RZ_2 property for the case when $m = -n$. Finally, since any group $BS(m,n)$ can be viewed as a HNN-extension, then we point out the Ribes-Zalesskii property of rank two on some HNN-extensions.

1. Introduction and results

Properties of the profinite topology were studied by M. Hall in [10]. A finitely generated subgroup H of a free group F *is closed* in the profinite topology of F if H is the intersection of subgroups of finite index that contain H. This is equivalent to the statement that for any finitely generated subgroup H of a free group F, and any element $g \in F \setminus H$, there exist a normal subgroup N of finite index in F such that $g \notin HN$. In connection with the result of Hall, some authors introduced the Ribes-Zalesskii property of rank k on an abstract group. An abstract group G satisfies the *Ribes-Zalesskii property of rank* k, or is RZ_k with k a natural number, if for any finitely generated subgroups $H_1, H_2, \cdots, H_k$ and any element $g \in G \setminus H_1 H_2 \cdots H_k$, there exist a normal subgroup N of finite index in G such that $g \notin H_1 H_2 \cdots H_k N$. A group is said to have the *Ribes-Zalesskii property* or is *RZ* if it is RZ_k for any natural number k. It is clear that finite groups and finitely generated abelian groups are RZ. See [6]. Also, a direct product of groups which are RZ is RZ. See [7]. Using the link between the profinite topology and finitely

2020 *Mathematics Subject Classification*: primary 20E06; secondary 20E26, 20F05, 22A05.

Key words and phrases: profinite topology, HNN-extension, Ribes-Zalesskii property of rank k, Baumslag-Solitar groups.

Received June 4, 2021, revised November 2021. Editor J. Rosický.

DOI: 10.5817/AM2022-1-35

approximable groups, C. Rosendal characterized countable discrete groups which are RZ. See [25].
RZ_0 means residually finite. Conditions under which a group G is RZ_0 or RZ_1 were established and some examples of groups RZ_0 and RZ_1 were given. See [9, 12, 13, 15]. It is easy to see that for any natural number k, RZ_{k+1} implies RZ_k. But the inverse is not true. For example $F_2 \times F_2$ cited by C. Rosendal in [26] is RZ_0 but not RZ_1, where F_2 is the free group of rank 2.
The original motivation for the study of the property RZ goes back to a problem posed by J. Rhodes on the existence of an algorithm to compute the closure of subset of finite semigroup. See [20]. Recently, M. Doucha and M. Malicki in [8] showed that the RZ_2 and RZ_3 properties form the lower and upper group theoretic bounds for finite appoximability of actions on triangle-free graphs and K_n-free graphs, $n \geq 3$.
Other authors have investigated on finding conditions under which the free constructions of groups inherit the RZ_k property of all the group factors. N.S. Romanovskii [24] has proved that the free product of groups which are RZ_1 is also RZ_1. Further, T. Coulbois [7] has proved that the free product of RZ groups is also RZ. Also, Ribes and Zalesskii have proved that, when $\mathcal{C}$ is a variety of finite groups closed under extensions, the free product of groups which are RZ_2 is also RZ_2 relatively to $\mathcal{C}$. See [22].
But for a free product with amalgamation $G = (G_1 * G_2; A = B, \varphi)$ (denoted also $G = G_1 \underset{A=B}{*} G_2$) of groups G_1 and G_2 with amalgamated isomorphic subgroups $A \leq G_1$ and $B \leq G_2$, a similar statement is not always true. Examples of free product with amalgamation of two RZ_1 groups which is not RZ_1 were given in the works of E. Rips [23] and R. Allenby and D. Doniz [1].
Moldavanskii and Uskova [18] proved that under some conditions, free products with amalgamation of two RZ_1 groups is RZ_1. Specifically, they proved

Proposition 1.1 ([18, Theorem 3])**.** *The group $G = (G_1 * G_2; A = B, \varphi)$ where A is a normal subgroup of G_1, B is a normal subgroup of G_2 and groups A and B satisfy the maximum conditions for subgroups, is RZ_1 if the groups G_1 and G_2 are RZ_1.*

In this paper we characterize groups which are RZ_2. We prove

Theorem 1.1. *Let G be a group and let U be a finitely generated subgroup contained in the center $Z(G)$ of G. G is RZ_2 if and only if the factor group G/U^n is RZ_2 for any nonzero natural number n.*

From this result, we obtain a result similar to that of Moldavanskii and Uskova for the property RZ_2 of groups with amalgamation. The case where the free factors in a free product amalgamated by a finite subgroup are RZ was studied by T. Coulbois in his thesis. See [6]. In this paper, we investigate the case where the amalgamated subgroup can be infinite. That is

Corollary 1.1. *Let $G = G_1 \underset{A=B}{*} G_2$ be a free product of groups G_1 and G_2 with amalgamated subgroups $A \leq G_1$ and $B \leq G_2$. If A and B are finitely generated* *subgroups contained in the centers $Z(G_1)$ and $Z(G_2)$ of G_1 and G_2 respectively, and groups G_1 and G_2 are RZ_2, then G is RZ_2.*

It is then easy to see that if G_1 and G_2 are two RZ_2 groups, and a and b are elements in G_1 and G_2 respectively with $a \in Z(A)$ and $b \in Z(B)$, then the group $G = G_1 \underset{a=b}{*} G_2$ is also RZ_2.
Also, we recall the class of two-generator one-relator groups, called the Baumslag-Solitar groups, given by the presentation $BS(m,n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$ where m and n are nonzero integers. This class of groups deeply studied by G. Baumslag and Solitar [4], were introduced to point out a class of finitely generated non-hopfian groups. Some residual properties of $BS(m,n)$ were studied [2, 3].

It is easily seen using the results of [21, 27]

Proposition 1.2. *For any nonzero integer n, group $BS(n,n)$ is RZ.*

Since for $|m| = n$ the group $BS(m,n)$ is RZ_0 and RZ_1 (see [16]), then the case where $m = -n$ is also for interest. Thus we investigate this case. We obtain

Theorem 1.2. *Let n be a nonzero natural number. If H_1 and H_2 are two finitely generated subgroups of $BS(n,-n)$ contained in the free factors of $BS(n,-n)$, then the product H_1H_2 is closed in the profinite topology on $BS(n,-n)$.*

Also, any Baumslag-Solitar group $BS(m,n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$ can be seen as an HNN-extension with associated subgroups $\langle b^m \rangle$ and $\langle b^n \rangle$. So, we also focus on Ribes-Zalesskii's property of rank k of some HNN-extensions. Let K be a finitely generated abelian group and let A, B be finitely generated isomorphic subgroups of K. Since finitely generated abelian groups are RZ, it follows immediately that if $A = B = K$, then the HNN-extension $G = \langle K, t \mid t^{-1}At = B \rangle$ is RZ as a finitely generated abelian group.
But if $A \neq B$ in the HNN-extension $G = \langle K, t \mid t^{-1}At = B \rangle$, then G is not RZ_1. See [17, Lemma 1]. Thus, G is not RZ_k for any natural number $k \geq 1$.

Using the result of G. Baumslag and M. Tretkoff that can be reformulated as

Proposition 1.3 ([2, Theorem 3.1])**.** *Let A be RZ_0 and let H, K be isomorphic finite subgroups of A. Then the HNN-extension $G = \langle A, t \mid t^{-1}Ht = K \rangle$ is RZ_0.*

It comes that if a group K is RZ and particularly RZ_0, A and B isomorphic finite normal subgroups of K, then the HNN-extension $G = \langle K, t \mid t^{-1}At = B \rangle$ is RZ_0. As in the proof of ([17, Lemma 2]), it can be pointed out a free product of RZ groups as a finite subgroup of finite index of G. Now, since any virtually RZ group is also RZ (see [7]), we obtain easily

Proposition 1.4. *Let K be RZ, and let A and B be isomorphic finite normal subgroups of K. Then, the HNN-extension $G = \langle K, t \mid t^{-1}At = B \rangle$ is RZ.*

From which we get by adding Theorem 1.1

Corollary 1.2. *Let K be a group and let A and B be isomorphic finitely generated subgroups of $Z(K)$, the center of K. Let $G = \langle K, t \mid t^{-1}At = B, \varphi \rangle$ be an HNN-extension with $\varphi(a) = t^{-1}at$ for any $a \in A$. If K is RZ_2 and contains a* *finitely generated subgroup of finite index U in both A and B such that $\varphi(u) = u$ for any $u \in U$, then G is RZ_2.*

2. Preliminaries

In this section we collect some notions, basic properties and facts about free products of groups with amalgamation, HNN-extensions and finitely generated groups. For more details see [14].

Let us recall some notions concerned with the construction of a free product $G = (G_1 * G_2, A = B, \varphi)$ of groups G_1 and G_2 with amalgamated subgroups $A \leq G_1$ and $B \leq G_2$ where $\varphi\colon A \to B$ is an isomorphism. The group $G = (G_1 * G_2, A = B, \varphi)$ can also be written as $G = G_1 \underset{\substack{\varphi \\ A=B}}{*} G_2$ or simply as $G = G_1 \underset{A=B}{*} G_2$ when there is no confusion. An element g in G can be written in a form $g = g_1 g_2 \cdots g_r$ $(r \geqslant 1)$ where for any $i = 1, 2, \ldots, r$ element g_i belongs to one of the free factor G_1 or G_2, and if $r > 1$ any successive g_i and g_{i+1} do not belong to the same factor G_1 or G_2 (nor to the amalgamated subgroups A and B). We say that g is written in a *reduced form.* In general, an element of the group $G = G_1 \underset{A=B}{*} G_2$ can have more than one reduced form. But any two reduced forms of an element g have the same number of components, which we will call the length of the element g and denote by $l(g)$.

About HNN-extensions, let G be a group and let A and B be its subgroups with $\varphi : A \to B$ an isomorphism. Let $\langle t \rangle$ be the infinite cyclic group generated by a new element t. The HNN-extension $G^\star$ of G relative to A, B and φ is the factor group $G * \langle t \rangle /N$, where N is the normal closure of the set $\{t^{-1}at(\varphi(a))^{-1}, a \in A\}$. The group G is called the basis of $G^\star$, t is its stable letter, and A and B are the associated subgroups. The notation $G^\star = \left\langle G, t; t^{-1}at = \varphi(a), a \in A \right\rangle$ is used.

Concerning finitely generated groups, it is not hard to obtain the following results.

Proposition 2.1. *Let G be a group and let N be a normal subgroup of G.*

(1) *If H is a finitely generated subgroup, then the subgroup $\overline{H} = HN/N$ of G/N is. Particularly, if G is a finitely generated, then G/N is.*

(2) *If N and G/N are finitely generated, then the group G is.*

Proof. Consider the canonical epimorphism $\pi\colon G \longrightarrow G/N$.

(1) Let H be subgroup and let X be its finitely generated subset. Then $\overline{H} = HN/N = \pi(H) = \pi(\langle X \rangle) = \langle \pi(X) \rangle$. Thus the subgroup HN/N is finitely generated.

(2) Since G/N is finitely generated, there exist elements $g_1, g_2, \ldots, g_r$ in G such that $G/N = \langle \overline{g_1}, \overline{g_2}, \ldots, \overline{g_r} \rangle$, where each $\overline{g_i}$ $(1 \leq i \leq r)$ represents the image by π of element g_i in G/N. Consider $g \in G$ such that $\overline{g} = \overline{g_1}^{s_1} \overline{g_2}^{s_2} \cdots \overline{g_r}^{s_r}$ where the s_k are integers. Then $\overline{g} = \overline{g_1^{s_1} g_2^{s_2} \cdots g_r^{s_r}}$, and there exists $n \in N$ with $g = g_1^{s_1} g_2^{s_2} \cdots g_r^{s_r} n$; that is $g \in \langle g_1, g_2, \ldots, g_r \rangle N$. Finally $G = \langle g_1, g_2, \ldots, g_r \rangle N$ is finitely generated since N is. □

Proposition 2.2. *Any quotient of a RZ_2 group by a finitely generated normal subgroup is also RZ_2.*

Proof. Let G be a RZ_2 group and let N be a finitely generated normal subgroup of the group G. We shall prove that the factor group G/N is RZ_2.
Consider two finitely generated subgroups $\overline{H_1} = H_1/N$ and $\overline{H_2} = H_2/N$ of G/N, where H_1 and H_2 are subgroups containing N. Let g be an element of G such that $\overline{g} \in G/N$ and $\overline{g} \notin \overline{H_1}\ \overline{H_2}$. It is clear that $g \notin H_1H_2$. Using Proposition 2.1, it is also clear that subgroups H_1 and H_2 are finitely generated. Therefore, since G is RZ_2, there exists a normal subgroup M of finite index in G such that $g \notin H_1H_2M$. Consequently we have $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{M}$ where $\overline{M} = MN/N$. If, on contrary $\overline{g} \in \overline{H_1}\ \overline{H_2}\ \overline{M}$, then $\overline{g} = \overline{h_1}\ \overline{h_2}\ \overline{t}$ with $h_1 \in H_1$, $h_2 \in H_2$ and $t \in MN$. And then there exist $m \in M$ and $n \in N$ such that $g = h_1h_2mn = h_1h_2(mnm^{-1})m$. Now, since $N \lhd G$ and $N \leq H_2$, it is obvious that $h = h_2(mnm^{-1}) \in H_2$. But this implies that $g = h_1hm \in H_1H_2M$ which contradicts the fact that $g \notin H_1H_2M$. So $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{M}$, with $\overline{M}$ a normal subgroup of finite index in G/N. Thus, the factor group G/N is RZ_2 as required. □

Proposition 2.3. *Let G be a group and let A be a finitely generated subgroup in G. If A is contained in $Z(G)$ the center of G, then for any nonzero natural number t, the subset $A^t = \{a^t, a \in A\}$ of G is a normal subgroup of finite index in A.*

Proof. Assume that the subgroup A is contained in $Z(G)$. Then A is a finitely generated abelian group. Therefore A is equal to a direct sum $\bigoplus_{i\leq l} A_i$, where each A_i is cyclic. For $i \leq l$, let a_i be a generator of A_i. So,

$$A = \langle a_1, a_2, \ldots, a_l\rangle$$

is generated by the elements $a_1, a_2, \ldots, a_l$. Let t be a nonzero natural number.
On one hand, since $Z(G)$ is commutative, it is obvious that $A^t = \{a^t, a \in A\}$ is a normal subgroup of A.
On the other hand the factor group

$$A/A^t = \left\langle \overline{a_1}, \overline{a_2}, \ldots, \overline{a_l} \mid \overline{a_1}^t = 1, \overline{a_2}^t = 1, \ldots, \overline{a_l}^t = 1 \right\rangle$$

is finitely generated where $\overline{a_i} = a_iA^t$ for any $i \in \{1, 2, \cdots, l\}$. Also, the group A/A^t is commutative, so it can be written as $\overline{A_t} = \left\langle \overline{a_1} \mid \overline{a_1}^t = 1\right\rangle \times \left\langle \overline{a_2} \mid \overline{a_2}^t = 1\right\rangle \times \cdots \times \left\langle \overline{a_l} \mid \overline{a_l}^t = 1\right\rangle$. Finally, since the order of each group $\left\langle \overline{a_i} \mid \overline{a_i}^t = 1\right\rangle$, $i \in \{1, 2, \cdots, l\}$ is at most t, it follows that the order of A/A^t is finite. □

3. Proof of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1. Since the subgroup $U \leq Z(G)$ is finitely generated, it comes that for any nonzero natural number t, the subgroup $U^t \leq G$ is normal and finitely generated. Thus, if G is RZ_2, then using Proposition 2.2 the factor group $G/U^t (t \geq 1)$ is.

Conversely, suppose that any factor group $G/U^t (t \geq 1)$ is RZ_2. Let prove that G is RZ_2. To do it, let H_1 and H_2 be two finitely generated subgroups of G, and let g be an element in G such that $g \notin H_1H_2$.
We need to determine a normal subgroup N of finite index in G ($N \lhd_f G$) such

that $g \notin H_1 H_2 N$. Consider for any nonzero natural number t, the factor group G/U^t and the canonical epimorphism

$$\vartheta_t\colon \quad G \longrightarrow \quad G/U^t\,.$$

Case 1. Assume that there exist a nonzero natural number t_0 such that $\vartheta_{t_0}(g) \notin \vartheta_{t_0}(H_1)\ \vartheta_{t_0}(H_2)$ in G/U^{t_0}. Since H_1 and H_2 are finitely generated, it follows using Proposition 2.1 that $\vartheta_{t_0}(H_1)$ and $\vartheta_{t_0}(H_2)$ are finitely generated. Now the group G/U^{t_0} is RZ_2. Therefore there exists $\overline{N} \lhd_f G/U^{t_0}$ such that $\vartheta_{t_0}(g) \notin \vartheta_{t_0}(H_1)\ \vartheta_{t_0}(H_2)\ \overline{N}$. Let N be the preimage of $\overline{N}$ by ϑ_{t_0}. Clearly, $g \notin H_1 H_2 N$. Thus G is RZ_2.

Case 2. Assume now that for any nonzero natural number t we have $\vartheta_t(g) \in \vartheta_t(H_1)\ \vartheta_t(H_2)$ in G/U^t. We need to prove that this case is not possible.

For $t = 1$, $\vartheta_1(g) = \vartheta_1(a)\vartheta_1(b)$ with $a \in H_1$ and $b \in H_2$. That is $gU = abU$ and then $g = abu$ with $u \in U$. Let $y = ab$. Then, we have $g = yu$.

For any $t \geq 2$, $\vartheta_t(g) = \vartheta_t(a_t)\vartheta_t(b_t)$, where $a_t \in H_1$ and $b_t \in H_2$; that is $g = a_t b_t u_t$ with the elements a_t, b_t and u_t fixed respectively in H_1, H_2 and U^t. Therefore for any $t \geq 2$ we have $g = a_t a^{-1} a b b^{-1} b_t u_t = h_t y k_t u_t$, where $h_t = a_t a^{-1} \in H_1$ and $k_t = b^{-1} b_t \in H_2$. Thus,

$$u = y^{-1} h_t y k_t u_t\,. \tag{3.1}$$

Set $S = \big\langle \{y^{-1} h_t y k_t \mid h_t \in H_1, k_t \in H_2, t \geq 2\}\big\rangle$ be the subgroup generated by the elements of the form $y^{-1} h_t y k_t$, with $h_t \in H_1$ and $k_t \in H_2$, $(t \geq 2)$. Since $y^{-1} h_t y k_t = u u_t^{-1} \in U$, then S is a subgroup of U. Also, for $s = y^{-1} h_t y k_t (t \geq 2)$, we have $s^{-1} = k_t^{-1} y^{-1} h_t^{-1} y \in S$; and it follows that $k_t s^{-1} = y^{-1} h_t^{-1} y$. From $U \leq Z(G)$ and $s^{-1} \in U$, we obtain $k_t s^{-1} = s^{-1} k_t = y^{-1} h_t^{-1} y$. The equality $s^{-1} = y^{-1} h_t^{-1} y k_t^{-1}$ then arises. Finally, $y^{-1} h_t^{\epsilon_t} y k_t^{\epsilon_t} \in S$ with $\epsilon_t = \pm 1$. Thus:

$$\begin{aligned}
(y^{-1} h_{t_1}^{\epsilon_{t_1}} y k_{t_1}^{\epsilon_{t_1}})(y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}}) &= (y^{-1} h_{t_1}^{\epsilon_{t_1}} y)(k_{t_1}^{\epsilon_{t_1}} \times y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}})\\
&= (y^{-1} h_{t_1}^{\epsilon_{t_1}} y)(y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} \times k_{t_1}^{\epsilon_{t_1}})\,, \quad \text{since}\\
y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} \in Z(G) &= y^{-1} h_{t_1}^{\epsilon_{t_1}} y y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} k_{t_1}^{\epsilon_{t_1}}\\
&= y^{-1} h_{t_1}^{\epsilon_{t_1}} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} k_{t_1}^{\epsilon_{t_1}}\,, \quad \epsilon_{t_i} = \pm 1\,.
\end{aligned}$$

It comes then that the elements of S have the form:

$$y^{-1} h_{t_1}^{\epsilon_{t_1}} \ldots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \ldots k_{t_1}^{\epsilon_{t_1}}\,, \quad \epsilon_{t_i} = \pm 1,\ i = 1, \ldots, n\,. \tag{3.2}$$

Subcase (a) Suppose that u belongs to subgroup S. So, from (3.2), we have $u = y^{-1} h_{t_1}^{\epsilon_{t_1}} \cdots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \ldots k_{t_1}^{\epsilon_{t_1}}$; that is $yu = h_{t_1}^{\epsilon_{t_1}} \ldots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \ldots k_{t_1}^{\epsilon_{t_1}} = h_{t_1}^{\epsilon_{t_1}} \ldots h_{t_n}^{\epsilon_{t_n}} a b k_{t_n}^{\epsilon_{t_n}} \ldots k_{t_1}^{\epsilon_{t_1}}$.

Then, since $h_{t_1}^{\epsilon_{t_1}} \ldots h_{t_n}^{\epsilon_{t_n}} a \in H_1$ and $b k_{t_n}^{\epsilon_{t_n}} \ldots k_{t_1}^{\epsilon_{t_1}} \in H_2$, it follows that $g = yu \in H_1 H_2$, and this result contradicts the assertion $g \notin H_1 H_2$.

Subcase (b) Now $u \notin S$. On one hand, since the group U is commutative and finitely generated, it possesses the maximal property for groups, that is, each of its subgroups is finitely generated. Thus, S is finitely generated. On the other hand, U as a commutative and finitely generated group is RZ_1. Therefore, U possesses a normal subgroup M of finite index such that $u \notin SM$.

Also, since $M \lhd_f U$, all the elements of the factor group U/M have finite order. Let U_0 be the finite set of representative classes modulo M in U. For any $g \in U_0$, there exists a natural number r_g such that $g^{r_g} \in M$. Also, for any $g \in U$, there exist $g_0 \in U_0$ such that $gg_0^{-1} \in M$. Thus, $(gg_0^{-1})^{r_{g_0}} = g^{r_{g_0}}(g_0^{r_{g_0}})^{-1}$ belongs to M, and it follows that $g^{r_{g_0}}$ also belongs to M. Let t' be the least common multiple of the r_g, with $g \in U_0$. We have $g^{t'} \in M$ for any $g \in U$, and then $U^{t'} \subseteq M$. If $t' = 1$, then any $g \in U$ belongs to M. Particularly, $u \in M$, and it contradicts the fact that $u \notin M$ since $u \notin SM$. So $t' \geq 2$, and $u = y^{-1}h_{t'}yk_{t'}u_{t'}$. Now, $y^{-1}h_{t'}yk_{t'} \in S$ and $u_{t'} \in U^{t'} \subseteq M$, thus $u \in SM$, which is again not possible.
Finally, **Case 2** is not possible as required, and we get only **Case 1**. Thus, the group G is RZ_2, and the theorem is completely demonstrated. □

We are now ready to prove Corollary 1.1.

Proof of Corollary 1.1. Suppose that all the assumptions of the corollary are satisfied. Since $A = B$ coincides with the center of the amalgamated group G (see [14, Corollary 4.5]), to prove that G is RZ_2, we prove that G/A^t is RZ_2 for any nonzero natural number t and conclude using Theorem 1.1. To do it, let t be a nonzero natural number and let $\overline{H_1}$ and $\overline{H_2}$ be two finitely generated subgroups of G/A^t. Let $\overline{g}$ be an element of G/A^t such that $\overline{g} \notin \overline{H_1}\ \overline{H_2}$.
We need to determine $\overline{N} \lhd_f G/A^t$ such that $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{N}$. We recall by Proposition 2.3 that the subgroups A^t and B^t are normal with finite index in A and B in respectively. Since A/A^t and B/B^t are finite and isomorphic, the canonical homomorphisms $G_1 \longrightarrow G_1/A^t$ and $G_2 \longrightarrow G_2/B^t$ can be extented to the epimorphism $G \longrightarrow G_1/A^t \underset{A/A^t = B/B^t}{*} G_2/B^t$ with kernel $A^t = B^t$. See ([19, Theorem 1.1]). This situation can be illustrated by the following diagram

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d] & G_1 \underset{A=B}{*} G_2 \arrow[d, dotted] & G_2 \arrow[l] \arrow[d] \\
G_1/A^t \arrow[r] & G_1/A^t \underset{A/A^t=B/B^t}{*} G_2/B^t & G_2/B^t \arrow[l]
\end{tikzcd}$$

Let $G(t) = G_1/A^t \underset{A/A^t=B/B^t}{*} G_2/B^t$. It is clear that the groups G/A^t and $G(t)$ are isomorphic.
Now, using the fact that the subgroups A^t and B^t have finite index respectively in A and B which are finitely generated, it follows by ([6, Proposition 1.1]) that A^t and B^t are finitely generated. Thus, by Proposition 2.2 the groups G_1/A^t and G_2/B^t are RZ_2. Also, the groups A/A^t and B/B^t are finite; thus, the group $G(t)$ is RZ_2 (see [6, Theorem 5.2]), and G/A^t is. Since G/A^t is RZ_2 for any arbitrary nonzero natural number t, we conclude by Theorem 1.1 that G is RZ_2. Hence Corollary 1.1 is demonstrated. □

4. Proof of Theorem 1.2

We recall a result of P. Stebe which will be used in some statement of the proof of the Theorem 1.2. It states that for any element h of a free group F and for any nonzero integer n, there exists a normal subgroup N of finite index in F such that $N \cap \langle h \rangle = \langle h^n \rangle$ (see [28]). We establish

Lemma 4.1. *Let n be a nonzero natural number. For any finitely generated subgroups H_1 and H_2 of $BS(n,-n) = \langle a, b \mid a^{-1}b^n a = b^{-n} \rangle$, and any normal subgroup U of finite index in $\langle b^n \rangle$ such that $(\langle b^n \rangle \cap H_1)U \neq \langle b^n \rangle$ and $(\langle b^n \rangle \cap H_2)U \neq \langle b^n \rangle$, there exists a normal subgroup N of finite index in $BS(n,-n)$ satisfying $N \cap \langle b^n \rangle = U$, $(N\langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$ and $(N\langle b^n \rangle) \cap NH_2 = N(\langle b^n \rangle \cap H_2)$.*

Proof. Let H_1 and H_2 be two finitely generated subgroups of $BS(n,-n)$, and let U be a normal subgroup of finite index t in $\langle b^n \rangle$ satisfying all the assumptions in the lemma. Consider $c_1, \cdots, c_t$ a system of left cosets representatives of U in $\langle b^n \rangle$ where $c_1 = 1$.
Since $\mathrm{BS}(n,-n)$ is RZ_1 and U is finitely generated as a finite index subgroup of the finitely generated group $\langle b^n \rangle$, there exists $N_1 \lhd_f BS(n,-n)$ such that $c_i \notin N_1U$ for any $i = 2, \ldots, t$. Also, there exists $i \in \{2, 3, \ldots, t\}$ such that $c_i \notin H_1U$. Indeed: assume in contrary that for any $i \in \{2, 3, \ldots, t\}$ $c_i \in H_1U$; that is $c_i = h_i u_i$ with $h_i \in H_1$ and $u_i \in U$. Therefore $h_i = c_i u_i^{-1} \in H_1 \cap \langle b^n \rangle$ for any $i \in \{2, 3, \ldots, t\}$. Thus, c_i belongs to the subgroup $(H_1 \cap \langle b^n \rangle)U$ of $\langle b^n \rangle$ for any $i \in \{1, 2, \ldots, t\}$. Consequently, it follows that $(H_1 \cap \langle b^n \rangle)U = \langle b^n \rangle$ and this contradicts the hypothesis $\langle b^n \rangle \neq (H_1 \cap \langle b^n \rangle)U$. So, there exists $i \in \{2, 3, \ldots, t\}$ such that $c_i \notin H_1U$. Similarly, there exists $j \in \{2, 3, \ dots, t\}$ such that $c_j \notin H_2U$.

It is easy to see that the groups H_1U and H_2U are finitely generated in $\mathrm{BS}(n,-n)$ and so, again using the fact that $\mathrm{BS}(n,-n)$ is RZ_1, there exist normal subgroups N_{2i} and N_{3j} of finite index in $\mathrm{BS}(n,-n)$ such that $c_i \notin N_{2i}H_1U$ and $c_j \notin N_{3j}H_2U$. Set $I = \{i \in \{2, \ldots, t\}, c_i \notin H_1U\}$ and $J = \{i \in \{2, \ldots, t\}, c_i \notin H_2U\}$. Thus, $N_2 = \bigcap\limits_{i \in I} N_{2i}$ and $N_3 = \bigcap\limits_{i \in J} N_{3i}$ are normal subgroups of finite index in $\mathrm{BS}(n,-n)$ as finite intersections of normal subgroups of finite index in $\mathrm{BS}(n,-n)$. Therefore, $c_i \notin N_2H_1U$ for any $i \in I$ and $c_j \notin N_3H_2U$ for any $j \in J$. Let:

$$N = N_1U \cap N_2U \cap N_3U\,.$$

For any $l \in \{1, 2, 3\}$, N_l is a normal subgroup of finite index in $\mathrm{BS}(n,-n)$, and N_lU is. Consequently, N is also a normal subgroup of finite index in $\mathrm{BS}(n,-n)$.

It is obvious that $U \subseteq N \cap \langle b^n \rangle$. Conversely, let $g \in N \cap \langle b^n \rangle$. There exist $n_1 \in N_1$ and $u \in U$ such that $g = n_1 u$. If $g \notin U$, then there exist $i \in \{2, 3, \ldots, t\}$ and c_i in $\langle b^n \rangle$ such that $gU = c_iU$. Thus $c_i \in gU = n_1uU = n_1U$, and this implies that $c_i \in N_1U$, but it contradicts the assumption that $c_i \notin N_1U$ for any $i \in \{2, 3, \ldots, t\}$. So $g \in U$ and $U = N \cap \langle b^n \rangle$.

Let us now prove that $(N\langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$. On one hand, it is easy to see that $(N\langle b^n \rangle) \cap NH_1 \supseteq N(\langle b^n \rangle \cap H_1)$. On the other hand, let $g \in (N\langle b^n \rangle) \cap NH_1$. Then $g = kb_1 = k'h_1$, where $k, k' \in N$, $b_1 \in \langle b^n \rangle$ and $h_1 \in H_1$. Since $\langle b^n \rangle = \bigcup\limits_{i=1}^{t} c_iU$

($c_i \in \langle b^n \rangle$), there exist $j \in \{1, 2, \ldots, t\}$ and $u \in U$ such that $b_1 = c_j u$. Thus $c_j = k^{-1}k'h_1u^{-1} \in NH_1U$. Since $U \subseteq H_1U$ implies $UH_1U = H_1U$, we have $NH_1U \subseteq N_2UH_1U \subseteq N_2H_1U$. Recalling that $c_i \notin N_2H_1U$ for any $c_i \notin H_1U$, we obtain $c_j \in H_1U$ since $c_j \in N_2H_1U$. Therefore, there exist $h_1' \in H_1$ and $u' \in U$ satisfying $c_j = h_1'u'$. From $U \leq \langle b^n \rangle$, we have $h_1' = c_ju'^{-1} \in \langle b^n \rangle$. Consequently, $h_1' \in \langle b^n \rangle \cap H_1$ and then

$$g = kb_1 = kc_ju = kh_1'u'u = k(h_1'u'uh_1'^{-1})h_1'.$$

Furthermore $U \leq N$ and $N \lhd BS(n, -n)$, so that $h_1'u'uh_1'^{-1} \in N$. Therefore $kh_1'u'uh_1'^{-1} \in N$ and then $g \in N(\langle b^n \rangle \cap H_1)$. Thus, $(N \langle b^n \rangle) \cap NH_1 \subseteq N(\langle b^n \rangle \cap H_1)$ and we get the equality $(N \langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$.
We prove similarly that $(N \langle b^n \rangle) \cap NH_2 = N(\langle b^n \rangle \cap H_2)$. Hence, the lemma is proven. □

Proof of Theorem 1.2. Let us recall that in the group $\mathrm{BS}(n, -n) = \langle b \rangle \underset{b^n=c}{*} BS(1, -1)$, the subgroups $\langle b \rangle$ and $BS(1, -1) = \langle a, c \mid a^{-1}ca = c^{-1} \rangle$ are the free factors. Let H_1 and H_2 be two finitely generated subgroups of $BS(n, -n)$ contained in the free factors, and let $g \in BS(n, -n) \setminus H_1H_2$. In order to prove that the product H_1H_2 is closed in the profinite topology of $BS(n, -n)$, we need to determine a normal subgroup N of finite index in $\mathrm{BS}(n, -n)$ such that $g \notin H_1H_2N$.
Case 1. Assume that H_1 and H_2 are subgroups of $\langle b \rangle$.
Since the group $\langle b \rangle$ is commutative, it comes that H_1H_2 is an infinite cyclic group. Also, $\mathrm{BS}(n, -n)$ is RZ_1 and $g \in BS(n, -n) \setminus H_1H_2$. Thus, there exists $M \lhd_f BS(n, -n)$ such that $g \notin H_1H_2M$. That is, the set H_1H_2 is closed in the profinite topology of $BS(n, -n)$.
Case 2. Next, consider that H_1 and H_2 are subgroups of $BS(1, -1)$.
Subcase (a) Suppose that $g \in BS(1, -1)$. Since the group $\mathrm{BS}(1, -1)$ is polycyclic, it is RZ_2. Thus, there exists a subgroup $M \lhd_f BS(1, -1)$ such that $g \notin H_1H_2M$. Let the factor groups $\overline{H_1} = H_1/H_1 \cap M$, $\overline{H_2} = H_2/H_2 \cap M$ and $\overline{BS(1, -1)} = BS(1, -1)/M$ be considered modulo M. By Proposition 2.1 (1), $\overline{H_1}$ and $\overline{H_2}$ are finitely generated subgroups of $\overline{BS(1, -1)}$. Let $\overline{g}$ be the class of g modulo M in $\overline{BS(1, -1)}$; then $\overline{g} \notin \overline{H_1}\ \overline{H_2}$ in $\overline{BS(1, -1)}$. Also, since $M \cap \langle c \rangle$ is generated by one element as a subgroup of a one generated group, there exists a natural number t such that $M \cap \langle c \rangle = \langle c^t \rangle = \langle b^{nt} \rangle$. Therefore, by the result of P. Stebe cited previously, there exists $L \lhd_f \langle b \rangle$ satisfying $L \cap \langle b^n \rangle = \langle b^{nt} \rangle = M \cap \langle c \rangle$.
Set $\overline{\langle b^n \rangle} = \langle b^n \rangle / (L \cap \langle b^n \rangle)$ and $\overline{\langle c \rangle} = \langle c \rangle / (M \cap \langle c \rangle)$ respectively subgroups of $\overline{\langle b \rangle} = \langle b \rangle / L$ and $\overline{BS(1, -1)}$. Clearly, the canonical epimorphisms $\langle b \rangle \longrightarrow \overline{\langle b \rangle}$ and $BS(1, -1) \longrightarrow \overline{BS(1, -1)}$ induce an epimorphism $\pi : BS(n, -n) \longrightarrow \overline{BS(n, -n)} = \overline{\langle b \rangle} \underset{\overline{b^n} = \overline{c}}{*} \overline{BS(1, -1)}$. Since the groups $\overline{\langle b \rangle}$ and $\overline{BS(1, -1)}$ are finite, it comes that the group $\overline{BS(n, -n)}$ is a free product of finite groups amalgamated by finite subgroups. Now, using the fact that Since $\overline{\langle b \rangle}$ and $\overline{BS(1, -1)}$ are finite, they are RZ_2. Thus $\overline{BS(n, -n)}$ is RZ_2 as a free product of RZ_2 groups amalgamated by finite subgroups. See [6, Theorem 5.3]. Also in $\overline{BS(n, -n)}$, we have $\overline{g} \notin \overline{H_1}\ \overline{H_2}$. Consequently, there exists a normal subgroup $\overline{N}$ of finite index in $\overline{BS(n, -n)}$ such

that $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{N}$. Taking N to be the preimage of $\overline{N}$ via π, we have $g \notin H_1H_2N$ as desired. Again the set H_1H_2 is closed in the profinite topology of $BS(n,-n)$.
Subcase (b) Suppose that $g \notin BS(1,-1)$. Let $g = g_1g_2\cdots g_r$ $(r \geqslant 1)$ be a reduced form of g in the amalgamated free product of groups $\mathrm{BS}(n,-n) = \langle b\rangle \underset{b^n=c}{*} BS(1,-1)$.
Suppose that $r=1$. That is $g \in \langle b\rangle \setminus \langle b^n\rangle$, since $g \notin BS(1,-1)$. Recall once again that $\mathrm{BS}(n,-n)$ is RZ_1. Then there exists $M \lhd_f BS(n,-n)$ such that $g \notin \langle b^n\rangle M$, and the factor group $BS(n,-n)/M$ is finite. Set $\overline{BS(n,-n)} = BS(n,-n)/M$, $\overline{\langle b\rangle} = \langle b\rangle/(\langle b\rangle \cap M)$, $\overline{BS(1,-1)}/(BS(1,-1)\cap M)$, $\overline{\langle b^n\rangle} = \langle b^n\rangle/(\langle b^n\rangle \cap M)$ and $\overline{\langle c\rangle} = \langle c\rangle/(\langle c\rangle \cap M)$. Let $\overline{g}$ be the class of g modulo M. It is clear that in $\overline{BS(n,-n)}$ we have $\overline{g} \notin \overline{H_1}\ \overline{H_2}$, where $\overline{H_1} = H_1/H_1\cap M$, $\overline{H_2} = H_2/H_2\cap M$. Since $\overline{BS(n,-n)}$ is finite, it is trivially RZ_2. Thus, there exists a normal subgroup $\overline{N}$ which is also trivial of finite index in $\overline{BS(n,-n)}$ and such that $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{N}$. Taking $N = M$ to be the preimage of $\overline{N}$ via π, we have $g \notin H_1H_2N$ as desired. Therefore, H_1H_2 is closed in the profinite topology of $BS(n,-n)$.
Suppose that $r > 1$. Let I and J be the subsets of $\{1,2,\cdots r\}$ consisting of indices of components of g which belong to $\langle b\rangle \setminus \langle b^n\rangle$ and $BS(1,-1)\setminus\langle c\rangle$ respectively. Since $BS(n,-n)$ is RZ_1, there exists a subgroup $M \lhd_f BS(n,-n)$ such that $g_i \notin \langle b^n\rangle M$ and $g_j \notin \langle c\rangle M$ for any $i \in I$ and any $j \in J$. Considering $\overline{BS(n,-n)} = BS(n,-n)/M$, $\overline{\langle b\rangle} = \langle b\rangle/(\langle b\rangle\cap M)$, $\overline{BS(1,-1)}/(BS(1,-1)\cap M)$, $\overline{\langle b^n\rangle} = \langle b^n\rangle/(\langle b^n\rangle\cap M)$ and $\overline{\langle c\rangle} = \langle c\rangle/(\langle c\rangle\cap M)$, we have $\overline{g} \notin \overline{BS(1,-1)}$ and $\overline{g} \notin \overline{H_1}\ \overline{H_2}$.
Using again the fact that $\overline{BS(n,-n)}$ is finite, and then trivially RZ_2, we obtain that there exists a normal subgroup $\overline{N}$ the trivial subgroup of finite index in $\overline{BS(n,-n)}$ such that $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{N}$. Thus, as in the previous case the desired result is obtained.
Case 3. Finally, suppose that $H_1 \leq \langle b\rangle$ and $H_2 \leq BS(1,-1)$. Let us recall that $g = g_1g_2\cdots g_r$ $(r \geqslant 1)$ is a reduced form of g in $\mathrm{BS}(n,-n) = \langle b\rangle \underset{b^n=c}{*} BS(1,-1)$.
Subcase (a) Suppose that $l(g) = 0$. That is $g \in \langle b^n\rangle = \langle c\rangle$. It is obvious that $g \notin (\langle b^n\rangle\cap H_1)(\langle c\rangle\cap H_2)$ since $g \notin H_1H_2$. Also, $(\langle b^n\rangle\cap H_1)(\langle c\rangle\cap H_2)$ can be viewed as a finitely generated subgroup of $\langle c\rangle$, and $\langle c\rangle$ is RZ_1. Therefore, there exists $U \lhd_f \langle c\rangle$ such that $g \notin (\langle b^n\rangle\cap H_1)(\langle c\rangle\cap H_2)U$; and it comes that $(\langle b^n\rangle\cap H_1)U \neq \langle b^n\rangle$ and $(\langle b^n\rangle\cap H_2)U \neq \langle b^n\rangle$. Thus, by Lemma 4.1, there exists a subgroup $M \lhd_f BS(n,-n)$ verifying $M\cap\langle c\rangle = U$, $(M\langle b^n\rangle)\cap(MH_1) = M(\langle b^n\rangle\cap H_1)$ and $(M\langle c\rangle)\cap(MH_2) = M(\langle c\rangle\cap H_2)$. Define the factor group $\overline{BS(n,-n)} == BS(n,-n)/M$, where $\overline{\langle b\rangle} = \langle b\rangle/(M\cap\langle b\rangle)$, $\overline{BS(1,-1)} = BS(1,-1)/(M\cap BS(1,-1))$, $\overline{\langle b^n\rangle} = \langle b^n\rangle/(M\cap\langle b^n\rangle)$ and $\overline{\langle c\rangle} = \langle c\rangle/(M\cap\langle c\rangle)$.

Since,

$$
\begin{aligned}
(MH_1/M)\cap(M\langle b^n\rangle/M) &= \{gM \mid g \in MH_1 \ \ and \ \ g \in M\langle b^n\rangle\}\\
&= \{gM \mid g \in MH_1 \cap M\langle b^n\rangle\}\\
&= (MH_1\cap M\langle b^n\rangle)/M\\
&= M(H_1\cap\langle b^n\rangle)/M\,,
\end{aligned}
$$

we have $\overline{H_1} \cap \overline{\langle b^n \rangle} = \overline{H_1 \cap \langle b^n \rangle}$ with $\overline{H_1} = H_1/(M \cap H_1 = MH_1/M$. Similarly, we obtain also $\overline{H_2} \cap \overline{\langle c \rangle} = \overline{H_2 \cap \langle c \rangle}$, with $\overline{H_2} = H_2/(M \cap H_2 = MH_2/M$.
We claim that $\overline{g} \notin \overline{H_1}\ \overline{H_2}$. Indeed: if $\overline{g} \in \overline{H_1}\ \overline{H_2}$, then $\overline{g} = \overline{h_1}\ \overline{h_2}$ with $\overline{h_1} \in \overline{H_1}$ and $\overline{h_2} \in \overline{H_2}$. Since $g \in \langle c \rangle$, $H_1 \leq \langle b \rangle$ and $H_2 \leq BS(1,-1)$, then $\overline{h_1} = \overline{g}\overline{h_2}^{-1} \in \overline{BS(1,-1)}$. Consequently, $\overline{h_1} \in \overline{H_1} \cap \overline{BS(1,-1)} \subseteq \overline{\langle b \rangle} \cap \overline{BS(1,-1)} = \overline{\langle b^n \rangle}$. Thus $\overline{h_1} \in \overline{H_1} \cap \overline{\langle b^n \rangle}$. Similarly, $\overline{h_2} \in \overline{H_2} \cap \overline{\langle c \rangle}$, so that $\overline{g} \in (\overline{H_1} \cap \overline{\langle b^n \rangle})(\overline{H_2} \cap \overline{\langle c \rangle}) = \overline{(H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)}$. Thus $g = h_1h_2m \in (H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)M$, where $m \in M$, and it follows that $m = h_2^{-1}h_1^{-1}g \in \langle c \rangle$. Therefore, $m \in M \cap \langle c \rangle = U$ so that $g \in (H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)U$. But this contradicts the assumption that $g \notin ((\langle b^n \rangle \cap H_1)(\langle c \rangle \cap H_2)U$. Thus $\overline{g} \notin \overline{H_1}\ \overline{H_2}$ in $\overline{BS(n,-n)}$. Since $\overline{BS(n,-n)}$ is RZ_2 as a finite group, there exists a subgroup $\overline{N} \lhd_f \overline{BS(n,-n)}$ such that $\overline{g} \notin \overline{H_1}\ \overline{H_2}\ \overline{N}$. And like in the previous cases, it comes that there exists a subgroup $N \lhd_f BS(n,-n)$ satisfying $g \notin H_1H_2N$. And the set H_1H_2 is closed in the profinite topology of $BS(n,-n)$ as desired.
Subcase (b) Suppose that $l(g) = 1$. That is $g \in BS(1,-1) \setminus \langle c \rangle$ (or $g \in \langle b \rangle \setminus \langle b^n \rangle$).
► Suppose in addition that $g \notin \langle c \rangle H_2$. Since $\langle c \rangle H_2$ is a finitely generated subgroup of $\mathrm{BS}(1,-1)$ which is RZ_2 as a polycylic group, there exists a subgroup $M \lhd_f BS(1,-1)$ such that $g \notin \langle c \rangle H_2M$. Thus $\overline{g} \notin \overline{\langle c \rangle}\overline{H_2}$ in $\overline{BS(1,-1)} = BS(1,-1)/M$, where $\overline{\langle c \rangle} = \langle c \rangle/(\langle c \rangle \cap M)$ and $\overline{H_2} = H_2/H_2 \cap M$. Since $M \cap \langle c \rangle$ can be viewed as a subgroup of $\langle b \rangle$, then using the P. Stebe's result cited previously, there exists $L \lhd_f \langle b \rangle$ satisfying $L \cap \langle b^n \rangle = M \cap \langle c \rangle$. Now, consider $\overline{\langle b \rangle} = \langle b \rangle/L$, $\overline{\langle b^n \rangle} = \langle b^n \rangle/L \cap \langle b^n \rangle = \langle c \rangle/M \cap \langle c \rangle = \overline{\langle c \rangle}$, and then $\overline{BS(n,-n)} = \overline{\langle b \rangle} \underset{\overline{b^n}=\overline{c}}{*} \overline{BS(1,-1)}$. In $\overline{BS(n,-n)}$, we have $\overline{H_1} = H_1/L \cap H_1$, $\overline{H_2} = H_2/M \cap H_2$ and $\overline{g} = gM \notin \overline{\langle c \rangle}\overline{H_2}$. Also, $\overline{g} \notin \overline{H_1}\ \overline{H_2}$. Indeed: if $\overline{g} \in \overline{H_1}\ \overline{H_2}$, then $\overline{g} = \overline{h_1}\ \overline{h_2}$ with $\overline{h_1} \in \overline{H_1}$ and $\overline{h_2} \in \overline{H_2}$. Thus $\overline{h_1} = \overline{g}\overline{h_2}^{-1} \in \overline{BS(1,-1)}$, and then $\overline{h_1} \in \overline{H_1} \cap \overline{BS(1,-1)} \subseteq \overline{\langle c \rangle}$. It comes that $\overline{g} = \overline{h_1h_2} \in \overline{\langle c \rangle}\overline{H_2}$, which contradicts the assumption that $\overline{g} \notin \overline{\langle c \rangle}\overline{H_2}$. Then $\overline{g} \notin \overline{H_1}\ \overline{H_2}$ in $\overline{BS(n,-n)}$. Using the fact that groups $\overline{\langle b \rangle}$ and $\overline{BS(1,-1)}$ are RZ_2 as finite groups, we obtain that $\overline{BS(n,-n)}$ is RZ_2 as a free product of RZ_2 groups amalgamated by finite subgroups. And the desired result is obtained like in **Case 2** *(b)*.
► Suppose now that $g \in \langle c \rangle H_2$. Hence $g = c^th_2$, with $t \in \mathbb{Z}$ and $h_2 \in H_2$. From $g \notin H_1H_2$ we have $c^t \notin H_1H_2$. Since $l(c^t) = 0$, so using **Case 3** *Subcase (a)* there exists $N \lhd_f BS(n,-n)$ such that $c^t \notin H_1H_2N$. Thus $g \notin H_1H_2N$ and the set H_1H_2 is closed in the profinite topology of $BS(n,-n)$.
The subcase $g \in \langle b \rangle \setminus \langle b^n \rangle$ is treated similarly, since $\langle b \rangle$ as a finitely generated abelian group is RZ and particularly RZ_2.
Subcase (c) Let finally examine the case $l(g) \geq 2$, with $g = g_1g_2 \dots g_r$ $(r \geqslant 2)$. Denote again by I and J the set of indices in $\{1, 2, \cdots, r\}$ of components of g belonging in $\langle b \rangle \setminus \langle b^n \rangle$ and $BS(1,-1) \setminus \langle c \rangle$ respectively. Since $\mathrm{BS}(n,-n)$ is RZ_1, the desired result is obtained like in *Case 2 (b)* $r > 1$. That is, the set H_1H_2 is closed in the profinite topology of $BS(n,-n)$. And the theorem is demonstrated. □

5. Proof of Corollary 1.2

Assume that K is RZ_2 and contains a finitely generated subgroup U of finite index in both A and B such that $\varphi(u) = u$ for any $u \in U$. Since $U \leq Z(K)$ and $t^{-1}ut = u$ for any $u \in U$, it comes that $U \leq Z(G)$. By Proposition 2.3, we have $U^n \leq_f U$ and consequently $U^n \leq_f A$ and $U^n \leq_f B$, for any nonzero natural number n. It is then obvious that U^n, for any nonzero natural number n, is finitely generated. Thus, K/U^n is RZ_2, by Proposition 2.2. Also, since A and B are isomorphic, so are the finite groups A/U^n and B/U^n for any nonzero natural number n. Thus, for any nonzero natural number n the HNN-extension $G_n = G/U^n = \left\langle K/U^n, \tau \mid \tau^{-1}A/U^n\tau = B/U^n \right\rangle$ is RZ by Proposition 1.4 and particularly RZ_2. Consequently G is RZ_2 by Theorem 1.1. So, the corollary is demonstrated.

Acknowledgement. The authors thank the anonymous referee for the throughout and carefull reading of the paper and for the very helpfull comments and suggestions that lead to the improvement of the paper.

References

[1] Allenby, R., Doniz, D., *A free product of finitely generated nilpotent groups amalgamating a cycle that is not subgroup separable*, Proc. Amer. Math. Soc. **124** (4) (1996), 1003–1005.

[2] Baumslag, B., Tretkoff, M., *Residually finite HNN extensions*, Comm. Algebra **6** (1978), 179–194.

[3] Baumslag, G., *On the residual finiteness of generalised free products of nilpotent groups*, Trans. Amer. Math. Soc. **106** (1963), 193–209.

[4] Baumslag, G., Solitar, D., *Some two-generator one-relator non-Hopfien groups*, Bull. Amer. Math. Soc. **38** (1962), 199–201.

[5] Cohen, D., *Combinatorial groups theory: a topological approach*, London Math. Soc. Stud. Texts, 1989, ISBN: 0-521-34133-7; 0-521-34936-2.

[6] Coulbois, T., *Propriètés de Ribes-Zalesskii, topologie profinie, produit libre et généralisations*, Ph.D. thesis, Universite Paris vii-Denis Diderot, 2000.

[7] Coulbois, T., *Free products, profinite topology and finitely generated subgroups*, Internat. J. Algebra Comput. **14** (2001), 171–184.

[8] Doucha, M., Malicki, M., *Generic representations of countable groups*, Trans. Amer. Math. Soc. **164** (2019), 105–114.

[9] Hall, M., *Cosets representations in free groups*, Trans. Amer. Math. Soc. **67** (1949), 421–432.

[10] Hall, M., *A topology for free groups and related groups*, Ann. of Math. **52** (1950), 127–139.

[11] Hall, P., *On the finiteness on certain soluble groups*, Proc. London Math. Soc. **164** (1959), 595–622.

[12] Hirsch, K., *On infinite soluble groups. III*, Proc. London Math. Soc. **49** (1946), 184–194.

[13] Lennox, J.C., Wilson, J.S., *On products of subgroups in polycyclic groups*, Arch. Math. (Basel) **33** (1979), 305–309.

[14] Magnus, W., Karass, A., Solitar, D., *Combinatorial Group Theory: Presentations of Groups in Term of Generators and Relations*, Interscience Publishers (John Wiley and Sons), New York, 1966.

[15] Mal'cev, A., *On homomorphisms onto finite groups*, Ivanov. Gos. Ped. Inst. Ucen. Zap. **18** (1958), 49–60.

[16] Meskin, S., *Nonresidually finite one relator groups*, Trans. Amer. Math. Soc. **164** (1972), 105–114.

[17] Metaftsis, V., Raptis, E., *Subgroups Separability of HNN-Extensions with Abelian Base Groups*, J. Algebra **245** (2001), 42–49.

[18] Moldavanskii, D., Uskova, A., *On the finitely separability of subgroups of generalized free products*, Arxiv: 1308.3955.[mathGR], (2013).

[19] Neumann, B.H., *An essay of free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 503–554.

[20] Pin, J.-E., Reutenaeur, C., *A conjecture on the Hall topology for free groups*, Bull. London Math. Soc. **23** (1991), 356–362.

[21] Ribes, L., *Profinite graphs and groups*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, Springer, Cham, 2017, ISBN: 978-3-319-61041-2; 978-3-319-61199-0.

[22] Ribes, L., Zalesskii, P.A., *Profinite topologies in free products of group*, Internat. J. Algebra Comput. **14** (2004), 751–772.

[23] Rips, E., *An example of a NON-LERF group which is a free product of LERF groups with an amalgamated cyclic subgroup*, Israel J. Math. Vol. **70** (1) (1990), 104–110.

[24] Romanovskii, N.S., *On the residual finiteness of free products with respect to subgroups*, Izv. Akad. Nauk SSSR Ser. Math. **33** (1969), 1324–1329.

[25] Rosendal, C., *Finitely approximable groups and actions Part I: The Ribes-Zalesskii property*, J. Symbolic Logic **76** (4) (2011), 1297–1306.

[26] Rosendal, C., *Finitely approximable groups and actions Part II: Generic representations*, J. Symbolic Logic **76** (4) (2011), 1307–1321.

[27] Shihong, You, *The product separability of the generalized free products of cyclic groups*, J. London Math. Soc.(2) **56** (1997), 91–103.

[28] Stebe, P., *Conjugacy separability of certain free products with amalgamation*, Trans. Amer. Math. Soc. **156** (1971), 119–129.

Department of Mathematics,
Higher Teacher's Training College, The University of Maroua,
P.O. Box 55 Maroua – Cameroon
E-mail: gilbertmantika@yahoo.fr

Department of Mathematics and Computer Science,
Faculty of Sciences, The University of Maroua,
P.O. Box 814 Maroua – Cameroon
E-mail: tematen@yahoo.fr

Department of Mathematics and Computer Science,
The University of Ngaoundere and AIMS Cameroon,
P.O. Box 454 Ngaoundere – Cameroon
E-mail: tieudjo@yahoo.com