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OSCILLATORY BEHAVIOR OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATION WITH MULTIPLE FUNCTIONAL DELAYS UNDER DERIVATIVE OPERATOR

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Abstract. In this article, we obtain sufficient conditions so that every solution of neutral delay differential equation

$$\Big(y(t)-\sum_{i=1}^{k}p_i(t)y(r_i(t))\Big)^{(n)}+v(t)G(y(g(t)))-u(t)H(y(h(t)))=f(t)$$

oscillates or tends to zero as $t\to\infty$, where, $n\geq 1$ is any positive integer, $p_i,\ r_i\in C^{(n)}([0,\infty),\mathbb{R})$ and p_i are bounded for each $i=1,2,\dots,k$. Further, $f\in C([0,\infty),\mathbb{R})$, $g,\ h,\ v,\ u\in C([0,\infty),[0,\infty))$, G and $H\in C(\mathbb{R},\mathbb{R})$. The functional delays $r_i(t)\leq t$, $g(t)\leq t$ and $h(t)\leq t$ and all of them approach ∞ as $t\to\infty$. The results hold when $u\equiv 0$ and $f(t)\equiv 0$. This article extends, generalizes and improves some recent results, and further answers some unanswered questions from the literature.

1. Introduction

In this article, we obtain sufficient conditions for every solution of the higher order neutral delay differential equation (NDDE in short)

$$\Big(y(t)-\sum_{i=1}^{k}p_i(t)y(r_i(t))\Big)^{(n)}+v(t)G\big(y(g(t))\big)-u(t)H\big(y(h(t))\big)=f(t)\,, \tag{1.1}$$

to oscillate or to tend to zero as $t\to\infty$, where, $n\geq 1$ is any positive integer, $p_i,\ r_i\in C^{(n)}([0,\infty),\mathbb{R})$ and p_i are bounded for each $i=1,2,\dots,k$. Further, $f\in C([0,\infty),\mathbb{R})$, $g,h,v,u\in C([0,\infty),[0,\infty))$, G and $H\in C(\mathbb{R},\mathbb{R})$. The functional delays $r_i(t)\leq t$, $g(t)\leq t$ and $h(t)\leq t$ and all of them approach ∞ as $t\to\infty$.

The results hold when $u\equiv 0$, $f(t)\equiv 0$, and $G(u)\equiv u$.

Some of the following assumptions would be needed later in this article.

(H0) $\liminf\limits_{t\to\infty} G(x(t))>0$ if $\liminf\limits_{t\to\infty} x(t)>0$ and $\limsup\limits_{t\to\infty} G(x(t))<0$ if $\limsup\limits_{t\to\infty} x(t)<0$.

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(H1) $xG(x) > 0$ for $x \neq 0$.

(H2) $v(t) > 0$, $\displaystyle\int_{t_0}^{\infty} v(s)\,ds = \infty$.

(H3) There exists a bounded function $F \in C^{(n)}([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$.

(H4) The function $F(t)$ in (H3) satisfies $\lim\limits_{t\to\infty} F(t) = 0$.

(H5) $\displaystyle\int_{t_0}^{\infty} t^{n-1}u(t)\,dt < \infty$.

(H6) H is bounded and $uH(u) > 0$ for $u \neq 0$.

(H7) $\displaystyle\int_{t_0}^{\infty} t^{n-1}v(t)\,dt = \infty$.

Note that if $\displaystyle\int_{t_0}^{\infty} t^{n-1}|f(t)|\,dt < \infty$ then, (H3) and (H4) hold.

In recent years there have been much interest in studying the oscillatory and asymptotic behaviour of neutral delay differential equations and it's applications. For some recent results, one may go through the publications [1, 2, 7, 8, 10, 11, 13] and references cited there in. However, study of NDDEs of the form (1.1) with several functional delays under the derivative operator; seems to be relatively scarce. It is found that the authors [1, 7, 8, 10, 11, 13] use the result [2, Lemma 1.5.2], as the main tool, to study NDDEs

$$\big(y(t) - p(t)y(t-\tau)\big)^{(n)} + q(t)G(y(t-\sigma)) = f(t) \tag{1.2}$$

where $n \geq 1$, is any positive integer. But this lemma cannot be applied to the study of (1.1) because of the presence of multiple functional delays under the derivative operator. In this article, by following the suggestion in [2, Notes 1.8, page 31], we extend [2, Lemma 1.5.2] from one delay to multiple delays for it's own sake and for it's application to study the oscillatory behavior of (1.1). Then these results are further applied to study the behavior of solutions of

$$\Big[y(t) - \sum_{i=1}^{k} p_i(t)y\big(r_i(t)\big)\Big]^{(n)} + q(t)G\big(y(g(t))\big) = f(t)\,, \tag{1.3}$$

where $q(t)$ changes sign. The paper [9] is concerned with the study of oscillatory and asymptotic behaviour of NDDE (1.2) with $q(t)$ having fixed sign. We considered the general case that $q(t)$ may change sign and generalized the results in [9] by dropping the conditions (i) G is non decreasing and $\liminf\limits_{|u|\to\infty} G(u)/u > \delta > 0$. Further, this article could address the proposed problems [2, Open problem 2.8.3, page 57, Open problem 10.10.2, page 287].

"Let t_1 be a fixed positive real number and

$$t_0 = \min\Big\{\inf_{t\geq t_1}(r_1(t), r_2(t), \ldots, r_k(t)),\ \inf_{t\geq t_1} g(t),\ \inf_{t\geq t_1} h(t)\Big\}\,.$$

By a solution of (1.1), we mean a function $y \in C([t_0, \infty), \mathbb{R})$ such that $y(t) - \sum_{i=1}^{k} p_i(t)y(r_i(t))$ is n times continuously differentiable on $[t_0, \infty)$ and the neutral equation (1.1) is satisfied by $y(t)$ for all $t \geq t_1$. It is known that (1.1) has a unique solution provided that an initial function $\phi \in C([t_0, t_1], \mathbb{R})$ is given to satisfy $y(t) = \phi(t)$ for all $t \in [t_0, t_1]$. Such a solution is said to be *non-oscillatory* if it is eventually positive or eventually negative, otherwise it is called *oscillatory*."

In this work we assume the existence of solutions of (1.1) and study only their qualitative behaviour. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all sufficiently large values of t.

2. Lemmas

In this section, some lemmas are presented which will be used to find sufficient conditions for oscillation of solutions of (1.1).

Lemma 2.1 ([5, p.193]). *"Let $y \in C^n([0, \infty), \mathbb{R})$ be of constant sign and $\not\equiv 0$ in any interval $[T, \infty), T \geq 0,$and $y^{(n)}(t)y(t) \leq 0.$Then there exists a number $t_0 \geq 0$ such that the functions $y^{(j)}(t), j = 1, 2, \ldots, n-1,$ are of constant sign on $[t_0, \infty)$ and there exists a number $m \in \{1, 3, \ldots, n-1\}$ when n is even or $m \in \{0, 2, 4, \ldots, n-1\}$ when n is odd such that*

$$y(t)y^{(j)}(t) > 0 \quad \textit{for} \quad j = 0, 1, 2, \ldots, m,\ t \geq t_0\,,$$
$$(-1)^{n+j-1}y(t)y^{(j)}(t) > 0 \quad \textit{for} \quad j = m+1, m+2, \ldots, n-1,\ t \geq t_0\,."$$

Lemma 2.2 ([12]). *"Let $u(t)$ and $v(t)$ be two real valued continuous functions defined for $t \geq t_0 \geq 0$. Then*

$$\begin{aligned}
\liminf_{t\to\infty} u(t) + \liminf_{t\to\infty} v(t) &\leq \liminf_{t\to\infty} \big(u(t)+v(t)\big) \\
&\leq \limsup_{t\to\infty} u(t) + \liminf_{t\to\infty} v(t) \quad \big(or\ \liminf_{t\to\infty} u(t) + \limsup_{t\to\infty} v(t)\big) \\
&\leq \limsup_{t\to\infty} \big(u(t)+v(t)\big) \\
&\leq \limsup_{t\to\infty} u(t) + \limsup_{t\to\infty} v(t) \qquad (2.1)
\end{aligned}$$

provided that no sum is of the form $\infty - \infty$."

Lemma 2.3 ([12]). *"Let $u(t)$ and $v(t)$ be two non negative real valued continuous functions defined for $t \geq t_0$. Then*

$$\begin{aligned}
&\liminf_{t\to\infty} u(t) \times \liminf_{t\to\infty} v(t) \\
&\qquad \leq \liminf_{t\to\infty} \big(u(t) \times v(t)\big) \\
&\qquad \leq \limsup_{t\to\infty} u(t) \times \liminf_{t\to\infty} v(t) \quad \big(or \ \liminf_{t\to\infty} u(t) \times \limsup_{t\to\infty} v(t)\big) \\
&\qquad \leq \limsup_{t\to\infty} \big(u(t) \times v(t)\big) \\
&\qquad \leq \limsup_{t\to\infty} u(t) \times \limsup_{t\to\infty} v(t) \qquad (2.2)
\end{aligned}$$

provided that no product is of the form $0 \times \infty$."

The following lemma generalizes an important result [2, Lemma 1.5.2].

Lemma 2.4 ([13]). *"Suppose that $\tau(t)$ is a continuous and strictly increasing unbounded function such that $\tau(t) \leq t$. Let u, v, p: $[t_0, \infty) \to R$ be such that*

$$u(t) = v(t) - p(t)v\big(\tau(t)\big)\,, \quad t \geq \tau_{-1}(t_0)\,. \tag{2.3}$$

Suppose that $p(t)$ is in one of the ranges

$$0 \leq p(t) \leq p_1\,, \tag{2.4}$$

$$-1 < -p \leq p(t) \leq 0\,, \tag{2.5}$$

or

$$-p_2 \leq p(t) \leq -p_1 < -1\,, \tag{2.6}$$

where p, p_1, p_2 are positive real numbers. If $v(t) > 0$ for $t \geq t_0 > 0$ and $\liminf_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} u(t) = L \in R$ exists, then $L = 0$."

By following the suggestion in [2, Note 1.8, page 31], we now extend the above lemma from single functional delay to several functional delays for its application to study the qualitative behaviour of solutions of (1.1).

Lemma 2.5. *Suppose that, for each $i = 1, 2, \ldots, k$, $p_i, r_i \in C([t_0, \infty), \mathbb{R})$, p_i are bounded, $r_i(t) \leq t$ and $\lim_{t\to\infty} r_i(t) = \infty$. Further, suppose that $y \in C([t_0, \infty), \mathbb{R})$. Assume $y(t) > 0$ for $t \geq t_0$. Let*

$$z(t) = y(t) - \sum_{j=1}^{k} p_j(t) y(r_j(t))\,, \quad t \geq t_1 > t_0\,. \tag{2.7}$$

If $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = \delta \in \mathbb{R}$ exists then the following statements are true.

(a) *If $p_j(t) \geq 0$ for each j then $\delta \leq 0$ and $p_j(t) \leq 0$ for each j then $\delta \geq 0$.*

(b) *Further, suppose that $y(t)$ is bounded and $p_j(t), j = 1, 2, \ldots, k$, satisfy one of the following four conditions.*

$$p_j(t) \geq 0 \quad \textit{for every } j = 1, 2, \ldots, k \quad \textit{and}$$
$$\sum_{j=1}^{k} \limsup_{t\to\infty} p_j(t) < p < 1\,. \tag{2.8}$$

$$p_j(t) \leq 0 \quad \textit{for every } j = 1, 2, \ldots, k \quad \textit{and}$$
$$\sum_{j=1}^{k} \liminf_{t\to\infty} p_j(t) > -p > -1\,. \tag{2.9}$$

$$p_j(t) < 0 \quad \textit{for every } j = 1, 2, \ldots, k \quad \textit{and there exists}, i \in \{1, 2, 3, \ldots, k\}$$
$$\textit{such that } \limsup_{t\to\infty} p_i(t) - \sum_{j\neq i} \liminf_{t\to\infty} p_j(t) < -1\,. \tag{2.10}$$

$$p_j(t) > 0 \quad \textit{for every } j = 1, 2, \ldots, k \quad \textit{and there exists}, i \in \{1, 2, 3, \ldots, k\}$$
$$\textit{such that } \liminf_{t\to\infty} p_i(t) - \sum_{j\neq i} \limsup_{t\to\infty} p_j(t) > 1\,. \tag{2.11}$$

Then $\delta = 0$ and $\lim_{t\to\infty} y(t) = 0$.

Proof. (a) Since $\lim_{t\to\infty} z(t) = \delta$ exists finitely then $\liminf_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) = \delta$. If $p_j(t) \geq 0$ then $z(t) \leq y(t)$ and $\liminf_{t\to\infty} z(t) \leq \liminf_{t\to\infty} y(t)$. This implies $\delta \leq 0$. Again if $p_j(t) \leq 0$ then $z(t) \geq y(t)$ and this implies $\delta \geq 0$. Hence the result follows.

(b) Consider case (i) i.e.; suppose $p_j(t)$ satisfy (2.8). As $p_i(t) \geq 0$, by part (a) above $\delta \leq 0$. Then applying Lemma 2.2 and 2.3 we have

$$\begin{aligned}
0 \geq \delta = \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty} \Big(y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} \Big(- \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\geq \limsup_{t\to\infty} y(t) - \limsup_{t\to\infty} \Big(\sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\geq \limsup_{t\to\infty} y(t) - \sum_{j=1}^{k} \limsup_{t\to\infty} \big(p_j(t) y\big(r_j(t)\big)\big) \\
&\geq \limsup_{t\to\infty} y(t) - \sum_{j=1}^{k} \limsup_{t\to\infty} p_j(t) \limsup_{t\to\infty} y\big(r_j(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{t\to\infty} y(t)\Big(1-\sum_{j=1}^{k}\limsup_{t\to\infty} p_j(t)\Big)\\
&\geq \limsup_{t\to\infty} y(t)(1-p) \geq 0\,.
\end{aligned}
$$

Hence $\delta = 0$ and $\limsup_{t\to\infty} y(t) = 0$ by (2.8). Then $\lim_{t\to\infty} y(t) = 0$.

Next consider case ii i.e; $p_j(t)$ satisfy (2.9). Clearly, $z(t) \geq 0$ due to (2.9) and this implies $\delta \geq 0$. Application of Lemmas 2.2 and 2.3 to (2.7) yields:

$$
\begin{aligned}
\delta = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}\Big(y(t)-\sum_{j=1}^{k} p_j(t)y\big(r_j(t)\big)\Big)\\
&\leq \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty}\Big(\sum_{j=1}^{k} -p_j(t)y\big(r_j(t)\big)\Big)\\
&\leq \sum_{j=1}^{k}\limsup_{t\to\infty}(-p_j(t))\limsup_{t\to\infty}\big(y(r_j(t))\big)\\
&= \sum_{j=1}^{k} -\liminf_{t\to\infty}(p_j(t)\limsup_{t\to\infty}\big(y(r_j(t))\big)\\
&\leq p\limsup_{t\to\infty}\big(y(t)\big) \leq p\alpha\,.
\end{aligned}
$$

Hence we get

$$
\alpha \geq \frac{\delta}{p} > \delta\,. \tag{2.12}
$$

Again

$$
\begin{aligned}
\delta = \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty}\Big(y(t)-\sum_{j=1}^{k} p_j(t)y\big(r_j(t)\big)\Big)\\
&\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\Big(\sum_{j=1}^{k} -p_j(t)y\big(r_j(t)\big)\Big)\\
&\geq \limsup_{t\to\infty} y(t) + \sum_{j=1}^{k}\liminf_{t\to\infty}\Big((-p_j(t))y\big(r_j(t)\big)\Big)\\
&\geq \limsup_{t\to\infty} y(t) + \sum_{j=1}^{k}\liminf_{t\to\infty}(-p_j(t))\liminf_{t\to\infty} y\big(r_j(t)\big)\\
&= \limsup_{t\to\infty} y(t) = \alpha\,.
\end{aligned}
$$

From this and (2.12) it follows that

$$
\alpha > \delta \geq \alpha\,,
$$

a contradiction. This implies $\delta = 0 = \alpha$. Then $\lim_{t\to\infty} y(t) = 0$.

Next consider case iii: i.e.; $p_j(t)$ satisfy (2.10). Then proceeding with the application of Lemmas 2.2 and 2.3 to (2.7) we obtain

$$\begin{aligned}
\delta = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty} \Big(y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\le \limsup_{t\to\infty} \Big(y(t) + \sum_{j\neq i} -p_j(t) y(r_j(t))\Big) + \liminf_{t\to\infty} \big(- p_i(t) y\big(r_i(t)\big)\big) \\
&\le \limsup_{t\to\infty} y(t) + \limsup_{t\to\infty} \sum_{j\neq i} -p_j(t) y\big(r_j(t)\big) \\
&\quad + \limsup_{t\to\infty} \big(- p_i(t)\big) \liminf_{t\to\infty} \big(y(r_i(t))\big) \\
&\le \limsup_{t\to\infty} y(t) + \sum_{j\neq i} \limsup_{t\to\infty} \big((-p_j(t)) y\big(r_j(t)\big)\big) \\
&\le \limsup_{t\to\infty} y(t) + \sum_{j\neq i} \limsup_{t\to\infty} \big(- p_j(t)\big) \limsup_{t\to\infty} \big(y(r_j(t))\big) \\
&\le \limsup_{t\to\infty} \big(y(t)\big) \Big[1 - \sum_{j\neq i} \liminf_{t\to\infty} p_j(t)\Big]. \qquad (2.13)
\end{aligned}$$

Again we have

$$\begin{aligned}
\delta = \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty} \Big(y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\ge \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty} \Big(\sum_{j=1}^{k} -p_j(t) y\big(r_j(t)\big)\Big) \\
&\ge \limsup_{t\to\infty} (-p_i(t) y(r_i(t)) + \liminf_{t\to\infty} \sum_{j\neq i} \big(- p_j(t) y(r_j(t))\big) \\
&\ge \limsup_{t\to\infty} y(r_i(t)) \liminf_{t\to\infty} (-p_i(t)) + \sum_{j\neq i} \liminf_{t\to\infty} \big((-p_j(t)) y(r_j(t))\big) \\
&\ge \limsup_{t\to\infty} y(t)\big(- \limsup_{t\to\infty} p_i(t)\big) + \sum_{j\neq i} \liminf_{t\to\infty} \big(-p_j(t)\big) \liminf_{t\to\infty} y\big(r_j(t)\big) \\
&\ge \limsup_{t\to\infty} y(t)\big(- \limsup_{t\to\infty} p_i(t)\big). \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14), it follows that

$$\limsup_{t\to\infty} y(t)\Big(\sum_{j\neq i} \liminf p_j(t) - 1 - \limsup p_i(t)\Big) \le 0 .$$

Using (2.10), we obtain $\alpha = \limsup_{t\to\infty} y(t) = 0$. Then $\lim_{t\to\infty} y(t) = 0$ and further, using (2.13) and (2.14) we obtain $\delta = \lim_{t\to\infty} z(t) = 0$.

Next consider case iv: i.e.; $p_j(t)$ satisfy (2.11). Then proceeding with the application of Lemmas 2.2 and 2.3 to (2.7) we obtain

$$\begin{aligned}
\delta = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty} \Big(y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty} \sum_{j=1}^{k} -p_j(t) y\big(r_j(t)\big) \\
&\le \limsup_{t\to\infty} y(t) - \limsup_{t\to\infty} \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big) \\
&\le \alpha - \liminf_{t\to\infty} \sum_{j\neq i} p_j(t) y(r_j(t)) - \limsup_{t\to\infty} \big(p_i(t)\big)\big(y(r_i(t))\big) \\
&\le \alpha - \sum_{j\neq i} \liminf_{t\to\infty} \big(-p_j(t)\big)\big(y(r_j(t))\big) - \liminf_{t\to\infty} \big(p_i(t)\big) \limsup_{t\to\infty} \big(y(r_i(t))\big) \\
&\le \alpha - \sum_{j\neq i} \liminf_{t\to\infty} p_j(t) \liminf_{t\to\infty} y\big(r_j(t)\big) - \liminf_{t\to\infty} \big(p_i(t)\big) \limsup_{t\to\infty} y(t) \\
&\le \alpha\big(1 - \liminf_{t\to\infty} p_i(t)\big)\,. \qquad (2.15)
\end{aligned}$$

Again we have

$$\begin{aligned}
\delta = \limsup_{t\to\infty} z(t) &= \limsup_{t\to\infty} \Big(y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\ge \liminf_{t\to\infty} y(t) + \limsup_{t\to\infty} \Big(\sum_{j=1}^{k} -p_j(t) y\big(r_j(t)\big)\Big) \\
&\ge -\liminf_{t\to\infty} \Big(\sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\Big) \\
&\ge -\liminf_{t\to\infty} \big(p_i(t) y\big(r_i(t)\big)\big) - \limsup_{t\to\infty} \sum_{j\neq i} \big(p_j(t) y\big(r_j(t)\big)\big) \\
&\ge -\liminf_{t\to\infty} y\big(r_i(t)\big) \limsup_{t\to\infty} p_i(t) - \sum_{j\neq i} \limsup_{t\to\infty} \big(\big(-p_j(t)\big) y\big(r_j(t)\big)\big) \\
&\ge -\sum_{j\neq i} \limsup_{t\to\infty} p_j(t) \limsup_{t\to\infty} y\big(r_j(t)\big) \\
&\ge -\limsup_{t\to\infty} y(t) \Big(\sum_{j\neq i} \limsup_{t\to\infty} p_j(t)\Big) \\
&= -\alpha\Big(\sum_{j\neq i} \limsup_{t\to\infty} p_j(t)\Big)\,. \qquad (2.16)
\end{aligned}$$

From (2.15) and (2.16), it follows that

$$-\alpha\Big(\sum_{j\neq i}\limsup_{t\to\infty} p_j(t)\Big) \le \delta \le \alpha\big(1-\liminf_{t\to\infty} p_i(t)\big)\,.$$

This implies

$$\alpha\Big(1-\liminf_{t\to\infty} p_i(t)+\sum_{j\neq i}\limsup_{t\to\infty} p_j(t)\Big)\ge 0\,.$$

By (2.11), we obtain $\alpha \le 0$. Since $y(t) > 0$ then $\alpha = 0$. This implies $\lim_{t\to\infty} y(t) = 0$. By (2.16), it follows that $\delta \ge 0$. Using (a), we obtain $\delta = 0$. Thus the lemma is proved. □

Remark 2.6. If $p_i(t) = p(t)$ and $p_j(t) = 0$, for $j \neq i$, then the conditions (2.8), (2.9), (2.10), (2.11) due to the boundedness of $p_j(t)$ reduces to the following conditions (i) $0 \le p(t) \le p < 1$, (ii)$-1 < -p \le p(t) \le 0$, (iii) $-p_1 \le p(t) \le -p < -1$ and (iv) $p_1 \ge p(t) \ge p > 1$ respectively. These conditions are assumed in [6, 8, 9, 10, 11].

Lemma 2.7. *Assume $y(t) < 0$ for $t \ge t_0$ and $\limsup_{t\to\infty} y(t) = 0$. Suppose that $z(t)$ is defined as in* (2.7)*. Further, assume and $\lim_{t\to\infty} z(t) = \delta$ exists finitely. Then*

(a) *If $p_j(t) \ge 0$ for each j then $\delta \ge 0$ and $p_j(t) \le 0$ for each j then $\delta \le 0$.*
(b) *Further, suppose that $y(t)$ is bounded and $p_j(t)$, $j = 1, 2, \ldots, k$, satisfy one of the conditions* (2.8), (2.9), (2.10) *or* (2.11)*. Then $\delta = 0$ and $\lim_{t\to\infty} y(t) = 0$.*

Proof. Proceeding as in the proof of above lemma with the substitution $x(t) = -y(t) > 0$, one may complete the proof of the lemma. □

Remark 2.8. Observe that $u(t)$ and $v(t)$ are not assumed to be bounded in Lemmas 2.5 or 2.7. However, we assume that $y(t)$ and $y(r_j(t))$ are bounded. This is done, only to avoid the statement, "provided that no sum is of the form $\infty - \infty$" in Lemma 2.2 and, "provided that no product is of the form $0 \times \infty$" in Lemma 2.3. However, if $p_j(t)$ satisfies (2.9) or (2.10), then the terms in $z(t)$ are positive when $y(t) > 0$. Hence in the limiting case the sum cannot yield $\infty - \infty$ form. Further, if $\liminf_{t\to\infty} p_j(t) > 0$ for each j in the case when $p_j(t)$ satisfies (2.9) then the product $-p_j(t)y(r_j(t))$ in the limiting case cannot be of the form $0 \times \infty$. Thus, if $p_j(t)$ satisfies (2.9) or (2.10), we can relax the condition of boundedness on $y(t)$ in the Lemma 2.5. We state this as a lemma.

Lemma 2.9. *Assume $y(t) > 0$ for $t \ge t_0$ with $\liminf_{t\to\infty} y(t) = 0$, and let $z(t)$ be defined as in* (2.7)*. Assume $\lim_{t\to\infty} z(t) = \delta$ exists is finite. Let $p_j(t)$ satisfy* (2.9) *or* (2.10)*. Assume $\liminf_{t\to\infty} |p_j(t)| > 0$ for the case $p_j(t)$ satisfying* (2.9)*. Then $\delta = 0$ and $\lim_{t\to\infty} y(t) = 0$.*

3. Oscillation of solutions under positive coefficients

Theorem 3.1. *Suppose that* (H1), (H3)–(H5), (H7) *hold. Assume that* $p_j(t)$ *for* $j = 1, 2, 3, \ldots, k$ *satisfies one of the conditions* (2.8)–(2.11). *Then every bounded solution of* (1.1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Assume y is a bounded and eventually positive solution of (1.1). Then there exists a t_0 such that for $t \geq t_0$: $y(t)$, $y(h(t))$, $y(g(t))$, $y(r_i(t))$ are positive.

Define

$$c(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_t^\infty (s-t)^{n-1} u(s) H\big(y(h(s))\big)\, ds \quad \text{for} \quad t \geq t_0\,. \tag{3.1}$$

By assumptions (H2) and (H4), the above integral converges, thus $c(t)$ is a well defined real-valued function, and

$$\lim_{t\to\infty} c(t) = 0\,. \tag{3.2}$$

Note that the nth derivative of c is $c^{(n)}(t) = -u(t)H(y(h(t)))$. For simplicity of notation, we define

$$z(t) = y(t) - \sum_{j=1}^{k} p_j(t) y\big(r_j(t)\big)\,, \tag{3.3}$$

and

$$w(t) = z(t) + c(t) - F(t), \tag{3.4}$$

where $F^{(n)}(t) = f(t)$. Since $v(t) > 0$, then from (3.4), (H1) and (1.1) it follows that

$$w^{(n)}(t) = -v(t)G(y(g(t))) \leq 0\,. \tag{3.5}$$

Then it follows from (3.5) that $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. As $y(t)$ is bounded, then $z(t)$ and $w(t)$ are bounded. Let $\lambda := \lim_{t\to\infty} w(t)$ which exists as a finite number because w is monotonic and bounded. Integrating on (3.5), n times,

$$w(t) - \lambda = \frac{(1)^{n-1}}{(n-1)!} \int_t^\infty (s-t)^{n-1} v(s) G\big(y(g(s))\big)\, ds\,.$$

Since w is bounded, the above integral is convergent. This in turn, by (H7), implies $\liminf_{s\to\infty} G\big(y(g(s))\big) = 0$. As $G(x) \neq 0$ for $x \neq 0$, $\liminf_{s\to\infty} y(g(s)) = 0$ and because $\lim_{t\to\infty} g(t) = \infty$, $\liminf_{t\to\infty} y(t) = 0$.

Since $\lim_{t\to\infty} w(t)$ exists, $c(t), F(t)$ approach zero, and each $p_j(t), j = 1, 2, \ldots, k$ are bounded, it follows that $\lim_{t\to\infty} z(t)$ exists as a finite number. Applying Lemma 2.5 we prove $\lim_{t\to\infty} y(t) = 0$. With the application of Lemma 2.7, the proof for the case $y(t) < 0$ is similar. Thus the theorem is proved. $\square$

Theorem 3.2. *Suppose that* (H0)–(H6) *hold. Assume that there exists positive scalars p and $p_j : j = 1, 2, \dots, k$ such that the functions $p_j(t)$ for $j = 1, 2, \dots, k$ satisfies the condition*

$$-1 < -p_j \le p_j(t) \le 0 \quad \text{for } j = 1, 2, \dots, k \quad \text{and } \sum_{j=1}^{k} p_j = p < 1\,. \tag{3.6}$$

Then every solution of (1.1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a solution of (1.1), which is eventually positive for $t \ge t_0$. Then proceeding as in Theorem 3.1, set $c(t)$, $z(t)$ and $w(t)$ by (3.1), (3.3) and (3.4) respectively to obtain (3.5). By (H2), $w^{(n)}(t)$ is not identically zero in any interval $[t_1, \infty)$. As in the proof of Lemma 2.1, we can show that there exists $t_1 \ge t_0$ such that $w, w', \dots, w^{(n-1)}$ are monotonic and of constant sign on $[t_1, \infty)$. However, we do not know yet that $w > 0$.

Suppose, if possible, y is unbounded. Then there exists an increasing sequence $\{a_j\}$ such that

$$\lim_{j\to\infty} a_j = \infty, \quad \lim_{j\to\infty} y(a_j) = \infty\,, \quad \text{with } y(a_j) = \max_{t_1 \le s \le a_j} y(s)\,.$$

By (3.2), for each $\epsilon > 0$, there exists N_0 such that

$$c(a_j) < \epsilon \quad \text{for } j \ge N_0\,.$$

Since $g(t)$, $h(t)$, $r_i(t)$ for each i, approach ∞ as $t \to \infty$, there exists $N_1 \ge N_0$ such that:

$$a_j, g(a_j), h(a_j), r_i(a_j) > t_1 \quad \text{for } \ j \ge N_1\,.$$

By (H3), there is an upper bound η for $|F|$. Using that $y(t) > 0$, the definition of $\{a_j\}$, and that each $r_i(t) \le t$, we have: by (3.6),

$$\begin{aligned} w(a_j) &= y(a_j) - \sum_{i=1}^{k} p_i(a_j) y\big(r_i(a_j)\big) + c(a_j) - F(a_j) \\ &\ge (1-p)y(a_j) - \epsilon - \eta\,, \quad j \ge N_1\,. \end{aligned}$$

Taking limits in the inequality above, we have $\lim_{j\to\infty} w(a_j) = \infty$. Since $w, w', \dots,$ $w^{(n-1)}$ are monotonic and of constant sign, it follows that $w > 0$ and $w' > 0$. Now by Lemma 2.1 $w^{(n)} \le 0$ and $w > 0$ imply $w^{(n-1)}(t) > 0$ for $t \ge t_1$.

Next we show that y is bounded below by a positive constant, which will be used for bounding the G term from below.

Using that w is positive and increasing, and that each $r_i(t) \le t$ and (3.6), we have:

$$
\begin{aligned}
(1-\sum_{i=1}^{k} p_i)\, w(t) &\le w(t) - \sum_{i=1}^{k} p_i w\big(r_i(t)\big) \\
&\le w(t) + \sum_{i=1}^{k} p_i(t) w\big(r_i(t)\big) \\
&\le y(t) - \sum_{i=1}^{k} p_i(t) y\big(r_i(t)\big) + c(t) - F(t) \\
&\quad + \sum_{i=1}^{k} p_i(t)\Big[y\big(r_i(t)\big) - \Big[\sum_{j=1}^{k} p_j\big(r_i(t)\big) y\big(r_j(r_i(t))\big)\Big] + c\big(r_i(t)\big) - F\big(r_i(t)\big)\Big] \\
&\le y(t) + c(t) - F(t) + \sum_{i=1}^{k} p_i(t)\big[c\big(r_i(t)\big) - F\big(r_i(t)\big)\big]
\end{aligned}
$$

Note that each $p_i(t)$ and $p_j(r_i(t))$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, k$ have the negative sign and $y > 0$ in the inequality above. This implies

$$
\Big(1 - \sum_{i=1}^{k} p_i\Big) w(t) \le y(t) + |c(t)| + |F(t)| + \sum_{i=1}^{k} |p_i(t)| \big[|c(r_i(t))| + |F(r_i(t))|\big] .
$$

This implies

$$
\Big(1 - \sum_{i=1}^{k} p_i\Big) w(t) \le y(t) + \epsilon + \eta + \sum_{i=1}^{k} p_i(\epsilon + \eta) \quad \text{for} \quad t \ge t_1 .
$$

Since $\lim_{t\to\infty} w(t) = \infty$, it follows that $\lim_{t\to\infty} y(t) = \infty$. Then there exists $t_2 \ge t_1$ such that for $t \ge t_2$: $y(t)$, $y(g(t))$, $y(h(t))$, $y(r_i(t))$ for each i, are bounded below by a positive constant. By (H0)–(H1), for $s \ge t_2$, $G(y(g(s)))$ is bounded below by a positive constant α. Integrating (3.5),

$$
w^{(n-1)}(t) = w^{(n-1)}(t_2) + \int_{t_2}^{t} -v(s) G\big(y(g(s))\big)\, ds \le w^{(n-1)}(t_2) - \alpha \int_{t_2}^{t} v(s)\, ds .
$$

Note that by (H2), the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the solution can not be unbounded and eventually positive. Hence $y(t)$ is bounded. Since (H2) implies (H7) then we proceed as in the proof of Theorem 3.1 to prove $\lim_{t\to\infty} y(t) = 0$. The proof for the case when $y(t)$ is eventually negative is similar. Thus the theorem is proved. □

Remark 3.3. The condition (3.6) is equivalent to the condition (2.9).

Theorem 3.4. *Suppose that* (H0)–(H6) *hold. Assume that the condition*

$$
0 \le p_i(t) \le p_i < 1, \textit{ for} i = 1, 2, \ldots, k \quad \textit{and} \quad \sum_{i=1}^{k} p_i = p < 1 . \tag{3.7}
$$

holds. Then every solution of (1.1) *oscillates or tends to zero as* $t \to \infty$.

Proof. By contradiction assume y is an eventually positive solution of (1.1), which does not tend to zero as $t \to \infty$. Then there exists a t_0 such that for $t \geq t_0$: $y(t)$, $y(h(t))$, $y(g(t))$, $y(r_i(t))$ are positive and $\limsup_{t\to\infty} y(t) > 0$. Define $c(t)$, $z(t)$ and $w(t)$ by (3.1),(3.3) and (3.4) respectively to obtain (3.5). By (H2), $w^{(n)}(t)$ is not identically zero in any interval $[t_1, \infty)$. Then from Lemma 2.1 it follows that $w, w', \dots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. We do not know that $w > 0$ yet. Since (3.7) holds,

$$w(t) \geq y(t) - \sum_{i=1}^{k} p_i y(r_i(t)) + c(t) - F(t)\,.$$

Taking the limit superior, using that w is monotonic and that $c(t)$ and $F(t)$ converge to zero, we have

$$\lambda = \lim_{t\to\infty} w(t) \geq (1 - \sum_{i=1}^{k} p_i) \limsup_{t\to\infty} y(t) > 0\,.$$

Then $w(t)$ is positive for t large enough. By [5, Lemma 5.2.1], $w^{(n)} \leq 0$ and $w > 0$ imply the existence of t_1 such that $w^{(n-1)}(t) > 0$ for $t \geq t_1$. Next we show that $\liminf_{t\to\infty} y(t) > 0$, which will be used for bounding $G(y(g(s)))$ from below by a positive constant. Using that $0 \leq p_j(t)$, $j = 1, 2, \dots, k$ and $y > 0$, we have

$$w(t) \leq y(t) + c(t) - F(t)\,.$$

Taking the limit inferior, using that w is monotonic and that $c(t)$ and $F(t)$ approach zero, we have

$$0 < \lambda = \lim_{t\to\infty} w(t) \leq \liminf_{t\to\infty} y(t)\,.$$

Then there exists a $t_2 \geq t_1$ such that for $t \geq t_2$: $y(t)$, $y(h(t))$, $y(g(t))$, $y(r_i(t))$ for each i, are bounded below by a positive constant. By (H0)–(H1), for $s \geq t_2$, $G(y(g(s)))$ is bounded below by a positive constant α. Integrating (3.5),

$$w^{(n-1)}(t) = w^{(n-1)}(t_2) + \int_{t_2}^{t} -v(s) G\big(y(g(s))\big)\, ds \leq w^{(n-1)}(t_2) - \alpha \int_{t_2}^{t} v(s)\, ds\,.$$

Note that by (H2), the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the solution can not be eventually positive without approaching zero.

The proof for the case when y is eventually negative and does not tend to zero as $t \to \infty$ is similar. This completes the proof. □

Remark 3.5. The condition (3.7) is equivalent to the condition (2.8).

Theorem 3.6. *Assume* (H0), (H1), (H3)–(H6). *Further, assume*

$$-p_j \leq p_j(t) \leq 0 \quad \textit{for} \quad j = 1, 2, \dots, k\,. \tag{3.8}$$

Suppose that there exists a real $\alpha > 0$ such that $r_i'(t) \geq \dfrac{1}{\alpha}$ for $i = 1, 2, \ldots, k$. Further Suppose that, the delay functions satisfy $g(r_j(t)) = r_j(g(t))$. for $j = 1, 2, \ldots, k$ and
(3.9)
$$\int_{t_0}^{\infty} v^*(t)\,dt = \infty\,, \quad \textit{where} \quad v^*(t) = \min\big\{v(t), v(r_1(t)), v(r_2(t)), \ldots, v(r_k(t))\big\}\,.$$

Let there exists a positive constant δ, such that for $x_i > 0$, $i = 1, 2, \ldots, k+1$ and $u > 0$

$$G(\sum_{i=1}^{k+1} x_i) \leq \delta \sum_{i=1}^{k+1} G(x_i) \quad \textit{and} \quad G(ux_i) \leq G(u)G(x_i) \tag{3.10}$$

and that for $x_i < 0$, $i = 1, 2, \ldots, k+1$ and $u > 0$,

$$G\Big(\sum_{i=1}^{k+1} x_i\Big) \geq \delta \sum_{i=1}^{k+1} G(x_i) \quad \textit{and} \quad G(ux_i) \geq G(u)G(x_i)\,. \tag{3.11}$$

Then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

Proof. By contradiction assume y is an eventually positive solution of (1.1), which does not tend to zero as $t \to \infty$. Then there exists a t_0 such that for $t \geq t_0$: $y(t)$, $y(h(t))$, $y(g(t))$ and $y(r_i(t))$: $i = 1, 2, \ldots, k$ are positive and $\limsup_{t\to\infty} y(t) > 0$. Define $c(t)$, $z(t)$ and $w(t)$ by (3.1), (3.3) and (3.4) respectively to obtain (3.5). Then, $w^{(n)} \leq 0$ and $w, w', \ldots, w^{(n-1)}$ are monotonic and of constant sign on some interval $[t_1, \infty)$. From $p_j(t) \leq 0$ for each $j = 1, 2, \ldots, k$ and $y > 0$, it follows that $w(t) \geq y(t) + c(t) - F(t)$. In the limit

$$\lambda = \lim_{t\to\infty} w(t) \geq \limsup_{t\to\infty} y(t) > 0\,.$$

Since $c(t)$ and $F(t)$ approach zero then $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} w(t) = \lambda > 0$. $z(t)$ is bounded below by a positive constant, for all t large. Using $y(t) + \sum_{i=1}^{k} p_i y(r_i(t)) \geq y(t) - \sum_{i=1}^{k} p_i(t) y(r_i(t))$, $\lim_{t\to\infty} g(t) = \infty$, and $g(r_i(t)) = r_i(g(t))$, $i = 1, 2, \ldots, k$, it follows that $y(g(t)) + \sum_{i=1}^{k} p_i y(g(r_i(t)))$ is also bounded below by a positive constant, on some interval $[t_2, \infty)$. Then by (H0)–(H1), there exist a positive constant α such that $\alpha \le G\big(y(g(t)) + \sum_{i=1}^{k} p_i y(g(r_i(t)))\big)$. Using (3.10)

$$\begin{aligned}\alpha &\le G\big(y(g(t)) + \sum_{i=1}^{k} p_i y(g(r_i(t)))\big)\\ &\le \delta\Big[G\big(y(g(t))\big) + \sum_{i=1}^{k} G\big((p_i) y(g(r_i(t)))\big)\Big]\\ &\le \delta\Big[G\big(y(g(t))\big) + \sum_{i=1}^{k} G(p_i) G\big(y(g(r_i(t)))\big)\Big]\end{aligned}$$

Since $r_i'(t) \ge \dfrac{1}{\alpha}$ and $w^{(n)}(r_i(t)) < 0$, it follows that $\alpha w^{(n)}(r_i(t)) r_i'(t) \le w^{(n)}(r_i(t))$ for $i = 1, 2, \ldots, k$. Using this, from (3.5), we obtain

$$\begin{aligned}&w^{(n)}(t) + \alpha \sum_{i=1}^{k} G(p_i) r_i'(t) w^{(n)}(r_i(t))\\ &\le w^{(n)}(t) + \sum_{i=1}^{k} G(p_i) w^{(n)}(r_i(t))\\ &\le -v^*(t)\big[G(y(g(t))) + \sum_{i=1}^{k} G(p_i) G\big(y(g(r_i(t)))\big)\big]\\ &\le -v^*(t)\alpha/\delta\,.\end{aligned}$$

Integrating,

$$\begin{aligned}&w^{(n-1)}(t) + \alpha \sum_{i=1}^{k} G(p_i) w^{(n-1)}\big(r_i(t)\big)\\ &\qquad \le w^{(n-1)}(t_2) + \alpha \sum_{i=1}^{k} G(p_i) w^{(n-1)}\big(r_i(t_2)\big) - (\alpha/\delta) \int_{t_2}^{t} v^*(s)\, ds\,.\end{aligned}$$

In the limit as $t \to \infty$, by (3.9), the right-hand side approaches $-\infty$ while the left-hand side is positive. This contradiction proves that eventually positive solutions must converge to zero. For eventually negative solutions,one may proceed as above to get the desired result. Thus the proof is complete. $\square$

Remark 3.7. The condition (3.8) is less restrictive than the condition (2.10).

Remark 3.8. The condition (3.9) implies (H2) but the converse is not necessarily true. However, if $v(t)$ is monotonic then both (3.9) and (H2) are equivalent. Indeed, if $v(t)$ is decreasing then $v^*(t) = v(t)$. Hence the equivalence of (3.9) and (H2) is immediate. On the other hand if $v(t)$ is increasing then assume that (H2) holds. As $v(t)$ is increasing, (3.9) implies $v^*(t) = v(r(t))$ where $r(t) = \min\{r_i(t) : i = 1, 2, \ldots, k\}$ for large t. Clearly, $r(t) \le t$ and $r(t) \to \infty$ as $t \to \infty$. $v(t)$ is increasing implies, there exists $\delta > 0$ such that $v(r(t)) > \delta$ for $t \ge t_1$. Hence

$\int_{t_1}^{\infty} v^*(t)dt = \int_{t_1}^{\infty} v(r(t))\,dt \geq \delta \int_{t_1}^{\infty} dt = \infty$. Hence (3.9) holds. Thus, (3.9) and (H2) are equivalent, when $v(t)$ is monotonic.

Theorem 3.9. *Assume* (H0)–(H6) *to hold. Further, assume that* $p_j(t)$ $j = 1, 2, \ldots, k$ *satisfy* (3.8). *Suppose that there exists a real* $\alpha > 0$ *such that* $r_i'(t) \geq \dfrac{1}{\alpha}$ *for* $i = 1, 2, \ldots, k$. *Further Suppose that, the delay functions satisfy* $g(r_j(t)) = r_j(g(t))$ *for* $j = 1, 2, \ldots, k$. *Suppose that* (3.10), (3.11) *hold and that* $v(t)$ *is monotonic. Then every solution of* (1.1) *oscillates or tends to zero as* $n \to \infty$.

Proof. The proof follows from that of Theorem 3.6 and the Remark 3.8. □

Remark 3.10. The prototype of the function G satisfying (H0), (H1), (3.10) and (3.11) is $G(u) = (\beta + |u|^{\mu})|u|^{\lambda} sgnu$, where $\lambda > 0, \mu > 0, \lambda + \mu \geq 1, \beta \geq 1$. For verification we may take help of the well known inequality(see [3, p.292])

$$u^p + v^p \geq \begin{cases} (u+v)^p\,, & 0 \leq p < 1\,, \\ 2^{1-p}(u+v)^p\,, & p \geq 1\,. \end{cases}$$

4. Oscillation of solutions under oscillatory coefficients

In this section, we find sufficient conditions so that every solution of the higher order ($n \geq 1$) neutral differential equation (1.3) oscillates or tends to zero as $t \to \infty$, where $q(t)$ is allowed to change sign. Let $q^+(t) = \max\{q(t), 0\}$ and $q^-(t) = \max\{-q(t), 0\}$. Then $q(t) = q^+(t) - q^-(t)$ and the equation (1.3) can be written as

$$\text{(4.1)} \quad \Big[y(t) - \sum_{i=1}^{k} p_i(t)y(r_i(t))\Big]^{(n)} + q^+(t)G\big(y(g(t))\big) - q^-(t)G\big(y(g(t))\big) = f(t)\,.$$

Now we proceed as in the previous section by setting $v(t) = q^+(t)$, $u(t) = q^-(t)$ and $H(x) = G(x)$. Assumptions (H2), (H5), (H6) and (H7) become
(B2): $\int_{t_0}^{\infty} q^+(t)\,dt = \infty$. (B5): $\int_{t_0}^{\infty} t^{n-1}q^-(t)\,dt < \infty$. (B6): G is bounded and (B7): $\int_{t_0}^{\infty} t^{n-1}q^+(t)\,dt = \infty$ respectively.

Therefore, the study of (1.3) reduces to the study of (1.1) in Theorems 3.1, 3.2 and 3.4. Thus, we have the following results for (1.3) where $q(t)$ changes sign.

Theorem 4.1. *Suppose that* (H1), (H3)–(H4),(B5),(B7) *hold. Assume that* $p_j(t)$ *for* $j = 1, 2, 3, \ldots, k$ *satisfies one of the conditions* (2.8)–(2.11). *Then every bounded solution of* (1.3), *where* $q(t)$ *changes sign, oscillates or tends to zero as* $t \to \infty$.

Theorem 4.2. *Suppose that* (H0), (H1), (B2), (H3), (H4), (B5), (B6) *hold. Assume that there exists a positive constant* p *such that the functions* $p_j(t)$ *for* $j = 1, 2, \ldots, k$ *satisfies the condition* (3.6). *Then every solution of* (1.3) *oscillates or tends to zero as* $t \to \infty$.

Theorem 4.3. *Suppose that* (H0), (H1), (B2), (H3), (H4), (B5), (B6) *hold. Assume that the condition* (3.7) *holds. Then every solution of* (1.3) *oscillates or tends to zero as* $t \to \infty$.

However, theorems in [4] can not be applied to (4.1) or to (1.3), because the condition "G is bounded"is not compatible to the condition that "$\liminf_{|u|\to\infty} G(u)/u > \delta > 0$."

For the results in this section, we need G to be bounded, continuous, and to satisfy (H0) and (H1). The prototype of such a function $G(y)$ is $y^{2n}\,\mathrm{sgn}(y)/(1+y^{2n})$.

5. Examples

The following examples illustrate Theorems 4.1 and 4.3.

Example 5.1. Consider the higher order NDDE

$$\big(y(t) - (1/2e)y(t-1) - (1/2e^2)y(t-2)\big)^{(n)} + q(t)y(t-3) = f(t) \tag{5.1}$$

where

$$q(t) = \begin{cases} \sin(t), & 2k\pi \le t \le (2k+1)\pi,\ k = 0,1,2,\dots \\ \dfrac{\sin(t)}{t^{n+2}}, & (2k+1)\pi \le t \le (2k+2)\pi,\ k = 0,1,2,\dots \end{cases} \tag{5.2}$$

and

$$f(t) = \begin{cases} \sin(t)e^{-t+3}, & 2k\pi \le t \le (2k+1)\pi,\ k = 0,1,2,\dots \\ \dfrac{\sin(t)e^{-t+3}}{t^{n+2}}, & (2k+1)\pi \le t \le (2k+2)\pi,\ k = 0,1,2,\dots \end{cases} \tag{5.3}$$

Clearly,

$$q^+(t) = \begin{cases} \sin(t), & 2k\pi \le t \le (2k+1)\pi,\ k = 0,1,2,\dots \\ 0, & (2k+1)\pi \le t \le (2k+2)\pi,\ k = 0,1,2,\dots \end{cases} \tag{5.4}$$

and

$$q^-(t) = \begin{cases} 0, & 2k\pi \le t \le (2k+1)\pi,\ k = 0,1,2,\dots \\ \dfrac{\sin(t)}{t^{n+2}}, & (2k+1)\pi \le t \le (2k+2)\pi,\ k = 0,1,2,\dots \end{cases} \tag{5.5}$$

It may be verified that the NDDE (5.1) satisfies all the conditions of theorem 4.1. Hence every bounded solution of (5.1) oscillates or tends to zero as $t \to \infty$. As such, it admits a positive solution $y(t) = e^{-t}$ which tends to zero as $t \to \infty$.

Example 5.2. The following higher order NDDE

$$\big(y(t) - (1/2e)y(t-1) - (1/2e^2)y(t-2)\big)^{(n)} + q(t)G\big(y(t-3)\big) = f(t) \tag{5.6}$$

where $G(u) = u^2\,\mathrm{sgn}\,(u)/(1+u^2)$ and $q(t)$ as in (5.2) and

$$f(t) = \begin{cases} e^6(e^{2t}+e^6)^{-1}\sin(t), & 2k\pi \le t \le (2k+1)\pi,\ k = 0,1,2,\dots \\ \dfrac{e^6\sin(t)}{(e^{2t}+e^6)t^{n+2}}, & (2k+1)\pi \le t \le (2k+2)\pi,\ k = 0,1,2,\dots \end{cases} \tag{5.7}$$

satisfies all the conditions of Theorem 4.3. Hence every solution of (5.6) oscillates or tends to zero as $t \to \infty$. As such, it admits a positive solution $y(t) = e^{-t}$ which tends to zero as $t \to \infty$.

Remark 5.3. The results of this article seems to be significant as no result in literature can be applied to the NDDEs (5.1) and (5.6).

6. Concluding remarks

The open problem [2, Problem 2.8.3, p.57] says:

Extend the following result to equations with oscillating coefficients.
Theorem 2.3.1 in [2]: Under the assumptions that $q(t) \geq 0$ and

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s)ds > e^{-1} \tag{6.1}$$

every solution of

$$y'(t) + q(t)y(t-\tau) = 0\,, \qquad t \geq t_0 \tag{6.2}$$

oscillates.

If we put $n = 1$, $f(t) = 0$, $G(y) = y$, $k = 1$ and $p_j(t)$ satisfying (2.8) $\equiv 0$ for each j in (1.3) then the following corollary follows from Theorem 4.1.

Corollary 6.1. *Suppose that* (B2) *hold . Assume*

$$\int_{t_0}^{\infty} q^{-}(t)\,dt < \infty\,. \tag{6.3}$$

Then every bounded solution of

$$y'(t) + q(t)y(g(t)) = 0\,. \tag{6.4}$$

oscillates or tends to zero as $t \to \infty$.

Again if we put $n = 1$, $f(t) = 0$, $k = 0$ in (1.3) then the following corollary follows from Theorem 4.3.

Corollary 6.2. *Suppose that* (H0), (H1), (B2), (B6) *and* (6.3) *hold. Then every solution of*

$$y'(t) + q(t)G(y(g(t))) = 0 \tag{6.5}$$

oscillates or tends to zero as $t \to \infty$. Or equivalently every unbounded solution of (6.5) *oscillates.*

Note that (6.1) implies

$$\int_{t_0}^{\infty} q(t)\,dt = \infty\,. \tag{6.6}$$

Further, (B2) is equivalent to (6.6) under the assumption (6.3). Thus, Corollaries 6.1, 6.2 answer the open problem [2, Problem 2.8.3, p.57] partially. Further, Theorem 3.6 answers the open problem [2, Problem 10.10.2, p.287]. Furthermore, as the condition "G is non decreasing"is not assumed, and $q(t)$ has no fixed sign, in our results, therefore, due to Remark 2.6, Theorems 4.3 and 4.2 of this article improve and generalize the [10, Theorem 2.2], and Theorem 3.6 improves and generalizes [10, Theorem 2.6], and Theorem 4.1 improves and generalizes [9, Theorem 3.5]. Last but not the least Lemmas 2.5 and 2.9 are two very important results of this paper.

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