ARCHIVUM MATHEMATICUM (BRNO)
Tomus 59 (2023), 11–19

GENERAL EXACT SOLVABILITY CONDITIONS FOR THE INITIAL VALUE PROBLEMS FOR LINEAR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Natalia Dilna

Abstract. Conditions on the unique solvability of linear fractional functional differential equations are established. A pantograph-type model from electrodynamics is studied.

1. Introduction

The fractional differential equations (FDEs) get a significant interest in modern literature on differential equations and are represented by numerous papers. Here referred to a few of them only [1, 2, 3, 4, 5, 6, 7, 8, 9].

The application scale of mentioned equations is quite broad. We want to accentuate the [9], where the authors made a complex overview of possible applications of FDE: the theories of differential, integral, and integro-differential equations, special functions of mathematical physics, and some present-day applications of fractional calculus, including fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics, and signal processing, and so on.

Conditions on the unique solvability of the boundary value problem for functional differential equations is a fundamental and non-trivial part of the study, and many publications are focused on them, for example, [10, 13, 14].

The main goal of our investigation is the exact conditions lookup of the unique solvability of the boundary value problem for the fractional functional differential equations (FFDEs). Some recent results [3, 4, 5, 6, 8] motivated us to continue in this direction.

2020 *Mathematics Subject Classification*: primary 26A33; secondary 34A08, 34B15.
Key words and phrases: fractional order functional differential equations, Caputo derivative, normal and reproducing cone, unique solvability.
This research was supported by the Slovak Grant Agency VEGA-SAV, Grant No. 2/0127/20.
Received August 31, 2022, accepted December 20, 2022. Editor Z. Došlá.
DOI: 10.5817/AM2023-1-11

2. Problem formulation

We consider fractional functional differential problem

$$D_a^q u(t) = (lu)(t) + f(t)\,, \quad t \in [a,b] \tag{2.1}$$

$$u(a) = c\,, \tag{2.2}$$

where D_a^q is the Caputo fractional derivative of order q, $0 < q < 1$, with the lower limit zero, operator $l = (l_k)_{k=1}^n \colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is the bounded linear operator, function $f \in C([a,b],\mathbb{R}^n)$ and $c \in \mathbb{R}^n$.

The main goal of our investigations is to find exact conditions sufficient for the unique solvability of the initial value problem (2.2) for systems of the linear FFDEs (2.1) presented by isotone operators (see Definition 2.3). A pantograph-type model from electrodynamics is studied as well.

Here are used spaces:

- $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions $[a,b] \to \mathbb{R}^n$ with the norm $C([a,b],\mathbb{R}^n) \ni u \to \max_{t\in[a,b]} |u(t)|_\infty = \max_{t\in[a,b]} \operatorname{ess\,sup} |u(t)|$;
- $AC([a,b],\mathbb{R}^n)$ is the Banach space of absolutely continuous functions $[a,b] \to \mathbb{R}^n$ with the norm $AC([a,b],\mathbb{R}^n) \ni u \to \int_0^l \|u'(\xi)\| d\xi + \|u(0)\|$.

Definition 2.1. By a *solution* of linear boundary-value problem (2.1), (2.2) we understand an absolutely continuous vector-function $u : [a,b] \to \mathbb{R}^n$ possessing property (2.2) and satisfying FFDE (2.1) for almost all t from the interval $[a,b]$.

Definition 2.2 ([2])**.** For a function u given on the interval $[a,b]$ *the Caputo derivative* of fractional order q is defined by

$$D_a^q u(t) = \frac{1}{\Gamma(1-q)} \Big(\frac{d}{dt}\Big) \int_a^t (t-s)^{-q} \big(u(s) - u(a)\big)\, ds\,,\ 0 < q < 1,$$

where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ is Gamma-function:

$$\Gamma(q) := \int_0^\infty t^{q-1} e^{-t}\, dt\,. \tag{2.3}$$

Definition 2.3 ([4])**.** For certain given $\{\sigma_1, \sigma_2, \dots, \sigma_n\} \subset \{-1, 1\}$

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{pmatrix} \tag{2.4}$$

an operator $l \colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is *σ-positive operator* if the fact that the relation

$$\sigma u(t) \geq 0\,, \quad t \in [a,b] \tag{2.5}$$

is true implies that

$$\sigma(lu)(t) \geq 0\,, \quad \text{for a.e.} \quad t \in [a,b]\,. \tag{2.6}$$

3. Auxiliary statements

Lemma 3.1 ([9, Lemma 2.21 and Lemma 2.22]). *Let* $0 < q < 1$ *and let* $u(t) \in C([a,b],\mathbb{R}^n)$ *or* $u(t)$ *belongs to the space of essentially bounded measurable functions* $L_\infty([a,b],\mathbb{R}^n)$, *then*

$$D_a^q I_a^q u(t) = u(t) \quad \text{almost everywhere on} \quad [a,b].$$

If $u(t) \in C^1([a,b],\mathbb{R}^n)$ *or* $u(t) \in AC([a,b],\mathbb{R}^n)$, *then*

$$I_a^q D_a^q u(t) = u(t) - u(a) \quad \text{almost everywhere on} \quad [a,b]\,,$$

where

$$I_a^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s)\, ds\,,$$

and Γ*-function is defined by* (2.3).

Taking into account Definition 2.1, Lemma 3.1 and relation (2.3) the next obvious Lemma is fulfilled.

Lemma 3.2. *The problem* (2.1), (2.2) *on* $[a,b]$ *is equivalent to the equation*

$$u(t) = u(a) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (lu)(s) ds + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)\, ds\,.$$

Lemma 3.3 ([12, the Fredholm alternative, Corollary from Theorem VI.14]). *The nonhomogeneous problem* (2.2) *for linear FFDE* (2.1) *is uniquely solvable if the corresponding homogeneous problem*

$$u(a) = 0 \tag{3.1}$$

for linear FFDE

$$D_a^q u(t) = (lu)(t)\,, \quad t \in [a,b]\,, \tag{3.2}$$

only has a trivial solution.

Let us fix $r \in \mathbb{N}$ and constants $\{h_1, h_2, \dots, h_r\} \in (0, +\infty)$ and introduce the sequence of functions

$$y_k(t) := \frac{\sum_{i=1}^r h_i}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l y_{k-i})(s)\, ds\,, \quad k \geq r\,, \quad t \in [a,b]\,, \tag{3.3}$$

where $\{y_0, y_1, \dots y_{r-1}\} \in AC([a,b],\mathbb{R}^n)$ chosen so that

$$\sigma y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad k = 0, 1, \dots, r-1\,, \tag{3.4}$$

and

$$y_k(a) = 0\,, \quad k = 0, 1, \dots, r-1\,. \tag{3.5}$$

Remark 3.4. If $r = 1$ and $h_1 = 1$, equality (3.3) takes the form

$$y_k(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l y_{k-1})(s)\, ds\,, \quad t \in [a,b]\,, \quad k \in \mathbb{N}\,, \tag{3.6}$$

and thus coincides with the sequence studied, e.g., in [4]. Formula (3.6) defines the standard iteration sequence used in studies of the uniqueness of the trivial solution of the integral fractional functional equation $y(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}(ly)(s)ds$, $t \in [a,b]$, which, because of Lemma 3.2, is equivalent to the homogeneous problem (3.1), (3.2).

Next, we will need the following technical Lemmas.

Lemma 3.5. *Suppose that the operator $l\colon AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is σ-positive. Then, for arbitrary absolutely continuous functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ satisfying conditions* (3.4)*,* (3.5)*, the corresponding functions $y_r, y_{r+1}, \dots$ defined by formulae* (3.3) *also satisfy conditions* (3.4)*,* (3.5)*:*

$$\sigma y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad y_k(a) = 0\,, \quad k \geq r\,.$$

Proof of Lemma 3.5. In view of (3.3), we have

$$y_r(t) := \frac{\sum_{i=1}^r h_i}{\Gamma(q)} \int_a^t (t-s)^{q-1}(ly_{r-i})(s)\,ds\,, \quad t \in [a,b]\,. \tag{3.7}$$

Taking into account the σ-positivity of the operator l and the non-negativeness of the coefficients $h_1, h_2, \dots, h_r$ in formula (3.3) and condition (3.4) yields $\sigma(ly_{r-i})(t) \geq 0$, $t \in [a,b]$. By induction, it is easy to show that (3.7) is fulfilled for all $k \geq r$. The property $y_k(a) = 0$ for all $k = 0,1,2,\dots,m$ is obvious from condition (3.4) and formula (3.5). □

Lemma 3.6. *For arbitrary vectors $x_0, x_1, \dots, x_m$ from $\mathbb{R}^n$, and some constants $\{\theta_k\}_{k=1}^m \subset [0,+\infty)$, the equality*

$$\sum_{k=r}^{m} \theta_k \sum_{i=1}^{r} h_i x_{k-i} = \sum_{j=0}^{m-1} \mu_j x_j \tag{3.8}$$

is fulfilled, where

$$\mu_k = \sum_{\nu \in T_{r,m}(k)} \theta_{\nu+k} h_\nu\,, \quad k = 0,1,\dots,m-1\,, \tag{3.9}$$

$$\text{and} \quad T_{r,m}(k) = \{\nu \in \mathbb{N} | \ \nu \leq r \leq \nu + k \leq m\}\,, \quad r \in \mathbb{N}\,.$$

4. General theorem

Theorem 4.1. *Suppose that operator l is σ-positive. Assume also that for some integers r and m, $m \geq r \geq 1$, a real number $\rho \in (1,+\infty)$, some constants $\{\theta_k\}_{k=1}^m \subset [0,+\infty)$ and $\{h_i\}_{i=1}^r \subset [0,+\infty)$, and certain absolutely continuous vector-functions $y_0, y_1, \dots, y_{r-1}$ satisfying conditions* (3.4)*,* (3.5)*, and the relation*

$$\sigma \sum_{k=0}^{r} \theta_k y_k(t) > 0 \quad \text{for all} \quad t \in (a,b] \tag{4.1}$$

such that the functional differential inequality

$$\sigma\Big(\sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{r}\Big(\sum_{j \in T_{r,m}(k)} \theta_{j+k} h_j - \rho\theta_k\Big)(l_k y)(t) - \rho\theta_m (l_m y)(t)\Big) \geq 0 \tag{4.2}$$

is fulfilled for a.e. t from $[a,b]$, $r \in \mathbb{N}$, $m \geq r$ and y_k, $k \geq r$ defined by (3.3).

Then, the homogenous linear initial value problem (3.1), (3.2) *only has a trivial solution and the nonhomogeneous linear Cauchy problem* (2.1), (2.2) *is uniquely solvable for an arbitrary $c \in \mathbb{R}$ and an arbitrary function $f \in C([a,b],\mathbb{R}^n)$.*

The unique solution of the problem (2.2) *for the equation* (2.1) *is representable in the form of a uniformly convergent on $[a,b]$ functional series*

$$u(t) = f_c(t) + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}(lf_c)(s)\,ds \\ + \frac{1}{\Gamma(q)}\int_a^t (t-\cdot)^{q-1} l\Big(\frac{1}{\Gamma(q)}\int_a^{\cdot}(t-s)^{q-1}(lf_c)(\xi)d\xi\Big)(s)\,ds + \dots,$$

where $f_c(t) := c + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s)\,ds$.

If, furthermore, the inequality $\sigma\Big(c + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s)\,ds\Big) \geq 0$, *is true a.e. on $[a,b]$, then the unique solution $u(\cdot)$ of the initial value problem* (2.2) *for FFDE* (2.1) *satisfy the condition* (2.5).

Proof. To prove Theorem 4.1 we need Theorem 4 from [4].

Theorem 4.2 ([4, Theorem 4]). *Assume that the linear operator $l = (l_k)_{k=1}^n$ in equation* (2.1) *is σ-positive. Suppose that there exist such a number $\rho > 1$ an function $y \in AC([a,b],\mathbb{R}^n)$ with properties*

$$y(a) = 0, \quad \sigma y(t) > 0 \quad \textit{for} \quad t \in (a,b]\,, \tag{4.3}$$

and a certain integer $k \geq 0$ that the components of the function $(y_{k,\nu})_{\nu=1}^n$ of the respective function y_k are continuous.

Additionally, the following fractional functional differential inequality

$$\sigma\Big(D_a^q y(t) - \rho(ly)(t)\Big) \geq 0 \textit{ for a.e. } t \in [a,b] \tag{4.4}$$

is fulfilled.

Then the assertion of Theorem 4.1 is true for the inhomogeneous (2.1), (2.2) *and homogeneous* (3.1), (3.2) *Cauchy problems.*

We consider certain absolutely continuous vector-functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ and construct the corresponding sequence of the functions $\{y_k\}_{k=0}^{r-1} : [a,b] \to \mathbb{R}^n$ according to formula (3.3) for $m \geq r$. Next, we introduce the function

$$y(t) = \sum_{k=0}^{m} \theta_k y_k(t)\,, \quad t \in [a,b]\,, \tag{4.5}$$

with the coefficients $\{\theta_k\}_{k=0}^m \in [0,+\infty)$ determined by the assumptions of the theorem. Note that, in view of (3.4), assumption (4.1) implies that (4.3) holds. Let us show that, under our assumptions, function (4.5) satisfies inequality (4.4). Taking into account (4.5), the corresponding function

$$\omega(t) := D_a^q y(t) - \rho(ly)(t) \tag{4.6}$$

has the form $\omega(t) := \sum_{k=0}^{m} \theta_k \big(D_a^q y_k(t) - \rho(l_k y)(t)\big)$, whence

$$\omega(t) := \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=r}^{m} \theta_k D_a^q y_k(t) - \rho \sum_{k=0}^{m} \theta_k (l_k y)(t)\,, \quad m \geq r\,. \tag{4.7}$$

In view of formula (3.3) and Lemma 3.1 for the functions $y_r, y_{r+1}, \dots,$ we have $D_a^q y_k(t) = \sum_{i=1}^{r} h_i(l y_{k-i})(t)$, $t \in [a,b]$, $k \geq r$, and, therefore, equality (4.7) can be rewritten

$$\omega(t) := \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=r}^{m} \theta_k \sum_{i=1}^{r} h_i (l y_{k-i})(t) - \rho \sum_{k=0}^{m} \theta_k (l y_k)(t)\,. \tag{4.8}$$

Taking into account Lemma 3.6, formula (4.8) can be rewritten as

$$\begin{aligned}\omega(t) &:= \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} \mu_k (l y_k)(t) - \rho \sum_{k=0}^{m} \theta_k (l y_k)(t) \\ &= \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} (\mu_k - \rho\theta_k)(l y_k)(t) - \rho\theta_m (l y_m)(t)\,,\end{aligned} \tag{4.9}$$

where $\mu_0, \mu_1, \dots \mu_{m-1}$ are given by relation (3.9). In view of (3.9), equality (4.9) is equivalent to the relation

$$\omega(t) = \sum_{k=0}^{r-1} \theta_k D_a^q y_k(t) + \sum_{k=0}^{m-1} \Big(\sum_{\nu \in T_{r,m}(k)} \theta_{\nu+k} h_\nu - \rho\theta_k \Big)(l y_k)(t) - \rho\theta_m (l y_m)(t)\,.$$

Hence, relation (4.2) guarantees that function (4.6) satisfies the condition $\sigma\omega \geq 0$ for a. e. $t \in [a,b]$, i. e., the fractional functional differential inequality (4.4) holds for the function y given by (4.5). It is obvious, that constructed in such way, y is a solution of the differential inequality (4.4).

To apply Theorem 4.2 we need to show that, under our assumptions, the solution mentioned possesses properties (3.4).

In view of Lemma 3.5, σ-positiveness of the operator l and non-negativity of all constants θ_k, $k = 0, 1, \dots, m$ the inequality

$$\sigma\theta_k y_k(t) \geq 0\,, \quad t \in [a,b]\,, \quad k = 0, 1, \dots, m\,, \tag{4.10}$$

is satisfied.

It follows from (4.10) that $\sigma\big(\sum_{k=0}^{m} \theta_k y_k(t) - \sum_{k=0}^{r-1} \theta_k y_k(t)\big) = \sigma \sum_{k=r}^{m} \theta_k y_k(t) \geq 0$, for $t \in [a,b]$, $k = 0, 1, \dots, m$, and, hence,

$$\sigma \sum_{k=0}^{m} \theta_k y_k(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t) \quad t \in [a,b]\,, \quad k = 0, 1, \dots, m\,. \tag{4.11}$$

Inequality (4.11) yields $\sigma y(t) = \sigma \sum_{k=0}^{m} \theta_k y_k(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t))$, $t \in [a,b]$ whence, under the assumption (4.1), we obtain $\sigma y(t) \geq \sigma \sum_{k=0}^{r-1} \theta_k y_k(t)) > 0$, $t \in (a,b]$ i.e., y satisfies condition (4.3). Thus, we have shown that function (4.5) satisfies the fractional functional differential inequality (4.4) and possesses properties (4.3) i. e., the assumptions of Theorem 4.2 are satisfied. Application of Theorem 4.2 leads us to the assertion required. □

Remark 4.3. Condition (4.2) appearing in the Theorem 4.1 presented are unimprovable in the sense that, generally speaking, that condition can not be assumed with $\rho = 1$. To check this, one can use, e.g., example 1 from [4].

5. Pantograph type model

Let us consider problem (2.1), (2.2) in view

$$D_0^q u(t) = \sum_{i=1}^{m} P_i(t)u(\lambda_i t) + f(t)\,, \quad t \in [0,1]\,, \quad u(0) = c\,, \tag{5.1}$$

where

$$(lu)(t) := \sum_{i=1}^{m} P_i(t)u(\lambda_i t)\,, \quad \text{and} \quad P_i(t) \quad := \begin{pmatrix} p_{11}^i(t) & p_{12}^i(t) & \dots & p_{1n}^i(t) \\ p_{21}^i(t) & p_{22}^i(t) & \dots & p_{2n}^i(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^i(t) & p_{n2}^i(t) & \dots & p_{nn}^i(t) \end{pmatrix} \tag{5.2}$$

have continuous components and $\lambda_i \in (0,1)$, $m \in \mathbb{N}$, $f \in C([0,1], \mathbb{R}^n)$.

Equation (5.1) is a famous equation called the pantograph type equation arising in electrodynamics [11]. The pantograph is a device used in electric trains to collect electric current from the overload lines.

Now let us establish exact conditions sufficient for the unique solvability of the initial value problem (5.1).

Theorem 5.1. *Suppose that*

$$\sigma P_i(t)\sigma \geq 0 \quad \textit{for almost all} \quad t \in [0,1], \quad 1 \leq i \leq m, \tag{5.3}$$

is fulfilled, where every P_i, $i = 1, \dots, n$, are defined by (5.2) *and have continuous components, σ is defined by* (2.4)*, and assume that there exists a real number $\rho > 1$ such that the functional differential inequality*

$$\sigma(y_0(t) + y_1(t) - \rho y_2(t)) \geq 0 \tag{5.4}$$

is satisfied for almost all t from $[0,1]$ and increasing functions y_0 and y_1 with properties (3.4)*,* (3.5)*,* (4.1) *and*

$$y_2(t) = \frac{\rho+1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sum_{i=1}^{m} P_i(s)(y_0(\lambda_i s) + y_1(\lambda_i s))ds. \tag{5.5}$$

Then, the assertion of Theorem 4.1 is true for the problem (5.1)*.*

Proof. Let us consider the function y from (4.5) in view:

$$y(t) = \theta y_0(t) + \theta y_1(t) + \theta y_2(t), \quad t \in [0,1] \quad \theta \in (0, +\infty),$$

where y_2 defined by (5.5) and increasing $\{y_0, y_1\}$ chosen so that (3.4), (3.5) and (4.1) are fulfilled. Obviously, $D_0^q y_2(t) = (\rho+1) \sum_{i=1}^{m} P_i(s)\ (y_0(\lambda_i t) + y_1(\lambda_i t))$, where

$h_0 = h_1 = \rho + 1$. Let us consider (4.8) with $r = 2$, $m = 2$, then

$$\omega(t) = \theta D_0^q y_0(t) + \theta D_0^q y_1(t) + \theta \sum_{i=1}^{m} P_i(s)\big(y_0(\lambda_i t) + y_1(\lambda_i t) - \rho y_2(\lambda_i t)\big)\,. \tag{5.6}$$

By the σ-positivity of the operator (5.2) (see condition (5.3) and Lemma 9 from [4]), inequality (5.4) and properties (3.4), (3.5), (4.1) for increasing functions y_0, y_1 we get that continuous function $\omega(t)$ from (5.6) implies the condition (4.2) from Theorem 4.1. The application of that theorem to the initial-value problem (5.1) and corresponding homogeneous problem implies the assertions required. Theorem 5.1 is proved. □

Remark 5.2. It is shown above that condition (5.4), which was obtained from complicated formula (4.2), is much more simple for an application then the inequality (25) from [4].

Acknowledgement. This research was supported in part by the Slovak Grant Agency VEGA-SAV, No. 2/0127/20. The author thanks the reviewer for the constructive comments.

References

[1] Aphithana, A., Ntouyas, S.K., Tariboon, J., *Existence and uniqueness of symmetric solutions for fractional differential equations with multi-point fractional integral conditions*, Bound. Value Probl. **2015** (68) (2015), 14 pp., https://doi.org/10.1186/s13661-015-0329-1.

[2] Diethelm, K., *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, 2010.

[3] Dilna, N., *Exact solvability conditions for the model with a discrete memory effect*, International Conference on Mathematical Analysis and Applications in Science and Engineering, Book of Extended Abstracts, 2022, 405–407 pp.

[4] Dilna, N., Fečkan, M., *Exact solvability conditions for the non-local initial value problem for systems of linear fractional functional differential equations*, Mathematics **10** (10) (2022), 1759, https://doi.org/10.3390/math10101759.

[5] Dilna, N., Gromyak, M., Leshchuk, S., *Unique solvability of the boundary value problems for nonlinear fractional functional differential equations*, J. Math. Sci. **265** (2022), 577–588, https://doi.org/10.1007/s10958-022-06072-8.

[6] Fečkan, M., Marynets, K., *Approximation approach to periodic BVP for fractional differential systems*, The European Physical Journal Special Topics **226** (2017), 3681–3692, https://doi.org/10.1140/epjst/e2018-00017-9.

[7] Fečkan, M., Wang, J.R., Pospíšil, M., *Fractional-Order Equations and Inclusions*, 1st. ed., Walter de Gruyter GmbH, Berlin, Boston, 2017.

[8] Gautam, G.R., Dabas, J., *A study on existence of solutions for fractional functional differential equations*, Collect. Math **69** (2018), 25–37.

[9] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier B.P., 2006.

[10] Opluštil, Z., *New solvability conditions for a non-local boundary value problem for nonlinear functional-differential equations*, Nonlinear Oscil. **11** (3) (2008), 365–386.

[11] Patade, J., Bhalekar, S., *Analytical solution of pantograph equation with incommensurate delay*, Phys. Sci. Rev. Inform. **9** (2017), 20165103 https://doi.org/10.1515/psr-2016-5103.

[12] Reed, M., Simon, B., *Methods of modern mathematical physics*, Acad. Press, New York-London, 1972.

[13] Rontó, A., Rontó, M., *Successive Approximation Techniques in Non-Linear Boundary Value Problems*, Handbook of Differential Equations: Ordinary Differential Equations, Elsevier, New York, 2009, pp. 441–592.

[14] Šremr, J., *Solvability conditions of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators*, Math. Bohem. **132** (2007), 263–295.

Institute of Mathematics, Slovak Academy of Sciences,
Štefánikova 49, 814 73 Bratislava, Slovakia
E-mail: dilna@mat.savba.sk