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A NOTE ON THE OSCILLATION PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH $p(t)$-LAPLACIAN

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Abstract. This paper deals with the oscillation problems on the nonlinear differential equation $(a(t)|x'|^{p(t)-2}x')'+b(t)|x|^{\lambda-2}x=0$ involving $p(t)$-Laplacian. Sufficient conditions are given under which all proper solutions are oscillatory. In addition, we give *a-priori* estimates for nonoscillatory solutions and propose an open problem.

1. Introduction

We consider the second-order nonlinear differential equation

$$(a(t)|x'|^{p(t)-2}x')' + b(t)|x|^{\lambda-2}x = 0, \quad t \geq t_0\,, \tag{1.1}$$

where $a(t)$, $b(t)$, and $p(t) > 1$ are positive continuous functions and $\lambda > 1$ is a constant. In addition, we assume

$$\limsup_{t\to\infty} a(t) < \infty, \tag{1.2}$$

that is, there exists $\alpha > 0$ such that $a(t) < \alpha$ for $t \geq t_0$.

Note that the differential operator in equation (1.1) is called $p(t)$-Laplacian. Such operator appears in mathematical models in a wide range of research fields such as nonlinear elasticity theory, electrorheological fluids, and image processing (see [2, 10, 14]). In recent years, increasing interest has been paid to the study of ordinary differential equations with $p(t)$-Laplacian. For example, we can find those results in [1, 4, 8, 9, 15, 16, 17, 18] and the references cited therein.

A function $x(t)$ is said to be a *solution* of equation (1.1) defined on (t_0, τ), if $x(t)$ and its quasiderivative

$$x^{[1]}(t) = a(t)|x'(t)|^{p(t)-2}x'(t)$$

are continuously differentiable, and $x(t)$ satisfies equation (1.1) on (t_0, τ). We study solutions of equation (1.1) which are defined on (t_0, τ); if $\tau < \infty$ then we suppose

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that $x(t)$ is nonextendable to the right, i.e.,

$$\limsup_{t \to \tau -} (|x(t)| + |x'(t)|) = \infty \,.$$

A nontrivial solution $x(t)$ of equation (1.1) is said to be a *singular solution of the first kind*, if there exists $T_x > t_0$ such that $x(t) \equiv 0$ for $t \geq T_x$. It is said to be a *singular solution of the second kind*, if $\tau < \infty$. It is said to be a *proper solution* if $x(t)$ is not singular. The existence of proper solutions for equation (1.1) can be referred to in [1]. A proper solution $x(t)$ of equation (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*.

Let $p(t) \equiv p > 1$. Then equation (1.1) becomes the so-called generalized Emden-Fowler differential equation

$$(a(t)|x'|^{p-2}x')' + b(t)|x|^{\lambda-2}x = 0, \quad t \geq t_0 \tag{1.3}$$

with the classical p-Laplacian. It is known that the study of equation (1.3) originates from gas dynamics in astrophysics. Moreover, asymptotic behavior of solutions of equation (1.3) corresponds to the concentration of a substance disappearing according to an isothermal reaction in an finite slab of catalyst (see [13]). Hence, a lot of papers have been devoted to the study of equation (1.3) (see [3, 5, 6, 7, 11, 12, 13]). Especially, on the oscillation problems, the following theorem is proved in [12].

Theorem A. All proper solutions of equation (1.3) with $a(t) \equiv 1$ are oscillatory if

$$\int^{\infty} b(t)\, dt = \infty \,. \tag{1.4}$$

According to the proof of Theorem A, we can easily get the analogue for equation (1.1) when $\liminf_{t\to\infty} p(t) > 1$ under (1.2). Here, a natural question now arises: Are all proper solutions of equation (1.1) oscillatory when $\lim_{t\to\infty} p(t) = 1$? The purpose of this paper is to answer the question. To be precise, we give sufficient conditions under which all proper solutions of equation (1.1) are oscillatory. Our main result is stated as follows.

Theorem 1.1. *Assume* (1.2). *Suppose that there exists a constant $c > 0$ such that*

$$p(t) \geq 1 + \frac{c}{\log\log t} \,. \tag{1.5}$$

Then, all proper solutions of equation (1.1) *are oscillatory if* (1.4) *holds.*

Remark 1.2. Theorem 1.1 contains not only the case of $\lim_{t\to\infty} p(t) = 1$, but also the case of $\liminf_{t\to\infty} p(t) > 1$.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we give some remarks and open problems.

2. Proof of the main theorem

In this section, we give the proof of Theorem 1.1. We begin with the following lemma.

Lemma 2.1. *Assume* (1.2) *and* (1.5). *Let* $y \in C^1[t_0, \infty)$ *be a function satisfying* $y(t) \neq 0$ *for* $t \geq t_0$. *Then, for any* $T \geq t_0$,

$$\limsup_{t\to\infty} \left\{ \frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} + (\lambda-1)\int_T^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds \right\} \geq 0 \tag{2.1}$$

holds.

Proof. Suppose, toward a contradiction, that (2.1) is false. Then, there exist constants $k > 0$ and $T' > t_0$ such that

$$\frac{k|y(T')|^{\lambda-1}}{\alpha} < 1 \tag{2.2}$$

and

$$\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} + (\lambda-1)\int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds \leq -k\,,$$

that is,

$$k + (\lambda-1)\int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds \leq -\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)} \tag{2.3}$$

for $t \geq T'$. Since the left-hand side of (2.3) is positive, we see that $y(t)y'(t) < 0$ for $t \geq T'$.

Dividing (2.3) by its left-hand side and multiplying by $-y'(t)/y(t)$, we get

$$-\frac{y'(t)}{y(t)} \leq \frac{a(t)|y'(t)|^{p(t)}/|y(t)|^{\lambda}}{k + (\lambda-1)\int_{T'}^t \left\{a(s)|y'(s)|^{p(s)}/|y(s)|^{\lambda}\right\}\,ds},$$

and therefore, we obtain

$$-(\log|y(t)|)' \leq \frac{1}{\lambda-1}\left(\log\left(k + (\lambda-1)\int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds\right)\right)'$$

for $t \geq T'$. Integrating the both sides of this inequality from T' to t, we have

$$\begin{aligned} &-\log|y(t)| + \log|y(T')| \\ &\leq \frac{1}{\lambda-1}\left(\log\left(k + (\lambda-1)\int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds\right) - \log k\right) \end{aligned}$$

for $t \geq T'$. Hence, we obtain

$$k\left|\frac{y(T')}{y(t)}\right|^{\lambda-1} \leq k + (\lambda-1)\int_{T'}^t \frac{a(s)|y'(s)|^{p(s)}}{|y(s)|^{\lambda}}\,ds \tag{2.4}$$

for $t \geq T'$.

From (2.3) and (2.4), we have

$$k\left|\frac{y(T')}{y(t)}\right|^{\lambda-1} \leq -\frac{a(t)|y'(t)|^{p(t)-2}y'(t)}{|y(t)|^{\lambda-2}y(t)}\,,$$

that is,

$$k|y(T')|^{\lambda-1} \leq -a(t)|y'(t)|^{p(t)-2}y'(t)\,\mathrm{sgn}\,y(t)$$

for $t \geq T'$. Using (1.2) and $y(t)y'(t) < 0$ for $t \geq T'$, we get

$$k|y(T')|^{\lambda-1} \leq a(t)|y'(t)|^{p(t)-1} \leq \alpha|y'(t)|^{p(t)-1}\,,$$

and therefore, we obtain

$$|y'(t)| \geq \left(\frac{k|y(T')|^{\lambda-1}}{\alpha}\right)^{1/(p(t)-1)} \tag{2.5}$$

for $t \geq T'$. We note that

$$c_0 := \frac{k|y(T')|^{\lambda-1}}{\alpha} < 1$$

from (2.2).

According to (1.5), we have

$$\frac{1}{p(t)-1} \leq \frac{\log\log t}{c},$$

and hence, we obtain

$$c_0^{1/(p(t)-1)} \geq c_0^{(\log\log t)/c} = (\log t)^{(\log c_0)/c} = \left(\frac{1}{\log t}\right)^{|\log c_0|/c}.$$

Together with (2.5), we get

$$|y'(t)| \geq \left(\frac{1}{\log t}\right)^{|\log c_0|/c}$$

for $t \geq T'$. In the case of $y(t) > 0$, this implies

$$y'(t) \leq -\left(\frac{1}{\log t}\right)^{|\log c_0|/c}.$$

Integrating the both sides of this inequality, we get

$$\lim_{t\geq\infty} y(t) - y(T') \leq -\int_{T'}^{\infty}\left(\frac{1}{\log t}\right)^{|\log c_0|/c} dt = -\infty\,.$$

This is a contradiction. In the case of $y(t) < 0$, as in the same manner in the previous case, we obtain

$$\lim_{t\to\infty} y(t) = \infty\,,$$

which is a contradiction. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose, toward a contradiction, that equation (1.1) has a nonoscillatory solution $x(t)$. Then, from Lemma 2.1, we have (2.1) with $y(t) = x(t)$. Without loss of generality, we may assume $x(t) \neq 0$ for $t \geq t_0$. Then, we can calculate

$$(|x(t)|^{\lambda-2}x(t))' = (\lambda-1)x'(t)|x(t)|^{\lambda-2}$$

and

$$\frac{(a(t)|x'(t)|^{p(t)-2}x'(t))'}{|x(t)|^{\lambda-2}x(t)} = \left(\frac{a(t)|x'(t)|^{p(t)-2}x'(t)}{|x(t)|^{\lambda-2}x(t)}\right)' + (\lambda-1)\frac{a(t)|x'(t)|^{p(t)}}{|x(t)|^{\lambda}}\,. \tag{2.6}$$

Dividing equation (1.1) by $|x(t)|^{\lambda-2}x(t)$, we have

$$\frac{(a(t)|x'(t)|^{p(t)-2}x'(t))'}{|x(t)|^{\lambda-2}x(t)} = -b(t)\,.$$

Integrating the both sides of this inequality from t_0 to t, we get

$$\int_{t_0}^{t} \frac{(a(s)|x'(s)|^{p(s)-2}x'(s))'}{|x(s)|^{\lambda-2}x(s)}\,ds = -\int_{t_0}^{t} b(s)\,ds\,.$$

Together with (2.6), we obtain

$$\begin{aligned}&\frac{a(t)|x'(t)|^{p(t)-2}x'(t)}{|x(t)|^{\lambda-2}x(t)} + (\lambda-1)\int_{t_0}^{t}\frac{a(s)|x'(s)|^{p(s)}}{|x(s)|^{\lambda}}\,ds\\ &= \frac{a(t_0)|x'(t_0)|^{p(t_0)-2}x'(t_0)}{|x(t_0)|^{\lambda-2}x(t_0)} - \int_{t_0}^{t} b(s)\,ds \to -\infty\end{aligned}$$

as $t\to\infty$. This is a contradiction to (2.1). □

3. Discussion and remarks

From Theorem 1.1, we see that if (1.5) holds (that is to say, $p(t)$ tends to 1 more slowly than $1/\log\log t$) then there are no nonoscillatory solutions. On the other hand, the nonexistence of nonoscillatory solutions is not guaranteed when $p(t)$ tends to 1 so rapidly. Hence, in this section, we consider the case when (1.5) is false.

If a nonoscillatory solution $x(t)$ of equation (1.1) is eventually negative, then $-x(t)$ is an eventually positive solution of equation (1.1). Hence, when we discuss nonoscillatory solutions, we focus only on eventually positive solutions, and let us simply call them *positive solutions*.

Let $x(t)$ be a positive solution. Then, from equation (1.1), $x^{[1]}(t)$ is decreasing. Therefore, we see that the sign of $x^{[1]}(t)$ is eventually constant, that is, $x(t)$ has a monotonicity for large t. The following proposition shows the *a-priori* estimate for nonoscillatory solutions for equation (1.1).

Proposition 3.1. *Assume* (1.4). *If there exists a nonoscillatory solution $x(t)$ of equation* (1.1)*, then $x(t)$ is decreasing to* 0 *as $t\to\infty$.*

Proof. We first suppose that $x(t)$ is nondecreasing. Integrating equation (1.1) from t_0 to ∞, we get

$$\begin{aligned}(x(t_0))^{\lambda-1}\int_{t_0}^{\infty} b(t)\,dt \le \int_{t_0}^{\infty} b(t)(x(t))^{\lambda-1}\,dt &= -\lim_{t\to\infty} x^{[1]}(t) + x^{[1]}(t_0)\\ &< x^{[1]}(t_0) < \infty\,,\end{aligned}$$

which is a contradiction.

We next suppose that there exists a constant $c_1>0$ such that $x(t)$ is decreasing to c_1. Then, we have

$$-x^{[1]}(t) > -x^{[1]}(t) + x^{[1]}(t_0) = \int_{t_0}^{t} b(s)(x(s))^{\lambda-1}\,ds \ge c_1^{\lambda-1}\int_{t_0}^{t} b(s)\,ds\,,$$

that is to say,

$$-x'(t) \geq \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\, ds \right)^{1/(p(t)-1)}$$

for $t \geq t_0$. Integrating the both sides of this inequality from t_0 to ∞, we obtain

$$\int_{t_0}^{\infty} \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\, ds \right)^{1/(p(t)-1)} dt \leq -\lim_{t\to\infty} x(t) + x(t_0) = -c_1 + x(t_0) < \infty\,.$$

On the other hand, from (1.2) and (1.4), we get

$$\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\, ds > 1$$

for t sufficiently large, which implies that

$$\int_{t_0}^{\infty} \left(\frac{c_1^{\lambda-1}}{a(t)} \int_{t_0}^{t} b(s)\, ds \right)^{1/(p(t)-1)} dt = \infty\,.$$

This is a contradiction. □

We finally propose the following open problem: Does equation (1.1) have a nonoscillatory solution which is decreasing to 0 as $t \to \infty$ in the case when (1.5) is false? If the nonexistence of such a solution is proved, then the condition (1.5) can be removed from Theorem 1.1. Otherwise, it will be a discrepancy between equations (1.1) and (1.3).

References

[1] Bartušek, M., Fujimoto, K., *Singular solutions of nonlinear differential equations with $p(t)$-Laplacian*, J. Differential Equations **269** (2020), 11646–11666.

[2] Berselli, L.C., Breit, D., Diening, L., *Convergence analysis for a finite element approximation of a steady model for electrorheological fluids*, Numer. Math. **132** (2016), 657–689.

[3] Chanturia, T.A., *On singular solutions of strongly nonlinear systems of ordinary differential equations*, Function theoretic methods in differential equations, Res. Notes in Math., no. 8, Pitman, London, 1976, pp. 196–204.

[4] Došlá, Z., Fujimoto, K., *Asymptotic behavior of solutions to differential equations with $p(t)$-Laplacian*, Commun. Contemp. Math. **24** (2022), 1–22, No. 2150046.

[5] Došlá, Z., Marini, M., *On super-linear Emden–Fowler type differential equations*, J. Math. Anal. Appl. **416** (2014), 497–510.

[6] Došlá, Z., Marini, M., *Monotonicity conditions in oscillation to superlinear differential equations*, Electron. J. Qual. Theory Differ. Equ. **2016** (2016), 1–13, No. 54.

[7] Došlý, O., Řehák, P., *Half-Linear Differential Equations*, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2005.

[8] Fujimoto, K., *Power comparison theorem for oscillation problems for second order differential equations with $p(t)$-Laplacian*, Acta Math. Hungar. **162** (2020), 333–344.

[9] Fujimoto, K., Yamaoka, N., *Oscillation constants for Euler type differential equations involving the $p(t)$-Laplacian*, J. Math. Anal. Appl. **470** (2019), 1238–1250.

[10] Harjulehto, P., Hästö, P., Lê, Ú.V., Nuortio, M., *Overview of differential equations with non-standard growth*, Nonlinear Anal. **72** (2010), 4551–4574.

[11] Kiguradze, I.T., Chanturia, T.A., *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[12] Kitano, M., Kusano, T., *On a class of second order quasilinear ordinary differential equations*, Hiroshima Math. J. **25** (1995), 321–355.

[13] Mehta, B.N., Aris, R., *A note on a form of the Emden-Fowler equation*, J. Math. Anal. Appl. **36** (1971), 611–621.

[14] Rajagopal, K.R., Růžička, M., *On the modeling of electrorheological materials*, Mech. Res. Comm. **23** (1996), 401–407.

[15] Şahiner, Y., Zafer, A., *Oscillation of nonlinear elliptic inequalities with* $p(x)$*-Laplacian*, Complex Var. Elliptic Equ. **58** (2013), 537–546.

[16] Shoukaku, Y., *Oscillation criteria for half-linear differential equations with* $p(t)$*-Laplacian*, Differ. Equ. Appl. **6** (2014), 353–360.

[17] Shoukaku, Y., *Oscillation criteria for nonlinear differential equations with* $p(t)$*-Laplacian*, Math. Bohem. **141** (2016), 71–81.

[18] Zhang, Q., *Oscillatory property of solutions for* $p(t)$*-Laplacian equations*, J. Inequal. Appl. **2007** (2007), 1–8, 58548.

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