ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 199–211

POSITIVE SOLUTIONS FOR A CLASS OF NON-AUTONOMOUS SECOND ORDER DIFFERENCE EQUATIONS VIA A NEW FUNCTIONAL FIXED POINT THEOREM

Lydia Bouchal[a], Karima Mebarki[a], and Svetlin Georgiev Georgiev[b]

Abstract. In this paper, by using recent results on fixed point index, we develop a new fixed point theorem of functional type for the sum of two operators $T+S$ where $I-T$ is Lipschitz invertible and S a k-set contraction. This fixed point theorem is then used to establish a new result on the existence of positive solutions to a non-autonomous second order difference equation.

1. Introduction

Cone fixed point theorems, particularly those of functional type, have provided several criteria for the existence and multiplicity of positive solutions for continuous and discrete boundary value problems. See [5, 16, 18, 21, 25, 26] for works on ordinary differential equations, and [2, 17, 20, 23, 24] for works on difference equations. The flexibility of using functionals instead of norms allows the theorems to be used in a wider variety of situations. The beginning of functional type fixed point theorems goes back to the original Leggett and Williams fixed point theorem [19] where the norm used in the lower boundary condition of Guo-Krasnosel'skii fixed point theorem [13, 15] was replaced by a positive concave functional. Many kinds of generalizations and variants of Leggett-Williams fixed point theorem have been obtained in different directions, such as the several Avery et al. fixed point theorems [4, 6, 7, 8, 9].

In [3, Theorem 10], Avery and Anderson generalized the Guo-Krasnosel'skii fixed point theorem. A generalization that allows the user to choose two functionals that satisfy certain conditions that are used instead of the norm. These functionals do not need to be concave or convex, which leaves more freedom, especially in applications to boundary value problems (BVPs for short). This is one of the reasons that motivated us to extend Avery-Anderson's theorem to the sum of two operators.

This paper is part of the literature devoted to applications of fixed point theorems of functional type for boundary value problems for finite difference equations. Firstly,

2020 *Mathematics Subject Classification*: primary 47H10; secondary 39A27.

Key words and phrases: fixed point, sum of operators, non-autonomous difference equations, positive solution.

Received June 14, 2022, revised September 2022. Editor G. Teschl.

DOI: 10.5817/AM2022-4-199

we use the fixed point index theory developed in [11, 12] to extend [3, Theorem 10] for operators that are sums of the form $T+S$, where $(I-T)$ is a Lipschitz invertible mapping with constant $\gamma>0$ and S is a k-set contraction with $k\,\gamma<1$. Then, this extension is applied to a class of BVPs for finite difference equations. We will present a technique that takes advantage of the flexibility of our fixed point theorem to obtain at least one positive solution to the following non-autonomous second order difference equation:

$$\triangle^2 u(k)+f(k,u(k))=0\,,\quad k\in\{0,1,\dots,N\},\ N\in\mathbb{N}, N>1\,,$$

with boundary conditions

$$u(0)=u(N+2)=0\,,$$

where $f\colon\{0,\dots,N+2\}\times[0,\infty)\to[0,\infty)$ is a continuous function and $\triangle^2$ is the second forward difference operator which acts on u by $\triangle^2 u(k)=u(k+2)-2u(k+1)+u(k)$, $k\in\{0,1,\dots,N\}$. By positive solution, we mean a function $u\colon\{0,\dots,N+2\}\to\mathbb{R}$ such that $u(k)\geq 0$ on $\{0,1,\dots,N+2\}$ and satisfies the posed BVP.
In [22], a compression-expansion fixed point theorem of functional type has been used to obtain at least one positive solution for the autonomous second order difference equation:

$$\triangle^2 u(k)+f(u(k))=0\,,\quad k\in\{0,1,\dots,N\},\ N\in\mathbb{N}, N>1\,, \tag{1.1}$$

with boundary conditions

$$u(0)=u(N+2)=0\,, \tag{1.2}$$

where $f\colon[0,\infty)\to[0,\infty)$ is a continuous function satisfying some conditions of monotonic type. In [23], the layered compression-expansion fixed point theorem was applied to show the existence of solutions to the problem (1.1)–(1.2), where the nonlinearity f is the sum of a monotonic increasing and a monotonic decreasing functions.

The paper is organized as follows: In Section 2, we give some auxiliary results. In Section 3, a new fixed point theorem of functional type for the sum $T+S$ in a cone is established, where $I-T$ is Lipschitz invertible and S a k-set contraction. In Section 4, we investigate the existence of at least one positive solution for a class of non-autonomous second order difference equations. We also give an example to illustrate our main results. The article ends with a conclusion.

2. Preliminary

Definition 2.1. A closed, convex set $\mathcal{P}$ in a Banach space E is said to be cone if

(1) $\lambda x\in\mathcal{P}$ for any $\lambda\geq 0$ and for any $x\in\mathcal{P}$,

(2) $x\in\mathcal{P}$, $-x\in\mathcal{P}$ implies $x=0$.

Let E be a real Banach space.

Definition 2.2. A mapping $K\colon E\to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 2.3 ([1])**.** *Let C be a closed subset of the class of continuous maps $u : \{0, \dots, N+1\} \to E$. If C is uniformly bounded and the set $\{u(k) :\ u \in C\}$ is relatively compact for each $k \in \{0, \dots, N+1\}$, then C is compact.*

The concept of set contraction is related to Kuratowski's measure of noncompactness χ. For the main properties of measure of noncompactness we refer the reader to [10, 14].

Definition 2.4. A mapping $A\colon E \to E$ is said to be k-set contraction if it is continuous, bounded and there exists a constant $k \geq 0$ such that

$$\chi(A(D)) \leq k\chi(D)\,,$$

for any bounded set $D \subset E$. The mapping A is said to be a strict set contraction if $k < 1$.

Obviously, if $A\colon E \to E$ is a completely continuous mapping, then A is 0-set contraction.

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|\cdot\|)$, Ω is a subset of $\mathcal{P}$ and U a bounded open subset of $\mathcal{P}$ and we will denote $\mathcal{P} \setminus \{0\}$ by $\mathcal{P}^*$.

The fixed point index $i_*(T+S, U\cap\Omega, \mathcal{P})$ defined by

$$i_*\left(T+S, U\cap\Omega, \mathcal{P}\right) = \begin{cases} i\left((I-T)^{-1}S, U, \mathcal{P}\right), & \text{if } U\cap\Omega \neq \emptyset \\ 0, & \text{if } U\cap\Omega = \emptyset\,, \end{cases} \tag{2.1}$$

is well defined whenever the mapping $T\colon \Omega \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $S\colon \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I-T)(\Omega)$. For details see [11, 12].

Proposition 2.5 ([12, Proposition 3.4])**.** *Let U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that $T\colon \Omega \subset \mathcal{P} \to E$ be a mapping such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $S : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I-T)(\Omega)$. If*

$$Sx \neq (I-T)(\lambda x)\,,\ \textit{for all } x \in \partial U,\ \lambda \geq 1 \textit{ and } \lambda x \in \Omega\,,$$

then

$$i_*(T+S, U\cap\Omega, \mathcal{P}) = 1\,.$$

Proposition 2.6 ([12, Proposition 3.9])**.** *Let U be a bounded open subset of $\mathcal{P}$. Assume that $T\colon \Omega \subset \mathcal{P} \to E$ be a mapping such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $S : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $S(\overline{U}) \subset (I-T)(\Omega)$. If there exists $u_0 \in \mathcal{P}^*$ such that*

$$Sx \neq (I-T)(x-\lambda u_0), \quad \textit{for all} \quad \lambda \geq 0 \quad \textit{and} \quad x \in \partial U \cap (\Omega + \lambda u_0)\,,$$

then

$$i_*(T+S, U\cap\Omega, \mathcal{P}) = 0\,.$$

Definition 2.7. A map α is said to be a nonnegative continuous functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha\colon \mathcal{P} \to [0, \infty)$ is continuous.

Let α and β be nonnegative continuous functionals on $\mathcal{P}$; and let r, R be two positive real numbers, we define the sets:

$$\mathcal{P}(\beta, R) = \{x \in \mathcal{P} :\ \beta(x) < R\}\,,$$

$$\mathcal{P}(\beta, \alpha, r, R) = \{x \in \mathcal{P} :\ r < \alpha(x) \text{ and } \beta(x) < R\}\,. \tag{2.2}$$

3. New fixed point theorem for sums of two operators

In the sequel, we will establish an extension of [3, Theorem 10] which guarantees the existence of at least one nontrivial positive solution to some equations of the form $Tx + Sx = x$ posed on cones of a Banach space.

Theorem 3.1. *Let E be a Banach space; $\mathcal{P} \subset E$ a cone; α and β be nonnegative continuous functionals on $\mathcal{P}$ and let $r < R$ be two positive real numbers. Let $T\colon \Omega \subset \mathcal{P} \to E$ be a mapping such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$ and $S\colon \overline{\mathcal{P}(\beta, R)} \to E$ be a k-set contraction mapping with $0 \le k < \gamma^{-1}$. Assume that $\mathcal{P}(\beta, \alpha, r, R) \cap \Omega \ne \emptyset$, $\overline{\mathcal{P}(\alpha, r)} \subset \mathcal{P}(\beta, R)$ and*

$$S(\overline{\mathcal{P}(\beta, R)}) \subset (I - T)(\Omega)\,. \tag{3.1}$$

If one of the two following conditions is satisfied

$(\mathcal{A}_1)$**:** *for all $x \in \partial\mathcal{P}(\alpha, r)$ and $\lambda > 1$ with $\lambda x \in \Omega$ and $T(\lambda x) + Sx \in \mathcal{P}$,*

$$\alpha(T(\lambda x) + Sx) \le r\,,\quad \lambda\alpha(x) \le \alpha(\lambda x) \quad \textit{and} \quad \alpha(0) < r\,, \tag{3.2}$$

and there exists $u_0 \in \mathcal{P}^*$, for all $\eta > 0$ and $x \in \partial\mathcal{P}(\beta, R) \cap (\Omega + \eta u_0)$ with $T(x - \eta u_0) + Sx + \eta u_0 \in \mathcal{P}$,

$$\beta(T(x - \eta u_0) + Sx + \eta u_0) \ne R\,, \tag{3.3}$$

or

$(\mathcal{A}_2)$**:** *for all $x \in \partial\mathcal{P}(\beta, R)$ and $\lambda > 1$ with $\lambda x \in \Omega$ and $T(\lambda x) + Sx \in \mathcal{P}$,*

$$\beta(T(\lambda x) + Sx) \le R\,,\quad \lambda\beta(x) \le \beta(\lambda x) \quad \textit{and} \quad \beta(0) < R\,, \tag{3.4}$$

and there exists $u_0 \in \mathcal{P}^*$, for all $\eta > 0$ and $x \in \partial\mathcal{P}(\alpha, r) \cap (\Omega + \eta u_0)$ with $T(x - \eta u_0) + Sx + \eta u_0 \in \mathcal{P}$,

$$\alpha(T(x - \eta u_0) + Sx + \eta u_0) \ne r\,, \tag{3.5}$$

then $T + S$ has at least one nontrivial fixed point $x^* \in \overline{\mathcal{P}(\beta, \alpha, r, R)} \cap \Omega$.

Proof. Suppose that $Tx + Sx \ne x$ for all $x \in \partial\mathcal{P}(\beta, \alpha, r, R)$, otherwise we are finished.

We will suppose that the condition $(\mathcal{A}_1)$ holds; the proof when $(\mathcal{A}_2)$ is satisfied is similar.

Claim 1: $Sx \ne (I - T)(\lambda x)$ for all $x \in \partial\mathcal{P}(\alpha, r)$, $\lambda > 1$ and $\lambda x \in \Omega$.

On the contrary, suppose that there exists a $x_0 \in \partial\mathcal{P}(\alpha, r)$, $\lambda_0 > 1$ and $\lambda_0 x_0 \in \Omega$ such that

$$T(\lambda_0 x_0) + Sx_0 = \lambda_0 x_0\,.$$

Then,

$$r \geq \alpha\left(T(\lambda_0 x_0) + Sx_0\right) = \alpha\left(\lambda_0 x_0\right) \geq \lambda_0 \alpha(x_0) > \alpha(x_0) = r\,,$$

which is a contradiction with (3.2).
Note that $0 \in \mathcal{P}(\alpha, r)$ by assumption. Hence, from Proposition 2.5,

$$i_*(T+S, \mathcal{P}(\alpha, r) \cap \Omega, \mathcal{P}) = 1\,.$$

Claim 2: $Sx \neq (I-T)(x - \eta u_0)$ for all $\eta > 0$ and $x \in \partial\mathcal{P}(\beta, R) \cap (\Omega + \eta u_0)$, for some $u_0 \in \mathcal{P}^*$. On the contrary, for any $u_0 \in \mathcal{P}^*$ there exist $\eta_0 > 0$ and $z_0 \in \partial\mathcal{P}(\beta, R) \cap (\Omega + \eta u_0)$ such that

$$Sz_0 = (I-T)(z_0 - \eta_0 u_0)\,.$$

So,

$$T(z_0 - \eta_0 u_0) + Sz_0 + \eta_0 u_0 = z_0\,.$$

Then,

$$\beta(T(z_0 - \eta_0 u_0) + Sz_0 + \eta_0 u_0) = \beta(z_0) = R\,,$$

which is a contradiction with (3.3).
As a result of Proposition 2.6, we arrive at

$$i_*(T+S, \mathcal{P}(\beta, R) \cap \Omega, \mathcal{P}) = 0\,.$$

Thus, from the additivity property of the fixed point index i_*, we have

$$\begin{aligned} i_*(T+S, \mathcal{P}(\beta, \alpha, r, R) \cap \Omega, \mathcal{P}) &= i_*(T+S, \mathcal{P}(\beta, R) \cap \Omega, \mathcal{P}) \\ &\quad - i_*(T+S, \mathcal{P}(\alpha, r) \cap \Omega, \mathcal{P}) \\ &= -1\,. \end{aligned}$$

By the existence property of the fixed point index the operator $T+S$ has at least one fixed point $x^* \in \mathcal{P}(\beta, \alpha, r, R) \cap \Omega$. Hence the desired result.

□

4. Applications

In this section, we will investigate the equation

$$\triangle^2 u(k) + f(k, u(k)) = 0\,, \quad k \in \{0, 1, \ldots, N\},\ N \in \mathbb{N}, N > 1 \tag{4.1}$$

with boundary conditions

$$u(0) = u(N+2) = 0\,, \tag{4.2}$$

where $f\colon \{0, \ldots, N+2\} \times [0, \infty) \to [0, \infty)$ is a continuous function satisfying:

$$(\mathcal{H}_1): \begin{cases} 0 \leq f(k, u(k)) \leq a(k) + b(k)|u(k)|^p,\ p \geq 0,\ a, b : \{0, \ldots, N+2\} \to [0, \infty) \\ \text{ be such that} \\ 0 \leq a(k),\ b(k) \leq B,\ k \in \{0, \ldots, N+2\} \text{ for some positive constant } B\,. \end{cases}$$

In what follows, by using our approach, we will establish sufficient criteria for the existence of positive solutions to BVP (4.1)–(4.2). Define the Banach space

$$E = \{u : \{0, \ldots, N+2\} \to \mathbb{R}\}$$

with the norm

$$\|u\| = \max_{k\in\{0,\ldots,N+2\}} |u(k)|\,.$$

Define the function

$$H(k,l) = \frac{1}{N+2}\begin{cases} k(N+2-l)\,, & k\in\{0,\ldots,l\}\,,\\ l(N+2-k)\,, & k\in\{l+1,\ldots,N+2\}\,,\end{cases}$$

for any $l\in\{0,\ldots,N+2\}$.

In [22] it is proved that if $u\in E$ is a solution to the BVP (4.1)–(4.2), then it is a solution to the sum equation

$$u(k) = \sum_{l=1}^{N+1} H(k,l) f(l,u(l))\,, \quad k\in\{0,\ldots,N+2\}\,,$$

and conversely. We have that

$$H(k,l)\le N+2\,, \quad k,l\in\{0,\ldots,N+2\}\,.$$

Let

$$S_1u(k) = \sum_{l=1}^{N+1} H(k,l) f(l,u(l))\,, \quad k\in\{0,\ldots,N+2\}\,.$$

Lemma 4.1. *Suppose that $(\mathcal{H}_1)$ holds. Let $u\in E$ and $\|u\|\le Q$ for some positive constant Q. Then*

$$S_1u(k)\le (N+2)(N+1)B(1+Q^p)\,, \quad k\in\{0,\ldots,N+2\}\,.$$

Proof. We have

$$\begin{aligned} S_1u(k) &= \sum_{l=1}^{N+1} H(k,l) f(l,u(l)) \\ &\le (N+2)\sum_{l=1}^{N+1}(a(l)+b(l)|u(l)|^p) \\ &\le (N+2)(N+1)B(1+Q^p), \quad k\in\{0,\ldots,N+2\}\,. \end{aligned}$$

This completes the proof. □

Suppose

$(\mathcal{H}_2)$: ϵ, A_1, B, B_1, R, R_1, r are positive constants such that

$$\begin{aligned} &\epsilon\in(0,1)\,, \quad \frac{B_1}{2} > A_1(N+3)\big((N+2)(N+1)B\,(1+R^p)+R\big)\,, \\ &\qquad\qquad \frac{r}{A_1} < R\,, \quad R_1>\max\{R,1\}, \quad A_1\in(0,1)\,, \\ &A_1(\epsilon+r+2B_1)\le r\,. \end{aligned}$$

For $u \in E$, define the operator

$$S_2u(k) = A_1 \sum_{m=0}^{k-1} (S_1u(m) - u(m)) \,, \quad k \in \{1, \dots, N+2\} \,.$$

Lemma 4.2. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Let $u \in E$ and*

$$S_2u(k) = C \,, \quad k \in \{1, \dots, N+2\} \,, \tag{4.3}$$

where C is a constant. Then u is a solution to the BVP (4.1)–(4.2).

Proof. We have

$$\sum_{m=0}^{k-1} \Big(\sum_{l=1}^{N+1} H(m,l) f(l,u(l)) - u(m) \Big) - \frac{C}{A_1} = 0 \,, \quad k \in \{1, \dots, N+2\} \,.$$

We take the Δ-operator of both sides of the last equation and we find

$$\begin{aligned} &\sum_{m=0}^{k} \Big(\sum_{l=1}^{N+1} H(m,l) f(l,u(l)) - u(m) \Big) - \sum_{m=0}^{k-1} \Big(\sum_{l=1}^{N+1} H(m,l) f(l,u(l)) - u(m) \Big) \\ &\quad = \sum_{l=1}^{N+1} H(k,l) f(l,u(l)) - u(k) = 0 \,, \end{aligned}$$

$k \in \{1, \dots, N+2\}$. This completes the proof. □

Lemma 4.3. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Let $u \in E$ and $\|u\| \le Q$ for some positive constant Q. Then*

$$\|S_2u\| \le A_1(N+3)\left((N+2)(N+1)B(1+Q^p) + Q\right) \,.$$

Proof. We have

$$\begin{aligned} \|S_2u\| &\le A_1 \sum_{m=0}^{k-1} (\|S_1u\| + \|u\|) \\ &\le A_1 \sum_{m=0}^{N+2} (\|S_1u\| + \|u\|) \\ &\le A_1(N+3)\left((N+2)(N+1)B(1+Q^p) + Q\right) \,. \end{aligned}$$

This completes the proof. □

The main result in this section is as follows.

Theorem 4.4. *Suppose that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Then the BVP* (4.1)–(4.2) *has at least one positive solution $u^* \in E$ such that $\frac{r}{A_1} \le \max_{k \in \{0, \dots, N+2\}} u^*(k) \le R$.*

Remark 4.5. Note that our conditions for existence of a positive solution to the BVP (4.1)–(4.2) is only a polynomial growth of f and then our main result can be considered as a complementary result of Theorem 4.1 in [22] in the autonomous case.

Proof. Let

$$\mathcal{P} = \{u \in E : u \geq 0\},$$

$$\Omega = \mathcal{P}\,.$$

For $u \in \mathcal{P}$, define the functionals

$$\alpha(u) = A_1 \max_{k\in\{0,\ldots,N+2\}} u(k)\,,$$

$$\beta(u) = \max_{k\in\{0,\ldots,N+2\}} u(k)\,,$$

and for $u \in E$, define the operators.

$$Tu(k) = -\epsilon \frac{u(k)}{R_1 + u(k)}\,,$$

$$S_3u(k) = \epsilon \frac{u(k)}{R_1 + u(k)} + u(k) + S_2u(k)\,,$$

$$Su(k) = S_3u(k) + B_1, \quad k \in \{1,\ldots,N+2\}\,.$$

Note that if $u \in \mathcal{P}$ is a fixed point of the operator $T + S$, then $Tu + Su = u$, whereupon $S_2u(k) = -B_1$, $k \in \{0,\ldots,N+2\}$, and then it is a positive solution to the BVP (4.1)-(4.2).

(1) Define the function

$$g(x) = \frac{x}{R_1 + x}\,, \quad x \geq 0\,.$$

Then

$$g'(x) = \frac{R_1}{(R_1 + x)^2}\,, \quad x \geq 0\,,$$

and

$$|g'(x)| \leq 1\,, \quad x \geq 0\,.$$

Hence,

$$|g(x) - g(y)| \leq |x - y|\,, \quad x, y \geq 0\,,$$

and

$$\left\| \frac{u}{R_1 + u} - \frac{v}{R_1 + v} \right\| \leq \|u - v\|\,, \quad u, v \in \mathcal{P}\,.$$

Therefore, for $u, v \in \mathcal{P}$, we have

$$\begin{aligned}\|(I-T)u-(I-T)v\| &\geq \|u-v\| - \epsilon\left\|\frac{u}{R_1+u} - \frac{v}{R_1+v}\right\| \\ &\geq (1-\epsilon)\|u-v\|\,.\end{aligned}$$

Thus, $I-T\colon \mathcal{P} \to E$ is Lipschitz invertible with a constant $\gamma = (1-\epsilon)^{-1}$.

(2) Let $u \in \overline{\mathcal{P}(\beta, R)}$. Then $\|u\| \leq R$ and by Lemma 4.3, it follows

$$\|S_2 u\| \leq A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right)\,,$$

and

$$\epsilon\frac{u(k)}{R_1+u(k)} \leq \epsilon\,, \quad k \in \{0, \dots, N+2\}\,.$$

Consequently

$$\|Su\| \leq \epsilon + R + A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right) + B_1\,,$$

Therefore, $S\colon \overline{\mathcal{P}(\beta, R)} \to E$ is a completely continuous operator. Thus, S is a 0-set contraction.

(3) Because $\frac{r}{A_1} < R$, we have that $\mathcal{P}(\beta,\alpha,r,R)\bigcap\Omega \neq \emptyset$ and $\overline{\mathcal{P}(\alpha,r)} \subset \mathcal{P}(\beta,R)$.

(4) Let $u \in \overline{\mathcal{P}(\beta, R)}$ be arbitrarily chosen. By Lemma 4.3 and $(\mathcal{H}_2)$, we find

$$\begin{aligned}\|S_2 u\| &\leq A_1(N+3)\left((N+2)(N+1)B(1+R^p)+R\right) \\ &< \frac{B_1}{2}\,.\end{aligned}$$

Therefore

$$S_2u(k) + \frac{B_1}{2} > 0\,, \quad k \in \{1, \dots, N+2\}\,.$$

Now, using that $u(k) \geq 0$, $k \in \{1, \dots, N+2\}$, we obtain

$$\begin{aligned}S_3u(k) + \frac{B_1}{2} &= \epsilon\frac{u(k)}{R_1+u(k)} + u(k) + S_2u(k) + \frac{B_1}{2} \\ &\geq S_2u(k) + \frac{B_1}{2} \\ &> 0\,, \quad k \in \{1, \dots, N+2\}\,.\end{aligned}$$

Hence,

$$\begin{aligned}Su(k) &= S_3u(k) + \frac{B_1}{2} + \frac{B_1}{2} \\ &> \frac{B_1}{2}\,, \quad k \in \{1, \dots, N+2\}\,.\end{aligned}$$

Next,

$$\begin{aligned} Su(k) &= \epsilon \frac{u(k)}{R_1 + u(k)} + S_2 u(k) + u(k) + B_1 \\ &\le \epsilon + R + B_1 + B_1 \\ &= \epsilon + R + 2B_1\,, \quad k \in \{1, \ldots, N+2\}\,. \end{aligned}$$

Take

$$v = \frac{-(R_1 + \epsilon - Su) + \sqrt{(R_1 + \epsilon - Su)^2 + 4R_1 Su}}{2}\,.$$

We have $v \ge 0$ and therefore $v \in \Omega$. Also,

$$0 = v^2 + (R_1 + \epsilon - Su)v - R_1 Su\,,$$

whereupon

$$v^2 + R_1 v + \epsilon v = Su\, v + R_1 Su$$

and

$$v(R_1 + v) + \epsilon v = Su(v + R_1)\,.$$

Thus,

$$\begin{aligned} Su &= v + \frac{\epsilon v}{R_1 + v} \\ &= (I - T)v\,. \end{aligned}$$

Therefore

$$S\left(\overline{\mathcal{P}(\beta, R)}\right) \subset (I - T)(\Omega)\,.$$

(5) Let $x \in \partial\mathcal{P}(\alpha, r)$ and $\lambda > 1$. Then

$$\begin{aligned} \alpha(T(\lambda x) + Sx) &= A_1 \max_{k \in \{0, \ldots, N+2\}} \left(-\frac{\epsilon \lambda x(k)}{R_1 + \lambda x(k)} + Sx(k) \right) \\ &\le A_1 \max_{k \in \{0, \ldots, N+2\}} Sx(k) \\ &\le A_1(\epsilon + r + 2B_1) \\ &\le r\,. \end{aligned}$$

(6) For any $x \in \partial\mathcal{P}(\alpha, r)$, $\lambda > 1$, we have

$$\begin{aligned} \alpha(\lambda x) &= A_1 \max_{k \in \{0, \ldots, N+2\}} (\lambda x(k)) \\ &= A_1 \lambda \max_{k \in \{0, \ldots, N+2\}} x(k) \\ &= \lambda\, \alpha(x) \end{aligned}$$

and

$$\alpha(0) < r\,.$$

(7) Let $\eta > 0$ and $u_0 \in \mathcal{P}^\star$ be arbitrarily chosen. Take
$$x \in \partial\mathcal{P}(\beta, R) \bigcap (\Omega + \eta u_0)\,.$$
Then $x(k) \leq R$, $k \in \{0, \dots, N+2\}$, and $x - \eta u_0 \in \Omega$ or
$$x(k) - \eta u_0(k) \geq 0\,, \quad k \in \{0, \dots, N+2\}\,.$$
Because
$$\frac{\epsilon(x(k) - \eta u_0(k))}{R_1 + x(k) - \eta u_0(k)} \leq \frac{\epsilon x(k)}{R_1 + x(k)}\,, \quad k \in \{0, \dots, N+2\}\,,$$
we get
$$\begin{aligned}\beta\left(\eta u_0 + T(x - \eta u_0) + Sx\right) &= \beta\left(\eta u_0 - \frac{\epsilon(x - \eta u_0)}{R_1 + x - \eta u_0} + \frac{\epsilon x}{R_1 + x} + x + S_2 x + B_1\right)\\ &\geq \beta\left(x + \frac{B_1}{2}\right)\\ &> \beta(x)\\ &= R\end{aligned}$$
and hence,
$$\beta\left(\eta u_0 + T(x - \eta u_0) + Sx\right) \neq R\,.$$

All conditions of $(\mathcal{A}_1)$ of Theorem 3.1 are then satisfied. Thus, we conclude that the BVP (4.1)–(4.2) has at least one solution $u^* \in \mathcal{P}$ such that $\frac{r}{A_1} \leq \|u^*\| \leq R$. This completes the proof. □

Example 4.6. Let
$$\epsilon = B = A_1 = \frac{1}{10^{500}}\,, \quad R = 1, \quad B_1 = \frac{2}{10^{400}}\,, \quad r = \frac{1}{10^{600}}\,,$$
$$N = 5\,, \quad p = 2\,, \quad R_1 = 100\,.$$
Then
$$\begin{aligned}A_1(N+3)\left((N+2)(N+1)B\left(1 + R^p\right) + R\right) &= \frac{1}{10^{500}} \cdot 8 \cdot \left(7 \cdot 6 \cdot \frac{1}{10^{500}}\,(1+1) + 1\right)\\ &< \frac{1}{10^{400}} = \frac{B_1}{2}\end{aligned}$$
and
$$R_1 = 100 > r\,, \quad \frac{r}{A_1} = \frac{1}{10^{100}} < R\,,$$
and
$$A_1(\epsilon + r + 2B_1) = \frac{1}{10^{500}}\left(\frac{1}{10^{500}} + \frac{1}{10^{600}} + \frac{4}{10^{400}}\right) < \frac{1}{10^{600}} = r\,.$$
Thus, $(\mathcal{H}_2)$ holds. Now, by our main result, it follows that the BVP
$$\Delta^2 u(k) = \frac{k}{10^{1000}(1 + k + k^2)} + \frac{1}{10^{500}}(u(k))^2\,, \quad k \in \{0, \dots, 5\}\,,$$
$$u(0) = u(7) = 0$$

has at least one positive solution.

5. Conclusion

(a) In this paper we have developed a new functional fixed point theorem on cones for the sum of two operators. The arguments are based upon recent fixed point index theory in cones of Banach spaces.

(b) By using our approach, sufficient conditions for the existence of at least one positive solution are established for a non-autonomous second order difference equation.

(c) The nonlinearity f considered in the BVP (4.1)–(4.2) is non-autonomous and satisfies a general growth condition, while in [22] the nonlinear term must be autonomous with some conditions of monotonic type. Moreover, one can easily give an example for the constants ϵ, A_1, B, B_1, R, R_1, r which satisfy the condition $(\mathcal{H}_2)$.

(d) The functionals α and β considered in this paper are more general than those in [22]. They are supposed to be only nonnegative and continuous, while in [22] the functionals α and β besides of being nonnegative and continuous were assumed concave and convex, respectively.

(e) For all the above reasons, our new topological approach developed in this article can be used to study other types of difference equations as well as dynamic equations.

Acknowledgement. The first and the second authors acknowledge support of "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria.

References

[1] Agarwal, R.P., O'Regan, D., Wong, P.J.Y., *Positive solutions of differential, difference and integral equations*, Springer Science and Business Media, 1998.

[2] Anderson, D.R., Avery, R.I., *Multiple positive solutions to a third-order discrete focal boundary value problem*, Comput. Math. Appl. **42** (3–5) (2001), 333–340.

[3] Anderson, D.R., Avery, R.I., *Fixed point theorem of cone expansion and compression of functional type*, J. Difference Equ. Appl. **8** (11) (2002), 1073–1083.

[4] Anderson, D.R., Avery, R.I., *A topological proof and extension of the Leggett-Williams fixed point theorem*, Commun. Appl. Nonlinear Anal. **16** (4) (2009), 39–44.

[5] Anderson, D.R., Avery, R.I., *Application of the omitted ray fixed point theorem*, Electron. J. Qual. Theory Differ. Equ. **2014** (17) (2014), 1–9.

[6] Anderson, D.R., Avery, R.I., Henderson, J., *Some Fixed point theorems of Leggett-Williams type*, Rocky Montain J. Math. **41** (2011), 371–386.

[7] Anderson, D.R., Avery, R.I., Henderson, J., *An extension of the compression-expansion fixed point theorem of functional type*, Electron. J. Differential Equations **2016** (253) (2016), 1–9.

[8] Anderson, D.R., Avery, R.I., Henderson, J., Liu, X., *Operator type compression-expansion fixed point theorem*, Electron. J. Differential Equations **2011** (2011), 1–11.

[9] Anderson, D.R., Henderson, J., Avery, R.I., *Functional compression-expansion fixed point theorem of Leggett-Williams type*, Electron. J. Differential Equations **2010** (2010), 1–9.

[10] Banas, J., Goebel, K., *Measures of noncompactness in Banach spaces*, Lect. Notes Pure Appl. Math., Marcel Dekker, Inc., New York, 1980.

[11] Djebali, S., Mebarki, K., *Fixed point index theory for perturbation of expansive mappings by k-set contractions*, Topol. Methods Nonlinear Anal. **54** (2A) (2019), 613–640.

[12] Georgiev, S.G., Mebarki, K., *On fixed point index theory for the sum of operators and applications to a class of ODEs and PDEs*, Appl. Gen. Topol. **22** (2) (2021), 259–294.

[13] Guo, D., *A new fixed point theorem*, Acta Math. Sinica **24** (3) (1981), 444–450.

[14] Guo, D., Cho, Y.J., Zhu, J., *Partial ordering methods in nonlinear problems*, Shangdon Science and Technology Publishing Press, Shangdon, 1985.

[15] Guo, D., Lakshmikantham, V., *Nonlinear problems in abstract cones*, vol. 5, Academic Press, Boston, Mass., USA, 1988.

[16] He, X., Ge, W., *Triple solutions for second-order three-point boundary value problems*, J. Math. Anal. Appl. **268** (1) (2002), 256–265.

[17] Henderson, J., Liu, X., Lyons, J.W., al. et, *Right focal boundary value problems for difference equations*, Opuscula Math. **30** (4) (2010), 447–456.

[18] Henderson, J., Thompson, H., *Multiple symmetric positive solutions for a second order boundary value problem*, Proc. Amer. Math. Soc. **128** (8) (2000), 2373–2379.

[19] Leggett, R.W., Williams, L.R., *Multiple positive fixed points of nonlinear operators on ordered Banach space*, Indiana Univ. Math. J. **28** (4) (1979), 673–688.

[20] Lyons, J.W., Neugebauer, J.T., *A difference equation with anti-periodic boundary conditions*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **22** (1) (2015), 47–60.

[21] Mohamed, A., *Existence of positive solutions for a fourth-order three-point BVP with sign-changing green's function*, Appl. Math. **12** (4) (2021), 311–321.

[22] Neugebauer, J., Seelbach, C., *A difference equation with Dirichlet boundary conditions*, Commun. Appl. Anal. **21** (2) (2017), 237–248.

[23] Neugebauer, J.T., *The role of symmetry and concavity in the existence of solutions of a difference equation with Dirichlet boundary conditions*, Int. J. Difference Equ. **15** (2) (2020), 483–491.

[24] Tian, Y., Ma, D., Ge, W., *Multiple positive solutions of four point boundary value problems for finite difference equations*, J. Difference Equ. Appl. **12** (1) (2006), 57–68.

[25] Yao, Q.L., *The existence and multiplicity of positive solutions for a third-order three-point boundary value problem*, Acta Math. Appl. Sin. **19** (1) (2003), 117–122.

[26] Zhang, H.E., Sun, J.P., *A generalization of the Leggett-Williams fixed point theorem and its application*, J. Appl. Math. Comput. **39** (1) (2012), 385–399.

[a]Laboratory of Applied Mathematics, Faculty of Exact Sciences,, University of Bejaia, 06000 Bejaia, Algeria
E-mail: `lydia.bouchal@univ-bejaia.dz`
`karima.mebarki@univ-bejaia.dz`

[b]Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia,
Sofia, Bulgaria
E-mail: `svetlingeorgiev1@gmail.com`