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TOPOLOGICAL ENTROPY AND DIFFERENTIAL EQUATIONS

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Abstract. On the background of a brief survey panorama of results on the topic in the title, one new theorem is presented concerning a positive topological entropy (i.e. topological chaos) for the impulsive differential equations on the Cartesian product of compact intervals, which is positively invariant under the composition of the associated Poincaré translation operator with a multivalued upper semicontinuous impulsive mapping.

1. Introduction and some preliminaries

The main aim of this short note is two-fold: (i) to describe briefly the recent state of the study of a topic at the title, (ii) to indicate the investigation of topological entropy for differential equations with multivalued impulses.

The first definition of topological entropy was given in 1965 by Adler, Konheim and McAndrew for (single-valued) continuous maps in compact topological spaces (see [1]). Another definition was introduced in 1971 by Bowen for uniformly continuous maps in not necessarily compact metric spaces (see [10]), who proved the equivalence of his definition with the one in [1] in compact metric spaces.

Definition 1 (cf. [10])**.** Let (X, d) be a metric space, K be a compact subset of X and $f : X \to X$ be a uniformly continuous map. A set $S \subset K$ is called (n, ε)-*separated* with respect to f, for a positive integer n and $\varepsilon > 0$, if for every pair of distinct points $x, y \in S$, $x \neq y$, there is at least one k with $0 \leq k < n$ such that $d(f^k(x), f^k(y)) > \varepsilon$. Then, denoting the number of different orbits of length n by

$$s(n, \varepsilon, f, K) := \max\{\#S : \ S \subset X \text{ is an } (n, \varepsilon)\text{-separated set with respect to } f\}\,,$$

the *topological entropy* $h(f)$ of f is defined as

$$h(f) := \sup_{K \subset X,\ K \text{ is compact}} \lim_{\varepsilon \to 0} \left[\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, f, K)\right] . \tag{1}$$

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The topological entropy in the sense of Definition 1 is, besides other things, a topological invariant, but not a homotopy invariant. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then according to [10, Theorem 15]

$$h(f) = \sum_{|\lambda_i|>1} \log|\lambda_i|\,, \tag{2}$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of f.

Definition 1 can be applied to differential equations via the associated Poincaré translation operators along their trajectories (see Section 2 below).

In this way, the upper and lower estimates of topological entropy were obtained for linear systems of ordinary differential equations in e.g. [9, 12, 23, 24].

For nonlinear systems, the obtained results can be characterized as either generic in $\mathbb{R}^2$ (lower estimates) by means of the Artin braid group theory (see e.g. [15, 19, 20, 27]) or rather implicit for higher-dimensional systems (upper and lower estimates, especially in $\mathbb{R}^3$) (see e.g. [16], [21], and the references therein) or just numerical (see e.g. [26]).

For systems with impulses, the results are rare. For nonlinear impulsive systems (see e.g. [2, 5, 8]), and for multivalued or discontinuous impulsive systems (see e.g. [3, 4, 7, 13, 14]).

Of course, if the systems or impulses are multivalued, then Definition 1 is insufficient, and must be appropriately changed. A suitable definition with this respect seems to be the following one by Kelly and Tennant (see [17]).

Definition 2 (cf. [17]). Let (X,d) be a compact metric space and $\varphi\colon X \to \mathcal{K}(X)$, where $\mathcal{K}(X) := \{K \subset X : K \text{ is a non-empty compact subset}\}$, be an upper semicontinuous map (i.e. φ has a closed graph $\Gamma_\varphi := \{(x,y) \in X \times X : y \in \varphi(x)\}$.

Let the *space of n-orbits* of φ be denoted as

$$\mathrm{Orb}_n(\varphi) := \{(x_1,\ldots,x_n) \in X^n : x_{i+1} \in \varphi(x_i), i = 1,\ldots,n-1\}\,.$$

We say that $S \subset \mathrm{Orb}_n(\varphi)$ is *(n,ε)-separated* for φ, for a positive integer n and $\varepsilon > 0$, if for every pair of distinct n-orbits $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ there is at least one k with $1 \le k \le n$ such that $d(x_k, y_k) > \varepsilon$.

The *topological entropy* $h_{\mathrm{KT}}(\varphi)$ of φ is defined as

$$h_{\mathrm{KT}}(\varphi) := \lim_{\varepsilon\to 0}\left[\limsup_{n\to\infty} \frac{1}{n}\log(s(n,\varepsilon,\varphi))\right]\,, \tag{3}$$

where $s(n,\varepsilon,\varphi)$ stands for the largest cardinality of an (n,ε)-separated subset of $\mathrm{Orb}_n(\varphi)$ for φ, i.e.

$$s(n,\varepsilon,\varphi) := \max\{\#S : S \subset \mathrm{Orb}_n(\varphi) \text{ is an } (n,\varepsilon)\text{-separated set for } \varphi\}\,.$$

One can readily check that, for single-valued continuous maps in compact metric spaces, Definition 2 reduces to Definition 1.

It will be convenient to prove the following lemma.

Lemma 3. *Let $\varphi : X \to \mathcal{K}(X)$ and $\psi : Y \to \mathcal{K}(Y)$ be upper semicontinuous maps with compact convex values in compact convex subsets X and Y of Banach spaces.*

Then the inequality

$$h_{\mathrm{KT}}(\varphi \times \psi) \geq \max\{h_{\mathrm{KT}}(\varphi), h_{\mathrm{KT}}(\psi)\}$$

holds for the Cartesian product $\varphi \times \psi : X \times Y \to \mathcal{K}(X \times Y)$, *where* $(\varphi \times \psi)(x,y) = \varphi(x) \times \psi(y)$, *for every* $(x,y) \in X \times Y$, *and* $X \times Y$ *is endowed with the maximum norm.*

Proof. According to the well known Kakutani-Fan's theorem (see e.g. [6, Corollary I.6.21]), there exist fixed points $\overline{x} \in \varphi(\overline{x})$ and $\overline{y} \in \psi(\overline{y})$ of φ and ψ, and subsequently $(\overline{x}, \overline{y}) \in (\varphi \times \psi)(\overline{x}, \overline{y})$ of $\varphi \times \psi$.

One can easily check that from Definition 2 it immediately follows (cf. [17, Theorem 4.2]):

$$\begin{aligned} h_{\mathrm{KT}}(\varphi \times \psi) &\geq \max\{h_{\mathrm{KT}}(\varphi \times \mathrm{id}\,|_{\overline{y}}), h_{\mathrm{KT}}(\mathrm{id}\,|_{\overline{x}} \times \psi)\} \\ &= \max\{h_{\mathrm{KT}}(\varphi), h_{\mathrm{KT}}(\psi)\}\,, \end{aligned}$$

which completes the proof. □

Remark 4. The inequality in Lemma 3 can be generalized to the equality

$$h_{\mathrm{KT}}(\varphi_1 \times \ldots \times \varphi_n) = \sum_{i=1}^{n} h_{\mathrm{KT}}(\varphi_i)\,,$$

where $\varphi_i : X_i \to \mathcal{K}(X_i)$ are upper semicontinuous maps in compact metric spaces X_i, $i = 1, \ldots, n$. Its proof is rather technical, but can be made quite analogously as in the single-valued case (see e.g. [25, Theorem 7.10]).

Let us note that in our papers [3, 4, 7] still formally another extension of Definition 1 was employed, matching with the Nielsen fixed point theory on tori $\mathbb{R}^n/\mathbb{Z}^n$. On the other hand, here we would like to follow rather the ideas from [8], where single-valued arguments were, however, exclusively applied on compact subsets in Euclidean spaces $\mathbb{R}^n$.

2. Some further preliminaries

Hence, as already pointed out, it will be also convenient to recall some properties of the Poincaré translation operators $T_\omega \colon \mathbb{R}^n \to \mathbb{R}^n$, associated with the differential equation

$$x' = F(t,x) \tag{4}$$

where $F \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $F(t,x) \equiv F(t+\omega, x)$, for some given $\omega > 0$, and is the *Carathéodory mapping*, i.e.

(i) $F(\cdot, x) \colon [0,\omega] \to \mathbb{R}^n$ is measurable, for every $x \in \mathbb{R}^n$,

(ii) $F(t, \cdot) \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous, for almost all (a.a.) $t \in [0,\omega]$.

Let, furthermore, (4) satisfy a uniqueness condition and all solutions of (4) entirely exist on the whole $(-\infty, \infty)$.

By a (*Carathéodory*) *solution* $x(\cdot)$ of (4), we understand a locally absolutely continuous function, i.e. $x \in AC_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, which satisfies (4) for a.a. $t \in \mathbb{R}$. For a continuous right-hand side F, we have obviously $x \in C^1(\mathbb{R}, \mathbb{R}^n)$.

The *Poincaré translation operator* $T_\omega : \mathbb{R}^n \to \mathbb{R}^n$ along the trajectories of (4) is defined as follows:

$$T_\omega(x_0) := \{x(\omega) : x(\cdot) \text{ is a solution of (4) such that } x(0) = x_0\}\,. \tag{5}$$

It is well known (see e.g. [18, 22]) that T_ω is an orientation-preserving homeomorphism such that $T_\omega^k = T_{k\omega}$, for every $k \in \mathbb{N}$. It is also isotopic to identity.

If $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, then T_ω is a diffeomorphism of class C^1 such that $\det \mathrm{D}\, T_\omega^k(x_0) > 0$ holds for every $x_0 = T_\omega^k(x_0)$ and any $k \in \mathbb{N}$, where the mapping $\mathrm{D}\, T_\omega(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the Fréchet derivative of T_ω at $x_0 \in \mathbb{R}^n$, which is a linear map corresponding to the Jacobian matrix of T_ω at x_0.

For $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$, we can consider the variation equation of (4) with respect to an ω-periodic solution $x(\cdot)$ of (4), namely

$$x' = \mathrm{D}_x\, F(t, x(t))x\,, \tag{6}$$

where $\mathrm{D}_x\, F(t,x)$ is the Jacobian matrix of $F(t,x)$ with respect to x. It is a linear differential equation with ω-periodic continuous coefficients. If $W(t)$ is its fundamental matrix, then $\mathrm{D}\, T_\omega(x_0) = W(\omega)W(0)^{-1}$.

Now, consider the vector linear equation

$$x' = A(t)x\,, \tag{7}$$

$A(t) = \{a_{ij}(t)\}_{i,j=1}^n$, with ω-periodic measurable coefficients $a_{ij}(t)$, where $\left|\int_0^\omega a_{ij}(t)\, \mathrm{d}t\right| < \infty$, for all $i, j = 1, \ldots, n$. It is well known that its solution $x(\cdot)$ with $x(0) = x_0$ can be expressed as $x(t) = W(t)x_0$, where $W(t)$ is the fundamental matrix of (7) such that $W(0) = W^{-1}(0)$ is a unit matrix. Thus, $T_\omega(x_0) = x(\omega) = W(\omega)x_0$. The operator $W(\omega)$ is called a *monodromy operator* and the eigenvalues of the related matrix are the *multiplicators* of (7). The same terminology is related to (6).

If, in particular, $A(t) \equiv A$ in (7) has constant coefficients $a_{ij}(t) \equiv a_{ij}$; $i, j = 1 \ldots, n$, then the solutions $x(\cdot)$ of (7) take the simple form $x(t) = e^{At}x_0$, i.e. $W(t) = e^{At}$, and so

$$T_\omega(x_0) = e^{A\omega}x_0 = W(\omega)x_0\,.$$

The multiplicators μ_i of the monodromy matrix $W(\omega)$ can be therefore easily expressed as

$$\mu_i = e^{\lambda_i \omega}\,, \quad i = 1, \ldots, n\,,$$

where λ_i, $i = 1, \ldots, n$, are the eigenvalues of A. Hence, in order to have $|\mu_i| > 1$, for some $i \in \{1, \ldots, n\}$, it is sufficient and necessary that $\operatorname{Re} \lambda_i > 0$ for such $i \in \{1, \ldots, n\}$.

Because of a uniqueness condition, there is, for any $k \in \mathbb{N}$, an evident one-to-one correspondence between the $k\omega$-periodic solutions $x(\cdot)$ of (4), $x(t) \equiv x(t + k\omega)$ and $x(t) \not\equiv x(t + j\omega)$ for $j < k$, and k-periodic points x_0 of T_ω such that $x(0) = x_0$, i.e. $x_0 = T_\omega^k(x_0)$ and $x_0 \neq T_\omega^j(x_0)$ for $j < k$.

Now, consider the impulsive differential equation

$$\begin{cases} x' = F(t,x),\ t \neq t_j := j\omega\,, \quad \text{for } \omega > 0\,, \\ x(t_j^+) = I(x(t_j^-))\,,\ j \in \mathbb{Z}\,, \end{cases} \tag{8}$$

where $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is as above in (4), and $I : \mathbb{R}^n \to K_0$ is a compact continuous impulsive mapping such that $K_0 := \operatorname{cl} I(\mathbb{R}^n)$ and $I(K_0) = K_0$. Its solutions will be also understood in the same Carathéodory sense, i.e. $x \in AC((t_j, t_{j+1}))$, $j \in \mathbb{Z}$.

One can readily check that there is again a one-to-one correspondence between $k\omega$-periodic solutions $x(\cdot)$ of (8) and k-periodic points $x_0 = x(0)$ of the composition $I \circ T_\omega$, for any $k \in \mathbb{N}$.

Therefore, we can naturally introduce the following definition of topological entropy for (4), (7) and (8).

Definition 5. We say that equation (4) (in particular (7)), resp. (8), has a *topological entropy* h if $h = h(T_\omega)$, resp. $h = h(I \circ T_\omega)$.

In view of Definitions 1, 5 and formulas (2), (5), equation (7) has a topological entropy $h = h(T_\omega) = h(W(\omega)\cdot \mathrm{id}) = \sum_{|\mu_i|>1} \log|\mu_i|$, where $W(\omega)$ is a fundamental matrix of (7) at $\omega > 0$ and μ_i, $i = 1,\dots,n$, are its eigenvalues. In particular, for $A(t) \equiv A$, we get that

$$h = h(e^{A\omega}\cdot \mathrm{id}) = \sum_{\operatorname{Re}\lambda_i>0} \log|e^{\lambda_i\omega}| = \omega \log e \sum_{\operatorname{Re}\lambda_i>0} \operatorname{Re}\lambda_i \, .$$

For $\omega = 1$, it is in accordance with the calculations in [9, 12, 23].

Similarly, equation (8), where $F(t,x) := A(t)x$ and I is a real $(n \times n)$-matrix, has a topological entropy

$$h = h(I \cdot W(\omega)\cdot \mathrm{id}) = \sum_{|\nu_i|>1} \log|\nu_i| \, ,$$

where $W(\omega)$ is as above and ν_i, $i = 1,\dots,n$, are the eigenvalues of the product $I \cdot W(\omega)$.

3. Topological chaos for impulsive differential equations

In spite of the above arguments, observe that e.g. the scalar equation $x' = ax$ with $I = \mathrm{id}|_{\mathbb{R}}$ possesses for $a = \frac{\log 2}{\omega \log e}$ the positive entropy $h = h(T_\omega) = \log 2$, but does not admit any nontrivial (nonzero) periodic solution. In the spirit of the "criticism" in [11], since the dynamics of $x' = ax$ have not a complicated behaviour, Definitions 1 and 5 are not suitable for any sort of deterministic chaos. In other words, to speak about topological chaos determined by a positive entropy requires here to be restricted to compact subsets of $\mathbb{R}^n$, which are positively invariant under the compositions $I \circ T_\omega$.

For multivalued impulses, the situation becomes still more delicate, because Definition 1 must be replaced e.g. by Definition 2.

For the sake of simplicity, we will consider just the linear homogeneous diagonal system of differential equations with special multivalued upper semicontinuous impulses, namely

$$\begin{cases} x' = (\operatorname{diag}[a_1,\dots,a_n])\, x \, , \ x = (x_1,\dots,x_n) \, , \\ x_i(t_j^+) \in I_i(x_i(t_j^-)) \, , \ i = 1,\dots,n, j \in \mathbb{Z} \, , \end{cases} \tag{9}$$

where $I_i : \mathbb{R} \to \mathcal{K}([0,1])$, $I_i(x_i) \equiv I_i(x_i+1)$, and

$$I_i|_{[0,1]}(x_i) := \begin{cases} [0,1]\,, & \text{for } x_i \in \{0,1\}\,, \\ x_i\,, & \text{otherwise (i.e. for } x_i \in (0,1))\,, \end{cases} \quad i = 1, \ldots, n.$$

Definition 6. We say that system (9) has a *topological entropy* h if $h := h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n})$, where h_{KT} denotes the topological entropy in the sense of Definition 2, $I = (I_1, \ldots, I_n)$ and $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ is the associated Poincaré translation operator along the trajectories of $x' = (\mathrm{diag}[a_1, \ldots, a_n])x$, defined for $\omega = 1$ in (5).

Theorem 7. *The topological entropy h of equation* (9) *satisfies the inequality $h \geq (\log\sqrt{2}) \sum_{a_i \geq 0} \mathrm{sgn}(1 + a_i)$. In particular, if at least one coefficient a_i is nonnegative (i.e. $a_i \geq 0$, for some $i \in \{1, \ldots, n\}$), then equation* (9) *exhibits on $[0,1]^n$ a topological chaos in the sense that $h > 0$.*

Proof. The associated Poincaré translation operator T_1 along the trajectories of $x' = (\mathrm{diag}[a_1, \ldots, a_n])x$, defined for $\omega = 1$ in (5), takes the form
$T_1(x) = (e^{a_1}, \ldots, e^{a_n})x = W(1)x$, where $\mu_i = e^{a_i}$, $i = 1, \ldots, n$, are the multiplicators of $W(1)$.

Since

$$I \circ T_1(x) = (I_1(e^{a_1}x_1), \ldots, I_n(e^{a_n}x_n))\ , \text{where } x = (x_1, \ldots, x_n) \in [0,1]^n,$$

$I_i(e^{a_i}x_i) \equiv I_i(e^{a_i}(x_i + e^{-a_i}))$, and

$$I_i|_{[0,e^{-a_i}]}(e^{a_i}x_i) = \begin{cases} [0,1]\,, & \text{for } x_i \in \{0, e^{-a_i}\}\,, \\ x_i\,, & \text{otherwise (i.e. for } x_i \in (0, e^{-a_i}))\,, \end{cases}$$

$i = 1, \ldots, n$, we obtain by means of the equality in Remark 4 (for a particular inequality, see Lemma 3) that

$$h := h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n}) = \sum_{i=1}^{n} h_{\mathrm{KT}}(I_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}))\,.$$

Checking the proof of [17, Theorem 6.2], it is straightforward to realize that $h_{\mathrm{KT}}(I_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]})) \geq \log\sqrt{2}$, provided $a_i \geq 0$ for some $i \in \{1, \ldots, n\}$. Summing up, $h = h_{\mathrm{KT}}(I \circ T_1|_{[0,1]^n}) \geq (\log\sqrt{2}) \sum_{a_i \geq 0} \mathrm{sgn}(1 + a_i)$, as claimed. □

Remark 8. If at least one component I_i, for some $i \in \{1, \ldots, n\}$, of the impulsive mapping I is replaced by $\hat{I}_i : \mathbb{R} \to \mathcal{K}([0,1])$, where $\hat{I}_i(x_i) \equiv \hat{I}_i(x_i + 1)$, and

$$\hat{I}_i|_{[0,1]}(x) := \begin{cases} [0,1]\,, & \text{for } x \in \{0,1\}\,, \\ \{0\}\,, & \text{otherwise (i.e. for } x_i \in (0,1))\,, \end{cases}$$

then (cf. Lemma 3) $h = h_{\mathrm{KT}}(\hat{I}_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]})) = \infty$, for any $a_i \in \mathbb{R}$, because (see [17, Theorems 5.4 and 7.1])

$$h_{\mathrm{KT}}(\hat{I}_i(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}) \geq \frac{1}{2} h_{\mathrm{KT}}(\hat{I}_i^{\,2}(e^{a_i} \cdot \mathrm{id}\,|_{[0,1]}) = h_{\mathrm{KT}}([0,1]|_{[0,1]}) = \infty\,,$$

where $[0,1]|_{[0,1]}$ denotes a constant multivalued mapping with values $[0,1]$.

Remark 9. In case of diagonalizable or weakly coupled systems, the situation becomes more complicated and requires a further technical elaboration. For the single-valued case, see e.g. [8].

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