ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 287–293

ON TOPOLOGICALLY DISTINCT INFINITE FAMILIES OF EXACT LAGRANGIAN FILLINGS

Roman Golovko

Abstract. In this note we construct examples of closed connected Legendrian submanifolds in high dimensional contact vector space that admit an arbitrary finite number of topologically distinct infinite families of diffeomorphic, but not Hamiltonian isotopic exact Lagrangian fillings.

1. Introduction and main results

Exact Lagrangian cobordisms between Legendrian submanifolds naturally arise from the Symplectic Field Theory (SFT) of Eliashberg-Givental-Hofer [15].

Within the last few years the question of existence of an infinitely many exact Lagrangian fillings (i.e. exact Lagrangian cobordisms with empty negative ends) for some given Legendrian submanifolds has become one of the central questions in contact and symplectic topology. First it has been positively answered by Casals-Gao [5], and then later it has been investigated using different methods in the works of An-Bae-Lee [1, 2], Casals-Zaslow [7], Gao-Shen-Weng [16, 17], Casals-Ng [6], Capovilla-Searle [4] and the author [19].

From the result of Chantraine [8] it follows that if L is an exact Lagrangian filling of a closed connected Legendrian Λ in the standard contact vector space $\mathbb{R}^3$ (i.e., Legendrian knot), then

$$tb(\Lambda) = -\chi(L)\,,$$

and therefore the topology of an exact Lagrangian filling of a Legendrian knot Λ is completely determined by $tb(\Lambda)$. Hence, we see that it is impossible to have several topologically distinct infinite families of diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings for Legendrian knots.

We show that in high dimensions there are examples of closed connected Legendrian submanifolds with arbitrary many topologically distinct infinite families of exact Lagrangian cobordisms each of which consists of diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings. These examples are constructed using the polyfillable Legendrian submanifolds coming from the work of Cao-Gallup-Hayden-Sabloff [3], family of Legendrian links in $\mathbb{R}^3$ considered by

2020 *Mathematics Subject Classification*: primary 53D12; secondary 53D42.
Key words and phrases: polyfillability, Legendrian submanifold, exact Lagrangian filling.
Received April 26, 2022, revised July 2022. Editor M. Čadek.
DOI: 10.5817/AM2022-5-287

Casals-Ng in [6] each element of which admits an infinite family of diffeomorphic and Hamiltonian non-isotopic exact Lagrangian fillings, and the high dimensional extension of the family of Casals-Ng from [19].

The result of this paper says the following:

Theorem 1.1. *For a given $n \geq 3$ and $K > 0$, there is a connected Legendrian submanifold $\Lambda \subset (\mathbb{R}^{2n+1}, dz - \sum_i y_i dx_i)$ which admits a collection of exact Lagrangian fillings*

$$\{L_k^j \mid 1 \leq j \leq K(\Lambda), k \in \mathbb{Z}_{\geq 0}\},$$

where $K(\Lambda) \geq K$, such that

- *$L_{k_1}^{j_1}$ is not homeomorphic to $L_{k_2}^{j_2}$ for $j_1 \neq j_2$ and k_1, $k_2 \in \mathbb{Z}_{\geq 0}$;*
- *for a fixed j, $\{L_k^j\}_{k=1}^{\infty}$ consists of an infinite number of diffeomorphic exact Lagrangian fillings that are pairwise distinct up to Hamiltonian isotopy.*

2. Exact Lagrangian cobordisms

Definition 2.1. Given two closed Legendrian submanifolds Λ_- and Λ_+ of the standard contact vector space $(\mathbb{R}^{2n+1}, \alpha_{st} := dz - \sum_i y_i dx_i)$. An *exact Lagrangian cobordism from Λ_- to Λ_+* is a properly embedded submanifold in the symplectization $L \subset (\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st}))$ such that for some $T > 0$

(i) $L \cap (-\infty, -T) \times \mathbb{R}^{2n+1} = (-\infty, -T) \times \Lambda_-$,
(ii) $L \cap (T, +\infty) \times \mathbb{R}^{2n+1} = (T, +\infty) \times \Lambda_+$,
(iii) $L \cap [-T, T] \times \mathbb{R}^{2n+1}$ is compact,
(iv) there is a function $f_L \in C^\infty(L)$ such that
  - $e^t \alpha_{st}|_{TL} = df_L$,
  - $f_L|_{(-\infty,-T)\times\Lambda_-}$, $f_L|_{(T,+\infty)\times\Lambda_+}$ are constant functions.

We call $(T, +\infty) \times \Lambda_+$ and $(-\infty, -T) \times \Lambda_-$ the *positive end* and the *negative end* of L, respectively. In addition, in case when $\Lambda_- = \emptyset$, L is called an *exact Lagrangian filling*. If Λ_+ admits at least two Hamiltonian non-isotopic exact Lagrangian fillings, then we say that Λ_+ is *polyfillable*.

If $L_{\Lambda_1}^{\Lambda_2}$ is an exact Lagrangian cobordism from Λ_1 to Λ_2, and $L_{\Lambda_2}^{\Lambda_3}$ is an exact Lagrangian cobordism from Λ_2 to Λ_3, then we can take a concatenation of $L_{\Lambda_1}^{\Lambda_2}$ and $L_{\Lambda_2}^{\Lambda_3}$ along Λ_2 that is an exact Lagrangian cobordism from Λ_1 to Λ_3 and we denote it by $L_{\Lambda_1}^{\Lambda_3} = L_{\Lambda_2}^{\Lambda_3} \circ L_{\Lambda_1}^{\Lambda_2}$. For more details we refer the reader to [14].

From the result of Ekholm [12]: an exact Lagrangian cobordism L from Λ_- to Λ_+ leads to a unital differential algebra morphism between the Chekanov-Eliashberg algebras

$$\Phi \colon (\mathcal{A}(\Lambda_+), \partial_+) \to (\mathcal{A}(\Lambda_-), \partial_-)$$

defined by the appropriate count of rigid pseudoholomorphic discs with boundary on the cobordism. The result of Ekholm [12] is written for $\mathbb{Z}_2$-coefficients only, but this result admits a natural extension to more general coefficients such as the fully noncommutative DGA with "Novikov coefficients", see [9, Section 8.1] for more details.

Following the work of Ekholm-Honda-Kálmán [14] observe that an exact Lagrangian filling L of a closed Legendrian submanifold $\Lambda \subset \mathbb{R}^3$ induces an augmentation of the Chekanov-Eliashberg algebra $(\mathcal{A}(\Lambda), \mathbb{Z}_2[H_1(L)])$ onto $\mathbb{Z}_2[H_1(L)]$. Following the observation of Karlsson from [20, Section 2.2], note that even though the original results of Ekholm-Honda,-Kálmán from [14, Section 3.5] are formulated for $n = 1$, tracing their proofs one sees that they can be extended word-by-word to arbitrary $n > 0$. Finally, following the work of Karlsson [20], one can extend the augmentation to be onto $\mathbb{Z}[H_1(L)]$. We will use the following version of this statement.

Theorem 2.2 ([14, 20])**.** *Let L be a spin exact Lagrangian filling of a closed Legendrian in the standard contact vector space $\Lambda \subset \mathbb{R}^{2n+1}$ such that both L and Λ have vanishing Maslov numbers. Then L (together with some extra data) induces an augmentation*

$$\varepsilon_L \colon (\mathcal{A}(\Lambda), \mathbb{Z}[H_1(L)]) \to \mathbb{Z}[H_1(L)]\,,$$

where $\mathbb{Z}[H_1(L)]$ lies entirely in grading 0. In addition, if L and L' are exact Lagrangian fillings of Λ that are isotopic through exact Lagrangian fillings of Λ, then there is a DGA homotopy between corresponding augmentations ε_L and $\varepsilon_{L'}$.

3. Proof of Theorem 1.1

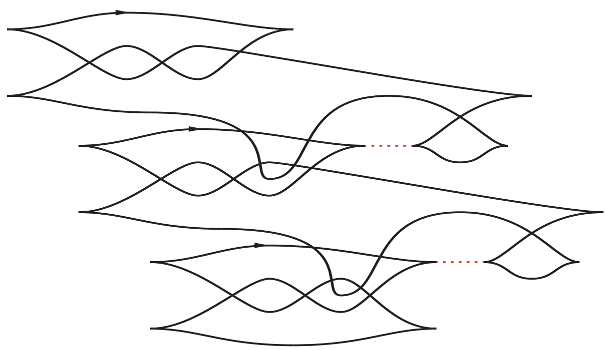

Fig. 1

First we slightly generalize the examples of high dimensional Legendrian submanifolds constructed by Cao-Gallup-Hayden-Sabloff from the proof of [3, Theorem 1.5, Proposition 4.4] with the property that they admit at least two non-homeomorphic exact Lagrangian fillings. Recall that such Legendrian submanifolds are obtained by first taking a polyfillable Legendrian link in the standard contact S^3 (or $\mathbb{R}^3$) from [3, Corollary 3.4] with K components, whose all components are Legendrian trefoils, make a small modification of it with a Reidemeister I move, see Figure 1, apply S^1-front spinning to the Legendrians and their exact Lagrangian fillings, see [13] and [19], and then attach $K-1$ copies of 1-handles, see Figure 2. This will lead to the polyfillable Legendrian surface of genus K. Then one applies the spherical spinning construction, see [18], and gets a version of polyfillable Legendrians diffeomorphic to $S^{k_1} \times \cdots \times S^{k_n} \times \Sigma_K$. We take Legendrian submanifolds obtained this way with Maslov number 0 and consider their non-homeomorphic exact Lagrangian fillings of Maslov number 0. The class of such n-dimensional Legendrian submanifolds will be denoted by $\mathcal{T}_n$.

Remark 3.1. Note that $\mathcal{T}_n$ is not empty, this follows from the construction of Cao-Gallup-Hayden-Sabloff [3], in particular in the proof of [3, Proposition 4.4.] Cao-Gallup-Hayden-Sabloff discuss Legendrians and their exact Lagrangian fillings obtained above that are gf-compatible, and hence have vanishing Maslov numbers.

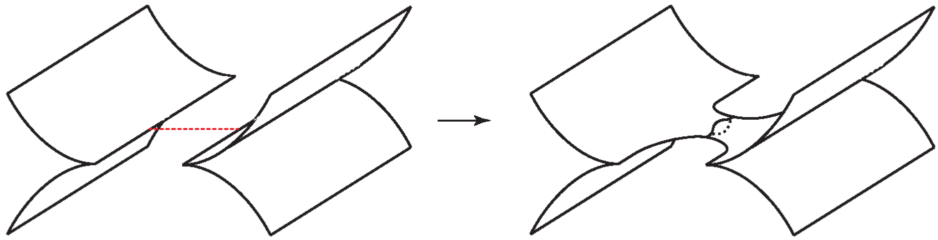

FIG. 2

Then we consider the class of Legendrian links $\mathcal{H}$ with an infinite number of diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings constructed by Casals-Ng in [6]. Note that in this class there are Legendrian knots. In particular, from [6, Corollary 1.3] it follows that the knot types 10_{139}, $m(10_{145})$, $m(10_{152})$, 10_{154}, and $m(10_{161})$ all have Legendrian representatives in the class discussed by Casals-Ng. We then take a Legendrian knot $\Lambda \in \mathcal{H}$, its multiple spherical spuns $\Sigma_{S^{l_1}} \dots \Sigma_{S^{l_m}} \Lambda$ and the multiple spherical spuns of the infinite family of exact Lagrangian fillings of $\Sigma_{S^{l_1}} \dots \Sigma_{S^{l_m}} \Lambda$, where $l_i \geq 2$ for all i. The class of such n-dimensional Legendrian submanifolds will be denoted by $\mathcal{I}_n$.

Remark 3.2. Legendrians in $\mathcal{I}_n$ and the corresponding fillings have Maslov numbers 0, this follows from the discussion in [6, 19].

Now we take $\Lambda_{\mathcal{T}} \in \mathcal{T}_n$ and $\Lambda_{\mathcal{I}} \in \mathcal{I}_n$. Let $\Lambda_{\mathcal{T}}$ be embedded by $i\colon \Lambda_{\mathcal{T}} \to \mathbb{R}^{2n+1}$, where i is an embedding of $\Lambda_{\mathcal{T}}$ such that $i(\Lambda_{\mathcal{T}}) \subset \{x_1 > \varepsilon\}$ and let $\Lambda_{\mathcal{I}}$ be embedded by $j\colon \Lambda_{\mathcal{I}} \to \mathbb{R}^{2n+1}$, where j is an embedding of $\Lambda_{\mathcal{I}}$ such that $j(\Lambda_{\mathcal{I}}) \subset \{x_1 < -\varepsilon\}$ for some $\varepsilon > 0$. From [3, Proposition 4.4] we know that for every $K > 0$ there is $\Lambda_{\mathcal{T}} \in \mathcal{T}_n$ which admits $K(\Lambda_{\mathcal{T}}) \geq K$ non-homeomorphic exact Lagrangian fillings $L_1^{\mathcal{T}}, \dots L_{K(\Lambda_{\mathcal{T}})}^{\mathcal{T}}$ with Maslov numbers 0.

From the discussion in [6] we know that $\Lambda_{\mathcal{I}}$ admits an infinite number of diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings that we denote by $\{L_m^{\mathcal{I}} \mid m \geq 0\}$.

Without loss of generality we assume that $L_1^{\mathcal{T}}, \dots, L_{K(\Lambda_{\mathcal{T}})}^{\mathcal{T}}$ are embedded in the symplectization by

$$i_l\colon L_l^{\mathcal{T}} \to \mathbb{R} \times \{(x_1, \dots, y_n, z) \mid x_1 > \varepsilon\} \subset \left(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st})\right),$$

and for every m, $L^{\mathcal{I}}_m$ is embedded in the symplectization by

$$j_m \colon L^{\mathcal{I}}_m \to \mathbb{R} \times \{(x_1, \dots, y_n, z) \mid x_1 < -\varepsilon\} \subset \left(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st})\right).$$

Remark 3.3. Note that from the construction of $L^{\mathcal{I}}_m$ from [6, 19] we know that $L^{\mathcal{I}}_m$ is given by the concatenation of one given exact Lagrangian filling L, concatenated m times exact Lagrangian concordance C and another fixed exact Lagrangian cobordism L' which does not depend on m, i.e. $L^{\mathcal{I}}_m = L' \circ C^m \circ L$. This, in particular implies that without loss of generality we can assume that

$$j_m(L^{\mathcal{I}}_m) \subset \mathbb{R} \times \{x_1 < -\varepsilon\} \subset \left(\mathbb{R} \times \mathbb{R}^{2n+1}, d(e^t \alpha_{st})\right)$$

for all $m \geq 0$.

Let Λ' be a Legendrian submanifold $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$ embedded by $i_{\Lambda'} \colon \Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}} \to \mathbb{R}^{2n+1}$ such that $i_{\Lambda'}|_{\Lambda_{\mathcal{T}}} = i|_{\Lambda_{\mathcal{T}}}$ and $i_{\Lambda'}|_{\Lambda_{\mathcal{I}}} = j|_{\Lambda_{\mathcal{I}}}$. Λ' admits an exact Lagrangian filling $L_{(l,m)} = L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m$ which is embedded by $i_{L_{(l,m)}} \colon L_{(l,m)} \to \mathbb{R} \times \mathbb{R}^{2n+1}$ with $i_{L_{(l,m)}}|_{L^{\mathcal{T}}_l} = i_l|_{L^{\mathcal{T}}_l}$, $i_{L_{(l,m)}}|_{L^{\mathcal{I}}_m} = j_m|_{L^{\mathcal{I}}_m}$. Since $\{L^{\mathcal{I}}_m \mid m \geq 0\}$ consists of diffeomorphic, but pairwise Hamiltonian non-isotopic exact Lagrangian fillings, $i_l(L^{\mathcal{T}}_l) \subset \mathbb{R} \times \{x_1 > \varepsilon\}$, $j_m(L^{\mathcal{I}}_m) \subset \mathbb{R} \times \{x_1 < -\varepsilon\}$, for a fixed l, $\{L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m \mid m \geq 0\}$ consists of diffeomorphic, but Hamiltonian non-isotopic exact Lagrangian fillings of $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$. In addition, since $L^{\mathcal{T}}_l$ is not homeomorphic to $L^{\mathcal{T}}_{l'}$ for $l \neq l'$ and $L^{\mathcal{I}}_m$ is diffeomorphic to $L^{\mathcal{I}}_{m'}$ for all m, m', $L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m$ is not homeomorphic to $L^{\mathcal{T}}_{l'} \sqcup L^{\mathcal{I}}_{m'}$ for $l \neq l'$.

We now perform an ambient 0-surgery defined by Dimitroglou Rizell in [10] to $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$. Legendrian ambient 0-surgery can be seen as a generalization of the cusp connected sum operation, and Dimitroglou Rizell in [10, Proposition 4.9] proved that it is well-defined. Legendrian 0-surgery leads to the exact Lagrangian cobordism from $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$ to the connected Legendrian Λ that we denote by $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$. Note that Λ admits a family of exact Lagrangian fillings that are obtained by concatenating $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$ with the exact Lagrangian fillings $L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m$ of $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$ along $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$, i.e. we have a family $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m)$ with $m \geq 0$, $1 \leq l \leq K(\Lambda_{\mathcal{T}})$.

Since $\Lambda_{\mathcal{T}}$ and $\Lambda_{\mathcal{I}}$ are separated by $\{x_1 = 0\}$ hyperplane and $L^{\mathcal{T}}_l$, $L^{\mathcal{I}}_m$ are separated by $\{x_1 = 0\}$ hyperplane, Maslov numbers of $\Lambda_{\mathcal{T}}$, $\Lambda_{\mathcal{I}}$, $L^{\mathcal{T}}_l$, $L^{\mathcal{I}}_m$ are zero, see Remarks 3.1, 3.2, from the construction of ambient 0-surgery and the corresponding cobordism $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$ [10] it follows that Maslov numbers of Λ, $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$, $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m)$ are zero, where $m \geq 0$, $1 \leq l \leq K(\Lambda_{\mathcal{T}})$.

Following the construction of Λ from [10] we see that the set of Reeb chords of Λ that we denote by $\mathcal{Q}(\Lambda)$ can be decomposed as $\mathcal{Q}(\Lambda) = \mathcal{Q}(\Lambda_{\mathcal{I}}) \sqcup \mathcal{Q}(\Lambda_{\mathcal{T}}) \sqcup \{c\}$, where c is a chord inside the attached handle and $|c| = n - 1$. From the proof of [10, Theorem 1.1, Corollary 1.2] there is a bijection between graded augmentations of Λ that we denote by $\mathrm{Aug}(\Lambda)$ and graded augmentations of $\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}$ that we denote by $\mathrm{Aug}(\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}})$ given by applying the pullback of a cobordism map $\Phi^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$ induced by $L^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}}$, i.e.

$$(\Phi^{\Lambda}_{\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}})^* \colon \mathrm{Aug}(\Lambda_{\mathcal{T}} \sqcup \Lambda_{\mathcal{I}}) \to \mathrm{Aug}(\Lambda)\,. \tag{3.1}$$

Note that since a DGA homotopy between augmentations of $\mathcal{A}(\Lambda)$ will vanish on c (by the grading reason), it will descend to DGA homotopies of the corresponding augmentations on $\mathcal{A}(\Lambda_{\mathcal{I}})$ and on $\mathcal{A}(\Lambda_{\mathcal{T}})$.

Let ε_m denote the augmentation induced by $L^{\mathcal{I}}_m$ and let $\varepsilon_{l,m}$ denote the augmentation induced by $L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m$. We see that $(\Phi^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}})^*(\varepsilon_{l,m})$ is not DGA homotopic to $(\Phi^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}})^*(\varepsilon_{l,m'})$ for $m \neq m'$, since from [19] we know that ε_m is not DGA homotopic to $\varepsilon_{m'}$. Then using Theorem 2.2 we see that for a fixed l and $m \neq m'$, $L^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m)$ and $L^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_{m'})$ are diffeomorphic, but not Hamiltonian isotopic. In addition, for $l \neq l'$ and fixed m, $L^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_l \sqcup L^{\mathcal{I}}_m)$ is not homeomorphic to $L^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_{l'} \sqcup L^{\mathcal{I}}_m)$. We now define

$$L^j_k := L^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}} \circ (L^{\mathcal{T}}_j \sqcup L^{\mathcal{I}}_k)\,.$$

This finishes the proof.

Remark 3.4. Another way to see that $(\Phi^{\Lambda}_{\Lambda_{\mathcal{T}}\sqcup\Lambda_{\mathcal{I}}})^*$ is injective at the level of homotopy classes of augmentations would be using the extension of the result of Pan [21, Theorem 1.6], which is based on the application of Cthulhu theory defined by Chantraine, Dimitroglou Rizell, Ghiggini and the author [9], and which at this stage is completely proven only for Legendrian knots in $\mathbb{R}^3$. The result of Pan is based on two main ingredient: the corresondence between gradient flow trees and holomorphic disks as in the works of Ekholm [11] for closed Legendrians and for exact Lagrangian cobordisms as in the work of Ekholm-Honda-Kalman [14] and the description of a cohomological unit in the augmentation category $\mathcal{A}ug_+$ as a sum of minima. Because of the geometric nature of the considered Legendrians, both ingredients used by Pan are expected to be extendable for the Legendrians that we consider, but this requires some additional work.

Acknowledgement. The author is supported by the GAČR EXPRO Grant 19-28628X. The author is grateful to the organizers of the 42nd Winter School on Geometry and Physics in Srni for the hospitality. In addition, the author is grateful to Frederic Bourgeois, Baptiste Chantraine, Georgios Dimitroglou Rizell, Noemie Legout and Yu Pan for the very helpful discussions.

References

[1] An, B.H., Bae, Y., Lee, E., *Lagrangian fillings for Legendrian links of affine type*, preprint 2021, arXiv:2107.04283.

[2] An, B.H., Bae, Y., Lee, E., *Lagrangian fillings for Legendrian links of finite type*, preprint 2021, arXiv:2101.01943.

[3] Cao, C., Gallup, N., Hayden, K., Sabloff, J.M., *Topologically distinct Lagrangian and symplectic fillings*, Math. Res. Lett. **21** (1) (2014), 85–99.

[4] Capovilla-Searle, O., *Infinitely many planar fillings and symplectic Milnor fibers*, preprint 2022, arXiv:2201.03081.

[5] Casals, R., Gao, H., *Infinitely many Lagrangian fillings*, Ann. of Math. **195** (1) (2022), 207–249.

[6] Casals, R., Ng, L., *Braid Loops with infinite monodromy on the Legendrian contact DGA*, J. Topol. **15** (4) (2022), 1927–2016.

[7] Casals, R., Zaslow, E., *Legendrian weaves*, preprint 2020, arXiv:2007.04943.

[8] Chantraine, B., *On Lagrangian concordance of Legendrian knots*, Algebr. Geom. Topol. **10** (2010), 63–85.

[9] Chantraine, B., Rizell, G. Dimitroglou, Ghiggini, P., Golovko, R., *Floer theory for Lagrangian cobordisms*, J. Differential Geom. **114** (3) (2020), 393–465.

[10] Dimitroglou Rizell, G., *Legendrian ambient surgery and Legendrian contact homology*, J. Symplectic Geom. **14** (3) (2016), 811–901.

[11] Ekholm, T., *Morse flow trees and Legendrian contact homology in 1-jet spaces*, Geom. Topol. **11** (2007), 1083–1224.

[12] Ekholm, T., *Rational symplectic field theory over* $\mathbb{Z}_2$ *for exact Lagrangian cobordisms*, J. Eur. Math. Soc. **10** (3) (2008), 641–704.

[13] Ekholm, T., Etnyre, J., Sullivan, M., *Non-isotopic Legendrian submanifolds in* $\mathbb{R}^{2n+1}$, J. Differential Geom. **71** (2005), 85–128.

[14] Ekholm, T., Honda, K., Kálmán, T., *Legendrian knots and exact Lagrangian cobordisms*, J. Eur. Math. Soc. (JEMS) **18** (11) (2016), 2627–2689.

[15] Eliashberg, Y., Givental, A., Hofer, H., *Introduction to symplectic field theory*, Geom. Funct. Anal. Special Volume **10** (2000), 560–673.

[16] Gao, H., Shen, L., Weng, D., *Augmentations, fillings, and clusters*, preprint 2020, arXiv:2008.10793.

[17] Gao, H., Shen, L., Weng, D., *Positive braid links with infinitely many fillings*, preprint 2020, arXiv:2009.00499.

[18] Golovko, R., *A note on the front spinning construction*, Bull. Lond. Math. Soc. **46** (2) (2014), 258–268.

[19] Golovko, R., *A note on the infinite number of exact Lagrangian fillings for spherical spuns*, Pacific J. Math. **317** (1) (2022), 143–152.

[20] Karlsson, C., *A note on coherent orientations for exact Lagrangian cobordisms*, Quantum Topol. **11** (1) (2020), 1–54.

[21] Pan, Y., *The augmentation category map induced by exact Lagrangian cobordisms*, Algebr. Geom. Topol. **17** (3) (2017), 1813–1870.

Faculty of Mathematics and Physics,
Charles University,
Sokolovská 83, 18000 Praha 8, Czech Republic
E-mail: `golovko@karlin.mff.cuni.cz`