ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 329–338

PSEUDO-RIEMANNIAN AND HESSIAN GEOMETRY RELATED TO MONGE-AMPÈRE STRUCTURES

S. Hronek and R. Suchánek

Abstract. We study properties of pseudo-Riemannian metrics corresponding to Monge-Ampère structures on four dimensional T^*M. We describe a family of Ricci flat solutions, which are parametrized by six coefficients satisfying the Plücker embedding equation. We also focus on pullbacks of the pseudo-metrics on two dimensional M, and describe the corresponding Hessian structures.

1. Introduction

Let $T^*M \xrightarrow{\pi} M$ be the cotangent bundle over a real, smooth, two dimensional manifold M. Let $\Omega \in \Omega^2(T^*M)$ be the canonical symplectic form, which in the Darboux (or canonical) coordinates is written as

$$\Omega = \mathrm{d}\,x \wedge \mathrm{d}\,p + \mathrm{d}\,y \wedge \mathrm{d}\,q\,. \tag{1}$$

Let us further consider a 2-form $\alpha \in \Omega^2(T^*M)$ given by

$$\begin{aligned} \alpha = A\,\mathrm{d}\,p \wedge \mathrm{d}\,y + B(\mathrm{d}\,x \wedge \mathrm{d}\,p - \mathrm{d}\,y \wedge \mathrm{d}\,q) + C\,\mathrm{d}\,x \wedge \mathrm{d}\,q \\ + D\,\mathrm{d}\,p \wedge \mathrm{d}\,q + E\,\mathrm{d}\,x \wedge \mathrm{d}\,y\,, \end{aligned} \tag{2}$$

where $A, B, C, D, E \in C^\infty(T^*M)$ are smooth functions. If $\Omega \wedge \alpha = 0$, then the pair (Ω, α) is called a Monge-Ampère (M-A) structure over T^*M [9, 10]. This terminology reflects the fact that the pairs (Ω, α) are in correspondence with a subclass of all nonlinear second-order PDEs, called Monge-Ampère equations, in the following way. Choose a function $f \in C^\infty(M)$. The deRham differential of f gives rise to a section, $\mathrm{d}\,f\colon M \to T^*M$, and one can consider the pullback $(\mathrm{d}\,f)^*\alpha \in \Omega^2(M)$. Then the equation

$$(\mathrm{d}\,f)^*\alpha = 0 \tag{3}$$

defines a nonlinear second order PDE with respect to f, where the nonlinearity is given by the determinant of the Hessian matrix of f. In the above chosen coordinates, the equation (3) writes

$$Af_{xx} + 2Bf_{xy} + Cf_{yy} + D\left(f_{xx}f_{yy} - {f_{xy}}^2\right) + E = 0\,, \tag{4}$$

2020 *Mathematics Subject Classification*: primary 53B20; secondary 83C15.
Key words and phrases: Hessian structure, Lychagin-Rubtsov metric, Monge-Ampère structure, Monge-Ampère equation, Plücker embedding.
Received May 23, 2022, revised September 2022. Editor M. Čadek.
DOI: 10.5817/AM2022-5-329

where $A, \ldots, E$ are the coefficients given in (2), but now depending on x, y, f_x, f_y, instead of x, y, p, q. The equation (4) is called a *2D (symplectic) Monge-Ampère equation* (shortly just M-A equation). M-A equations arise and have rich applications, for example, in differential geometry of surfaces, integrability of geometric structures, hydrodynamics, acoustics, variational calculus, Riemannian, CR, or complex geometry [4, 9, 10, 12, 16]. For a detailed exposition of some of these applications and geometric treatment of M-A equations, especially in 2D and 3D, see [8, 9].

From a different perspective, M-A structure (Ω, α) yields other geometric structures on T^*M, for example, complex, product, Kähler, or nearly Calabi-Yau structures (depending on various assumptions on the coefficients $A, \ldots, E$) [3, 4, 8, 9, 10, 15, 16]. We call a M-A structure *non-degenerate*, if the *Pfaffian of a M-A structure*, which is given by the equation

$$\alpha \wedge \alpha = \mathrm{Pf}(\alpha)\Omega \wedge \Omega \,, \tag{5}$$

satisfies $\mathrm{Pf}(\alpha) \neq 0$. The sign of the Pfaffian decides whether M-A structure gives rise to a complex or product structure, and whether the M-A equation corressponding to the M-A structure is elliptic or hyperbolic [3, 9, 10]. We have found that non-degeneracy of M-A structures on T^*M is equivalent to non-degeneracy of certain bilinear forms on M.

Inspired by applications in theoretical meteorology [6, 12, 14], we are interested in a specific family of symmetric bilinear forms, parameterized by 2-forms α as given in (2) (or, equivalently, by the corresponding coefficients $A, \ldots, E$). More concretely, there is a map

$$\begin{aligned} \Omega^2(T^*M) &\to S^2(T^*M)\,, \\ \alpha &\mapsto g_\alpha\,, \end{aligned}$$

given locally by

$$g_\alpha(X, Y) := \frac{2(\iota_X \alpha \wedge \iota_Y \Omega + \iota_Y \alpha \wedge \iota_X \Omega) \wedge \pi^* \,\mathrm{vol}}{\Omega \wedge \Omega}\,, \tag{6}$$

where X, $Y \in \Gamma(TT^*M)$, ι is the interior product, and $\pi^*\,\mathrm{vol} \in \Omega^2(T^*M)$ is the pullback of a locally chosen top form $\mathrm{vol} \in \Omega^2(M)$ along the cotangent bundle projection. Under a mild assumption on α, the symmetric form g_α is a pseudo-metric on T^*M, which is called *Lychagin-Rubtsov metric* [2, 5, 7].

The L-R metrics are the main objects of our interest. They find applications, for example, in theoretical meteorology [6, 13, 15, 16]. In particular, we want to mention a paper in progress of I. Roulstone, V. Rubtsov, and M. Wolf, where an approach motivated by general principles of topological fluid dynamics is employed. Specifically, the curvature of L-R metric (6) is used to study geometric aspects of semi-geostrophic incompressible flows in 2D and 3D, which are associated with the accumulation of vorticity of the flow. These results were discussed on a series of lectures *Monge-Ampère Geometry and the Navier-Stokes Equations*, given by I. Roulstone during *Winter School and Workshop Wisla 22* held in the beginning of February 2022.

1.1. **Structure of the paper.** After the introduction we focus on properties of L-R metrics over the 4D cotangent bundle T^*M and investigate the Ricci flat case. Using the Plücker embedding, we show that there is a correspondence between the set given by Ricci flat g_α and points in projective space. We then consider the pullback of g_α along sections of $T^*M \to M$ and describe their basic features, as well as some particular cases, among which are Hessian structures. Hessian structures are connected to and have applications in, for example, affine differential geometry, homogeneous spaces, cohomology theory, statistical manifolds, string theory, etc. [1, 17, 18]. We also find a special subclass of Hessian structures given by the solutions of certain Monge-Ampère equations. In this case, we compute the Koszul forms and comment on the related Kähler structure and its Ricci tensor. We note that more general M-A structures (i.e. A, B, C not necessarily zero) naturally lead to deformations of the Hessian structures on M. We show that the pullback metrics behave rather independently of the original ones, and that the non-degeneracy of a special subset of the pullback metrics is equivalent to non-degeneracy of the corresponding M-A structures.

2. Pseudo-Riemannian Lychagin-Rubtsov metric on T^*M

We start with some basic information about the L-R metric. In the canonical coordinates, the matrix of g_α defined by (6) is

$$G_\alpha = \begin{pmatrix} 2C & -2B & D & 0 \\ -2B & 2A & 0 & D \\ D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \end{pmatrix}. \tag{7}$$

We note the independency of G_α on E. From $\det G_\alpha = D^4$ further follows the non-degeneracy condition is

$$\det G_\alpha \neq 0 \iff D \neq 0\,.$$

Lemma 2.1. *Let g_α be the L-R metric given by* (6)*. Suppose that α given by* (2) *satisfies $D \neq 0$. Then g_α is a pseudo-Riemannian metric field on T^*M with signature* $(2,2)$*.*

Proof. From (7) we see that the vector fields $\frac{\partial}{\partial p}, \frac{\partial}{\partial q}$ span a 2-dimensional totally isotropic subspace. Since $\dim T_pT^*M = 4$, the signature of g_α is necessarily $(2,2)$. □

2.1. **L-R metric and Ricci flatness condition.** We search for the conditions on α so that the L-R metric (6) satisfies the Ricci flatness condition

$$R_{ij} = 0\,. \tag{8}$$

The Ricci curvature R_{ij} and the Ricci scalar R of g_α are, in general, complicated expressions with the second order derivatives of the coefficients $A, B, \ldots$. We focused on the special case $A = B = C = 0$. This means that α reduces to

$$\alpha = D\,\mathrm{d}\,p \wedge \mathrm{d}\,q + E\,\mathrm{d}\,x \wedge \mathrm{d}\,y\,, \tag{9}$$

and the corresponding Monge-Ampère equation is

$$D \det \operatorname{Hess}(f) = -E\,.$$

Although we have restricted our considerations significantly by this choice of A, B, C, the above M-A equation has interesting properties. For example, it naturally emerges in the context of incompressible fluid dynamics, and is related to rich geometric structures [6, 8, 9, 16]. Moreover, the pullback of the L-R metric yields a Hessian structure on M, which will be further discussed in the next section. We proceed with the following lemma, which will be used in the subsequent result.

Lemma 2.2. *Let g_α be the L-R metric with $A = B = C = 0$. Then the Ricci scalar is*

$$R = \frac{3}{D^3}\left(-D_q D_y + 2DD_{yq} - D_p D_x + 2DD_{xp}\right)\,. \tag{10}$$

Proof. The proof is given by a direct computation. Note that the factor $\frac{3}{D^3}$ is well-defined, since G_α corresponds to a metric only if $D \neq 0$, due to Lemma (2.1). □

Proposition 2.3. *Let $\alpha \in \Omega^2(T^*M)$ be given by* (9)*, where $E \in C^\infty(T^*M)$ is arbitrary. Then g_α is a Ricci flat pseudo-Riemannian metric if and only if,*

$$D(x,y,p,q) = \left(c_1 + c_2 x + c_3 y + c_4 p + c_5 q + c_6(xp + yq)\right)^{-2}\,, \tag{11}$$

where the constants $c_i \in \mathbb{R}$, $i = 1, 2, \dots, 6$, satisfy

$$c_2 c_4 + c_3 c_5 - c_1 c_6 = 0\,. \tag{12}$$

Proof. The Ricci curvature is a symmetric tensor and thus, in general, has 10 independent components. We will split the set of equations corresponding to $R_{ij} = 0, 1 \leq i, j \leq 4$ into three subsets. The first subset contains four equations

$$3D_i^2 - 2DD_{ii} = 0, \quad i \in \{x, y, p, q\}\,, \tag{13}$$

as well as the second subset

$$\begin{aligned} 3D_x D_y - 2DD_{xy} &= 0\,,\\ 3D_x D_q - 2DD_{xq} &= 0\,,\\ 3D_p D_y - 2DD_{py} &= 0\,,\\ 3D_p D_q - 2DD_{pq} &= 0\,. \end{aligned} \tag{14}$$

The last two equations are

$$\begin{aligned} -2DD_{yq} + 3D_x D_p - 4DD_{xp} &= 0\,,\\ -2DD_{xp} + 3D_y D_q - 4DD_{yq} &= 0\,. \end{aligned} \tag{15}$$

Now we use the following scheme for looking for solutions of the above system of PDEs. We start with the first equation $3D_x^2 - 2DD_{xx} = 0$, which can be easily transformed by multiplying both sides of the equation with the factor $\frac{1}{DD_x}$

$$\frac{3}{2}\frac{D_x}{D} = \frac{D_{xx}}{D_x}\,,$$

thus obtaining an equation, which can be integrated to a separable equation

$$\frac{3}{2}\ln D = \ln D_x + c\,.$$

The solution to the last equation is

$$D = \frac{1}{(c_1(y,p,q) + xc_2(y,p,q))^2}\,,$$

where $c_1(y,p,q), c_2(y,p,q)$ are unknown functions. We use the remaining equations in (13) and (14) to fix the dependencies of $c_1(y,p,q)$ and $c_2(y,p,q)$ on the variables y,p,q. This leads to

$$D(x,y,p,q) = (c_1 + c_2x + c_3y + c_4p + c_5q + c_6(xp+yq))^{-2}\,.$$

Plugging this in the remaining two equations (15), we get the condition (12), which finishes the proof. □

2.1.1. *Plücker embedding.* Following [11], let V be a real vector space, $\dim V = 4$. Denote by $\mathrm{Gr}(2,V)$ the Grassmanian of $2D$ subspaces in V, and by $\mathbb{P}(\Lambda^2 V)$ the projectivization of the space of exterior 2-forms on V. Let $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}$ be coordinates on $\mathbb{P}(\Lambda^2 V)$. The Plücker embedding is a map

$$\mathrm{Gr}(2,V) \to \mathbb{P}(\Lambda^2 V) \tag{16}$$

with image given by the equation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0\,. \tag{17}$$

Now we are ready to formulate a corollary of Proposition 2.3, which describes a link between the solutions g_α and points on quadric in the 5D real projective space.

Corollary 2.3.1. *There is a correspondence between the set of metrics $g_\alpha \in S^2(T^*M)$, which satisfy* (11)*,* (12)*, and the quadric in $\mathbb{R}P^5$ given by the image of the Plücker embedding* (16)*. The condition* (12) *is equivalent to vanishing of the scalar curvature of g_α.*

Proof. Define

$$c_1 = -p_{14}\,,\quad c_2 = p_{12}\,,\quad c_3 = -p_{13}\,,\quad c_4 = p_{34}\,,\quad c_5 = p_{24}\,,\quad c_6 = p_{23}\,.$$

Then (17) is satisfied only if (12) holds. If g_α is given by (11), then its scalar curvature is

$$R = -24(c_2c_4 + c_3c_5 - c_1c_6)\,.$$

Thus (12) holds only if $R = 0$. □

3. The pullback metric

Given a section of the cotangent bundle $\mathrm{d}\,f\colon M \to T^*M$, we can pullback the metric from T^*M to M and induce a (pseudo-)Riemannian structure $(\mathrm{d}\,f)^* g_\alpha$ on M. In this section we explore the properties of this metric.

3.1. **General case.** We start with the matrix of the pullback metric $(\mathrm{d}\, f)^* g_\alpha$ in canonical coordinates, which we will denote G^*_α

$$G^*_\alpha = \begin{pmatrix} 2C & -2B \\ -2B & 2A \end{pmatrix} + 2D\,\mathrm{Hess}(f)\,. \tag{18}$$

We remind the reader that we use the same notation for the functions $A, B, C, D \in C^\infty(T^*M)$ (i.e. depending on x, y, p, q) as we do for the precompositions with the section of $T^*M \to M$ (i.e. depending on x, y, f_x, f_y). The determinant is

$$\det G^*_\alpha = 4(AC - B^2) + 4D\left(Af_{xx} + 2Bf_{xy} + Cf_{yy} + Df_{xx}f_{yy} - Df_{xy}^2\right)\,. \tag{19}$$

We observe that the non-degeneracy of g_α, which was governed by the condition $D \neq 0$, is not directly related to non-degeneracy of $(\mathrm{d}\, f)^* g_\alpha$, which is much more complicated. This allows for all four situations of degeneracy/non-degeneracy of the pair $(g_\alpha, (\mathrm{d}\, f)^* g_\alpha)$.

3.2. **Special choices of $A, \dots, E$.** We will now discuss certain choices of the coefficients $A, \dots, E$, leading to further simplifications and interesting properties of $(\mathrm{d}\, f)^* g_\alpha$.

3.2.1. *Hessian structures.* Suppose that $A = B = C = 0$ and $D = \frac{1}{2}$. Then the pullback metric is of the form

$$G^*_\alpha = \mathrm{Hess}(f)\,, \tag{20}$$

which is called *a Hessian structure* on M. The theory of Hessian structures is well studied, for example, see [17], where the Riemannian geometry of the Hessian structures is investigated. Interesting properties of these structures are also described, for example, in [1, 17, 18].

In [17], the author defines so-called first and second Kozsul forms, a_i, b_{ij}, of the Hessian structure, which are derived from the Christoffel symbols of the corresponding metric as follows

$$a_i = \Gamma^k_{ki}, \qquad b_{ij} = \partial_j \alpha_i\,. \tag{21}$$

We can employ the condition that the M-A equation $(\mathrm{d}\, f)^*\alpha = 0$ is satisfied, which yields

$$\det G^*_\alpha = -E\,.$$

In this case, the first and second Kozsul forms simplify to the following nice form

$$a_i = \frac{1}{2}\frac{\partial \ln \det G^*_\alpha}{\partial x^i} = \frac{1}{2}\frac{\partial \ln |E|}{\partial x^i}\,,$$
$$b_{ij} = \frac{1}{2}\frac{\partial^2 \ln |E|}{\partial x^i \partial x^j} = \frac{1}{2}\,\mathrm{Hess}(\ln |E|)\,.$$

Especially the second Koszul form is of great importance. Following [17], given a Hessian structure on M, one can define a Kähler structure on TM. Let (x^i, ξ^i) be a coordinate system on TM and $z^j = x^j + i\xi^j$. Then the Hessian structure is

$$g^T = (g_{ij} \circ \pi)\,\mathrm{d}\, z^i\,\mathrm{d}\,\overline{z}^j\,.$$

Then the Ricci tensor of g^T is given by the second Koszul form

$$R^T_{ij} = -\frac{1}{2} b_{ij} \circ \pi \,.$$

The authors of this paper are interested in the relationship between the complex structure given by the above construction, and the complex structure naturally associated with M-A structures with negative Pfaffian (see (5) for the definition of the notion).

3.2.2. *Deformations of the Hessian structure.* Let us now suppose only $D = \frac{1}{2}$, then

$$G^*_\alpha = \begin{pmatrix} 2C & -2B \\ -2B & 2A \end{pmatrix} + \operatorname{Hess}(f) \,.$$

It is possible to choose suitable functions A, B, C such that the matrix

$$\begin{pmatrix} 2C & -2B \\ -2B & 2A \end{pmatrix}$$

is a Hessian matrix of some function ϵg, where $\epsilon > 0$ is a scalar. Such a choice gives rise to a *deformation of the Hessian structure*, since

$$G^*_\alpha = \operatorname{Hess}(f) + \epsilon \operatorname{Hess}(g) = \operatorname{Hess}(f + \epsilon g) \,.$$

Notably, all the results of the previous paragraphs holds.

3.2.3. *Pfaffian and non-degeneracy.* So far we did not need to assume that the M-A equation related to the M-A structure (Ω, α) is satisfied by f. Nevertheless, if we assume that the M-A equation $(\mathrm{d}\, f)^* \alpha = 0$ holds, then it allows us to significantly simplify the expression (19).

Proposition 3.1. *Let $f \in C^\infty(M)$ and consider a M-A structure $(\Omega, \alpha) \in \Omega^2(T^*M) \times \Omega^2(T^*M)$. Suppose that f satisfies the corresponding M-A equation $(\mathrm{d}\, f)^* \alpha = 0$. Then $(\mathrm{d}\, f)^* g_\alpha$ is a metric on M only if $\operatorname{Pf}(\alpha) \neq 0$, which is equivalent to non-degeneracy of the M-A structure (Ω, α). Moreover, the eigenvalues of the matrix of $(\mathrm{d}\, f)^* g_\alpha$ in canonical coordinates are*

$$\lambda_{1,2} = C + A + D(f_{xx} + f_{yy}) \pm 2\sqrt{\operatorname{Pf}(\alpha)} \,. \tag{22}$$

Proof. A coordinate description of the M-A equation $(\mathrm{d}\, f)^* \alpha = 0$ is (4). Thus, if f satisfies the equation, then

$$A f_{xx} + 2B f_{xy} + C f_{yy} + D \left(f_{xx} f_{yy} - {f_{xy}}^2 \right) = -E \,.$$

This implies that (19) becomes

$$\det G^*_\alpha = 4(AC - B^2) - 4DE \;.$$

Using the equation (5) and the coordinate descriptions (1), (2) of the M-A structure, one can easily compute $\operatorname{Pf}(\alpha) = -B^2 + AC - DE$. Therefore

$$\det G^*_\alpha = 4 \operatorname{Pf}(\alpha) \,.$$

The eigenvalues are solutions of the following equation

$$\lambda^2 - \lambda\,(2C + 2A + 2Df_{yy} + 2Df_{xx}) - 4(B^2 - AC) - 4DE = 0 \tag{23}$$

and their determination is a direct computation. □

We observe that the signature of $(\mathrm{d}\,f)^* g_\alpha$ depends on two M-A equations. Firstly, to obtain the previous result, we had to assume that f solves the M-A equation $(\mathrm{d}\,f)^*\alpha = 0$. Secondly, we see that the eigenvalue equation (23) contains the Laplacian expression $2D(f_{yy} + f_{xx})$. But vanishing of this expression amounts to the Laplace equation $f_{yy} + f_{xx} = 0$, which, in general, is a different M-A equation then $(\mathrm{d}\,f)^*\alpha = 0$.

4. Conclusions and outlook

Motivated by the results of V. Lychagin et. al. [10, 12], and I. Roulstone et. al. [6, 13, 14], we were interested in (pseudo-)Riemannian and Hessian structures related to Monge-Ampère structures and the corresponding $2D$ Monge-Ampère equations. More concretely, this paper was focused on the following questions.

(1) Are there any non-constant Lychagin-Rubtsov metrics with vanishing Ricci curvature?

(2) If the answer to the first question is positive, can we classify all such Monge-Ampère structures and the corresponding M-A equations?

(3) What kind of geometry on M yields the pullback of the L-R metric along sections of $T^*M \to M$?

We have presented partial answers to the above questions in the case of 4D metrics on T^*M and their 2D pullbacks on M.

The first question is positively answered in proposition 2.3. There is a family of non-constant L-R metrics, depending on five real parameters satisfying conditions (11), (12), with $R_{ij} = 0$. Considering the second question, we have found definite answers for the case

$$D \det \operatorname{Hess} f = -E\ , \tag{24}$$

This amounts to suppressing three out of five degrees of freedom of a general 2D (symplectic) M-A equation (3) by choosing $A = B = C = 0$. The answer to the last question is also based on this choice, which naturally leads to 2D Hessian structures on M, their deformations, and a subclass of Hessian structures determined by the requirement that f is a solution of the M-A equation (24). We want to emphasize that only some of our results depend on the requirement that f satisfies certain PDE. The L-R metric and its pullback can be introduced without any mentioning of the correspondence between M-A structures and M-A equations, so (24) (or possibly other M-A equation) is an additional assumption. Aside from the above three questions, we observed an interesting relation between a family of solutions of the vacuum field equations and the Plücker embedding, which is described in Corollary 2.3.1. Regarding the future work, we are interested in the following.

– Further properties of the curvature tensor of g_α and the relation between the curvature and the corresponding M-A equations.

– The Hessian structures determined by (24) and aim at establishing further links between M-A structures and Hessian structures (as well as their deformations) in dimensions greater then 2.
– Comparison of the complex structure related to the Koszul form with the complex structures associated with M-A equations satisfying $\mathrm{Pf}(\alpha) > 0$.

Acknowledgement. Both authors are grateful for the valuable comments and suggestions of the reviewer. R. Suchánek is grateful to J. Slovák and V. Rubtsov for their discussions and advices, as well as to the Masaryk University and to the University of Angers for their hospitality. The work of R. Suchánek was financially supported under the project GAČR EXPRO GX19-28628X, and project MUNI/A/1092/2021, and partly by the Barrande Fellowship program organized by The French Institute in Prague (IFP) and the Czech Ministry of Education, Youth and Sports (MYES). Authors are also grateful to the organizers of the 42nd Winter School Geometry and Physics for a wonderful event, which enabled the authors to work on this subject in a very inspiring atmosphere.

References

[1] Amari, S., Armstrong, J., *Curvature of Hessian manifolds*, Differential Geom. Appl. **33** (2014), 1–12.

[2] Banos, B., *Integrable geometries and Monge-Ampèere equations*, arXiv: Differential Geometry (2006).

[3] Banos, B., *Monge-Ampère equations and generalized complex geometry –the two-dimensional case*, J. Geom. Phys. **57** (3) (2007), 841–853.

[4] Banos, B., *Complex solutions of Monge-Ampère equations*, J. Geom. Phys. **61** (2011), no. 11, 2187–2198.

[5] Banos, B., *Complex solutions of Monge-Ampère equations*, J. Geom. Phys. **61** (2011), no. 11, 2187–2198.

[6] Banos, B., Rubtsov, V., Roulstone, I., *Monge–Ampère structures and the geometry of incompressible flows*, J. Phys A **49** (2016).

[7] Delahaies, S., *Complex and contact geometry in geophysical fluid dynamics*, Ph.D. thesis, 01 2009.

[8] Kosmann-Schwarzbach, Y., Rubtsov, V., *Compatible structures on Lie algebroids and Monge-Ampère operators*, Acta Appl. Math. **109** (2010), no. 1, 101–135.

[9] Kushner, A., Lychagin, V., Rubtsov, V., *Contact geometry and nonlinear differential equations*, Encyclopedia Math. Appl., Cambridge University Press, 2006.

[10] Lychagin, V.V., *Contact geometry and non-llnear second-order differential equations*, Russian Math. Surveys **34** (1979), no. 1, 149–180.

[11] Lychagin, V.V., Roubtsov, V., *Monge–Ampère Grassmannians, characteristic classes and all that*, pp. 233–257, Springer International Publishing, Cham, 2019.

[12] Lychagin, V.V., Rubtsov, V.N., Chekalov, I.V., *A classification of Monge-Ampère equations*, Ann. Sci. Éc. Norm. Supér. (4) **Ser. 4, 26** (1993), no. 3, 281–308 (en). MR 94c:58229

[13] Roulstone, I., Banos, B., Gibbon, J.D., Roubtsov, V.N., *Kähler geometry and Burgers' vortices*, (2009).

[14] Rubtsov, V., Roulstone, I., *Holomorphic structures in hydrodynamical models of nearly geostrophic flow*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), 1519–1531.

[15] Rubtsov, V. N., Roulstone, I., *Examples of quaternionic and Kähler structures in Hamiltonian models of nearly geostrophic flow*, J. Phys. A **30** (1997), no. 4, L63–L68.

[16] Rubtsov, Volodya, *Geometry of Monge-Ampère structures*, pp. 95–156, Springer International Publishing, Cham, 2019.

[17] Shima, Hirohiko, *The geometry of Hessian structures*, World Scientific, 2007.

[18] Totaro, B., *The curvature of a Hessian metric*, Internat. J. Math. **15** (2004), no. 04, 369–391.

Department of Theoretical Physics and Astrophysics,
Faculty of Science, Masaryk University,
611 37 Brno, Czech Republic

Department of Mathematics and Statistics,
Faculty of Science, Masaryk University,
611 37 Brno, Czech Republic
and
Angevin Laboratory of Mathematical Research - UMR CNRS 6093,
University of Angers, 2 Boulevard Lavoisier,
49045, Angers CEDEX 0, France
E-mail: r.suchanek.r@gmail.com