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STABLE PERIODIC SOLUTIONS IN SCALAR PERIODIC DIFFERENTIAL DELAY EQUATIONS

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Abstract. A class of nonlinear simple form differential delay equations with a T-periodic coefficient and a constant delay $\tau > 0$ is considered. It is shown that for an arbitrary value of the period $T > 4\tau - d_0$, for some $d_0 > 0$, there is an equation in the class such that it possesses an asymptotically stable T-period solution. The periodic solutions are constructed explicitly for the piecewise constant nonlinearities and the periodic coefficients involved, by reduction of the problem to one-dimensional maps. The periodic solutions and their stability properties are shown to persist when the nonlinearities are "smoothed" at the discontinuity points.

1. Introduction

Differential delay equations serve as mathematical models of various phenomena in numerous applications where delays are intrinsic features of their functioning. An extensive list of applications can be found in e.g. monographs [4, 8, 10] with further references therein. The theoretical basics of the equations are given in monographs [3, 6].

The scalar differential delay equation

$$x'(t) = -\mu x(t) + g\big(x(t-\tau)\big), \quad g \in C(\mathbb{R},\mathbb{R}), \quad \mu \geq 0 \tag{1.1}$$

is one of the simplest by its form and still exhibiting a variety of quite complex dynamics and a broad range of applications. Depending on the particular form of the nonlinearity g it is well-known under particular names such as Mackey-Glass model [2, 9], Lasota-Wazewska equation [12], Nicholson's blowflies model [1], some other named models [10].

Equations of form (1.1) were studied in numerous publications primarily with respect to the property of global asymptotic stability of the equilibrium and the existence of nontrivial periodic solutions. In the presence of the negative feedback property for the nonlinearity g and the instability of the linearized equation about

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the unique equilibrium it is shown that the equation typically possesses a nontrivial periodic solution slowly oscillating about the equilibrium. The standard techniques used to prove the existence of periodic solutions are the ejective fixed point theory with its modifications [3, 6]. The exact value of the periods for such periodic solutions is generally not known; it can be arbitrary and varies continuously under continuous changes of g and μ.

A natural extension of model (1.1) is the following equation with the periodic coefficient $a(t)$:

$$x'(t) = -\mu x(t) + a(t) f\big(x(t-\tau)\big)\,, \qquad t \geq 0\,, \tag{1.2}$$

where f, $a \in C(\mathbb{R}, \mathbb{R})$ and $a(t+T) \equiv a(t)$ for some positive period $T \geq \tau > 0$. The presence of the non-autonomous periodic input $a(t)$ can be justified in corresponding biological models by various factors, for example, by seasonal changes in the negative feedback [4, 8, 10].

Given equation (1.2) with the T-periodic coefficient $a(t)$ one can ask a natural question whether it admits periodic solutions with the same period. The primary objective of this note is to show that such periodic solutions exist for a wide continuous range of periods $T \geq 4\tau - \delta_0$ for some small $\delta_0 > 0$. The periodic solutions are constructed explicitly in terms of piece-wise constant functions f and a, and their continuous approximations. The special case of $\mu = 0$ is considered in this paper. The extension to the case of $\mu > 0$ is straightforward, however, it requires substantial additional space for adequate exposition and will be treated in a separate paper.

2. Preliminaries

Consider the scalar differential delay equation

$$x'(t) = a(t) f\big(x(t-\tau)\big)\,, \qquad t \geq 0\,, \tag{2.1}$$

with a T-periodic coefficient $a(t) \geq 0$, $a(t+T) \equiv a(t)$, $T \geq \tau$, and a nonlinearity $f(x)$ satisfying the negative feedback assumption $x \cdot f(x) < 0 \; \forall x \in \mathbb{R}, x \neq 0$.

For the continuous functions f and a the standard choice of the initial set for equation (2.1) is the Banach space of continuous functions on the initial interval $[-\tau, 0]$: $\mathbb{X} = C([-\tau, 0], \mathbb{R})$. For an arbitrary initial function $\phi \in \mathbb{X}$ there exists a unique solution $x(t) = x(t; \phi)$ to equation (2.1) defined for all $t \geq 0$. It is obtained by forward integration. At every $t \geq 0$ the solution $x(t)$ can be viewed as an element of space $\mathbb{X}$ by the following representation: $\mathbb{X} \ni x_t(s) := x(t+s), s \in [-\tau, 0]$.

For the initial basic construction of periodic solutions in Section 3 the functions f and a are piecewise constant. For arbitrary $\phi \in \mathbb{X}$ the corresponding solution $x(t; \phi)$ is explicitly built for all $t \geq 0$ by direct integration. It turns out to be a piecewise affine function differentiable everywhere except at discrete isolated set of points (in fact, it is a finite set of points on every finite interval $[0, t_0], t_0 > 0$).

We are interested in oscillation of solutions about the equilibrium $x(t) \equiv 0$. Sufficient conditions for the oscillation can be easily found in relevant available publications on the issue (see e.g. the monograph [5] and further references therein).

In particular, when $\sup_{t\in[0,T]}\int_t^{t+\tau} a(s)\,ds > -1/f'(0)$ all solutions of equation (2.1) oscillate. We shall assume this condition to hold throughout the paper.

Due to the negative feedback assumption the important role in the dynamics is played by the slowly oscillating solutions. A solution is called slowly oscillating if the distance between its any two zeros is greater than the delay $\tau > 0$. Any initial function $\phi \in \mathbb{X}$ such that $\phi(s) > 0\ \forall s \in [-\tau, 0]$ gives rise to a slowly oscillating solution, under the assumption of oscillation of all solutions.

Define two sets

$$K_+ := \{\phi \in \mathbb{X} \mid \phi(s) \geq 0\ \forall s \in [-\tau, 0],\ \phi \not\equiv 0\}$$

and

$$K_- := \{\phi \in \mathbb{X} \mid \phi(s) \leq 0\ \forall s \in [-\tau, 0],\ \phi \not\equiv 0\}\,.$$

It is a straightforward calculation to verify that for arbitrary $\phi \in K_+, \phi(0) \neq 0$, the corresponding solution $x(t;\phi)$ has an increasing sequence of zeros $0 < z_1 < z_2 < z_3 < \cdots$ such that $z_{k+1} - z_k > \tau, k \in \mathbb{N}$, and

$$x(t) < 0\ \ \forall t \in (z_{2k-1}, z_{2k}) \quad \text{and} \quad x(t) > 0\ \ \forall t \in (z_{2k}, z_{2k+1}).$$

Therefore, the solution $x(t;\phi)$ is slowly oscillating [3, 6]. Similar property is valid for any solution $x(t;\psi), \psi \in K_-, \psi(0) \neq 0$.

3. Piecewise constant nonlinearities

In this section we consider the particular case of equation (2.1) when $\tau = 1$:

$$x'(t) = a(t) f\big(x(t-1)\big)\,, \qquad t \geq 0\,. \tag{3.1}$$

Note that the case of general delay $\tau > 0$ can always be normalized to $\tau = 1$ by time rescaling $t = \tau \cdot s$. We start with the case when the nonlinearity f is the negative sign function

$$f(x) = f_0(x) = -\mathrm{sign}(x) = \begin{cases} +1 & \text{if } \ x < 0 \\ \ \ 0 & \text{if } \ x = 0 \\ -1 & \text{if } \ x > 0\,, \end{cases}$$

and the T-periodic coefficient $a(t)$ is a piecewise constant function defined by two positive constants a_1, a_2 as

$$a(t) = A_0(t) = \begin{cases} a_1 & \text{if } \ t \in [0, p_1) \\ a_2 & \text{if } \ t \in [p_1, p_1 + p_2) \\ \text{periodic extension on } \mathbb{R} \text{ outside interval } [0, T), T = p_1 + p_2\,, \end{cases}$$

where a_1, a_2, p_1, p_2 are all positive constants.

Due to the piecewise constant values of both $f(x)$ and $a(t)$ the forward solutions to (3.1) can be calculated explicitly; they are piecewise affine continuous functions for $t \geq 0$ differentiable everywhere except at a countable set of isolated points (where the solution changes slope). We will consider initial functions $\phi(s) \in C([-1, 0], \mathbb{R})$ which give rise to slowly oscillating solutions. Without loss of generality one can assume that $\phi \in K_+$ and $\phi(s) > 0\ \forall s \in [-1, 0]$. The corresponding solution

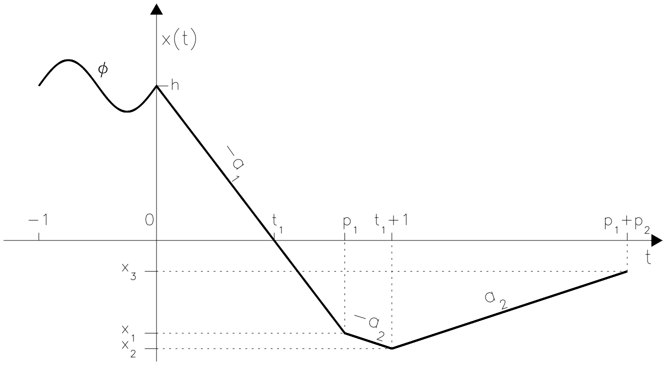

FIG. 1: Solution of Eq. (3.1) for $t \in [0, T]$.

$x(t) = x(t;\phi), t \geq 0$, depends only on the value $\phi(0) := h > 0$ and does not depend on the remaining values $\phi(s) > 0, s \in [-1, 0)$ on the initial interval.

Given $\phi \in K_+$ and its corresponding solution $x(t;\phi)$ we would like to find conditions such that its segment $x_{p_1+p_2}(s)$ belongs to the set K_-, in other words, such that the translation operator by the period $T = p_1 + p_2$ along the solutions maps the set K_+ into K_-.

The following is the explicit calculation of the solution $x(t;h)$, $t \geq 0$, for $h > 0$. See Figure 1 for the geometric representation of the solution.

On the interval $[0, p_1]$ the solution is given by $x(t) = h - a_1 t$. We assume that $x_1 := x(p_1) = h - a_1 p_1 < 0$. Therefore, there exists the unique value $t_1 = h/a_1 < p_1$ such that $x(t_1) = 0$.

We also assume that $p_1 - t_1 < 1$, implying $t_1 + 1 > p_1$. Then on the interval $[p_1, t_1 + 1]$ the solution x is given by $x(t) = x_1 - a_2(t - p_1)$. Set $x_2 := x(t_1 + 1) = (1 - a_2/a_1)h - a_2 + p_1(a_2 - a_1)$. Since $x_1 < 0$ and $a_2 > 0$ we also have that $x_2 < 0$ is valid. Therefore, the segment of the solution $x(t), t \in [t_1, t_1 + 1]$, belongs to the set K_-.

We next suppose that $t_1 + 1 < p_1 + p_2$. On the interval $[t_1 + 1, p_1 + p_2]$ the solution x is found as $x(t) = x_2 + a_2[t - (t_1 + 1)]$. Set $x_3 := x(p_1 + p_2)$ and additionally assume that $x_3 < 0$. The value of x_3 is easily calculated as

$$x_3 = x(p_1 + p_2) = \left(1 - 2\frac{a_2}{a_1}\right) h - a_1 p_1 - a_2(2 - 2p_1 - p_2) := F_1(h) = mh - b\,.$$

We note that the piecewise affine solution $x(t;h)$ is continuous on $[0, p_1 + p_2]$ and differentiable everywhere except at points $t = p_1$ and $t = t_1 + 1$.

Likewise, when $\psi \in K_-$ and $\psi(0) = h < 0$ analogous calculations yield

$$x_3 = x(p_1 + p_2;\psi) = \left(1 - 2\frac{a_2}{a_1}\right) h + a_1 p_1 + a_2(2 - 2p_1 - p_2) := F_2(h) = mh + b\,.$$

We would like to guarantee that the slope $m = 1 - 2a_2/a_1$ of the affine maps F_1, F_2 satisfies $|m| < 1$, and the y-intercept $b = a_1 p_1 + a_2(2 - 2p_1 - p_2) > 0$ is positive. An easy calculation leads to the following conclusion:

Proposition 3.1. *Suppose that* $a_2 < a_1$. *Then* $|m| = |1 - 2a_2/a_1| < 1$. *If in addition* $(a_1/a_2 - 2)p_1 > p_2 - 2$ *then* $b = a_1p_1 + a_2(2 - 2p_1 - p_2) > 0$ *is positive.*

Define the piecewise affine map F by

$$F(h) = \begin{cases} F_1(h) & \text{if } \ h > 0 \\ F_2(h) & \text{if } \ h < 0\,. \end{cases} \tag{3.2}$$

Under the assumptions of Proposition 3.1 map F has a unique attracting two-cycle $\{h_1^*, h_2^*\} = \{-b/(1+m), b/(1+m)\}$ which attracts all initial values $h \in \mathbb{R}$ when $-1 < m < 0\,(a_2 < a_1 < 2a_2)$ and all $h \in (-b/m, b/m)$ when $0 < m < 1\,(2a_2 < a_1)$. This two-cycle of F corresponds to asymptotically stable slowly oscillating periodic solution of equation (3.1) with the period $T = 2(p_1+p_2)$. It is also easy to see that the periodic solution has the following symmetry property $x(t+p_1+p_2) = -x(t) \ \ \forall t \in \mathbb{R}$. This is due to the fact that $F_2(-h) = -F_1(h), h > 0$.

The above consideration immediately implies the following

Corollary 3.2. *Suppose that* a_1, a_2, p_1, p_2 *satisfy the assumptions of Proposition 3.1. Then the corresponding equation* (3.1) *has a unique asymptotically stable slowly oscillating symmetric periodic solution* $x_*(t)$ *with the period* $T = 2(p_1 + p_2)$.

Consider next the case of equation (3.1) when $f(x) = f_0(x)$ and the piecewise constant coefficient $a(t)$ is defined by four constants as

$$a(t) = A_1(t) = \begin{cases} a_1 & \text{if } \ t \in [0, p_1) \\ a_2 & \text{if } \ t \in [p_1, p_1 + p_2) \\ a_3 & \text{if } \ t \in [p_1 + p_2, p_1 + p_2 + p_3) \\ a_4 & \text{if } \ t \in [p_1 + p_2 + p_3, p_1 + p_2 + p_3 + p_4) \\ \multicolumn{2}{l}{\text{periodic extension on } \mathbb{R} \text{ outside interval } [0, T),} \\ \multicolumn{2}{l}{T := p_1 + p_2 + p_3 + p_4\,,} \end{cases} \tag{3.3}$$

where a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4 are all positive constants.

Proposition 3.3. *Assume that the two quadruples* a_1, a_2, p_1, p_2 *and* a_3, a_4, p_3, p_4 *each satisfy the conditions of Proposition 3.1 and that in addition the inequalities* $m_1 = 1 - 2a_2/a_1 < b_1/b_2$, $m_2 = 1 - 2a_4/a_3 < b_2/b_1$ *are valid. Then the differential delay equation* (3.1) *possesses a unique asymptotically stable slowly oscillating periodic solution with the period* $T = p_1 + p_2 + p_3 + p_4$.

The validity of Proposition 3.3 is seen from the analogous construction of the piecewise affine map F similar to that given my (3.2), where F_1 is built of the quadruple a_1, a_2, p_1, p_2 while F_2 is derived from the quadruple a_3, a_4, p_3, p_4. The periodic solution is defined by the unique attracting 2-cycle of the interval map F, which is given explicitly by $(h_1^*, h_2^*) = ((m_1b_2 - b_1)/(1 - m_1m_2), (b_2 - m_2b_1)/(1 - m_1m_2))$, $h_1^* > 0$, $h_2^* < 0$. Such 2-cycle exists if the additional assumptions of the proposition on values of m_1, m_2, b_1, b_2 are met. Note that one of the inequalities for m_1, m_2 is satisfied by default.

Next we would like to see what values of the period T can be achieved for the stable periodic solutions, based on values $a_i, p_i, 1 \le i \le 4$, defining the piecewise constant coefficient $a(t)$. According to Proposition 3.1, one must first have that $a_1 > a_2$ and $a_3 > a_4$ are satisfied so that both $|m_1| < 1$, $|m_2| < 1$ are valid. Since we also require that $b_1 = p_1(a_1-2a_2)+a_2(2-p_2) > 0$, $b_2 = p_3(a_3-2a_4)+a_4(2-p_4) > 0$, both can be achieved if $a_1 > 2a_2$, $a_3 > 2a_4$ and $p_2 < 2$, $p_4 < 2$. To get arbitrarily large values of the period T one can proceed in several ways. One is to keep values $p_2 < 2$, $p_4 < 2$ fixed and increase either or both values of a_1, a_3 indefinitely. Another way is to define the coefficient $a(t)$ by modifying $A_1(t)$ in (3.3) beyond the initial period $T_1 = p_1 + p_2 + p_3 + p_4$ by $A_2(t) := A_1(t)$ for $t \in [0, T_1)$ and $A_2(t) \equiv 0$ for $t \in [T_1, T_1 + p_5)$, for some $p_5 > 0$ (which can be any). By increasing the value of p_5 the new period $T = p_1 + p_2 + p_3 + p_4 + p_5$ of the periodic solution can be made continuously arbitrarily large.

In view of the construction and consideration above we arrive at the following statement

Theorem 3.4. *There is a constant $T_0 > 2$ such that for arbitrary period T within the range $T_0 \leq T < \infty$ there are choices of values a_i, p_i, $1 \leq i \leq 4$, such that equation* (3.1) *with nonlinearity $f = f_0$ and the respective coefficient $a = A_2$ has an asymptotically stable slowly oscillating periodic solution with the period T.*

Since by the very construction the periodic solutions are slowly oscillating each semi-cycle is of the length greater than 1. Therefore the period of any such periodic solution is always greater than 2. Period $T = 4$ is achieved when each of the semi-periods $T_1 = p_1 + p_2$ and $T_2 = p_3 + p_4$ is 2. We obtain the period 4 solution for this particular choice of the constants' values $a_1 = 7$, $a_2 = 3$, $a_3 = 6$, $a_4 = 2.5$ and $p_1 = p_2 = 1$, $p_3 = 1.2$, $p_4 = 0.8$. Due to the continuous dependence of the period T on the constants' values the smaller period can be achieved by their perturbation. We numerically observed stable periodic solutions by changing the above values proportionally up to when $T_1 = T_2 = 1.8$, thus making the period $T = 3.6$ (one can choose this value as T_0 in the statement). It would be of interest to derive a sharper estimate for T_0.

4. Smoothed nonlinearities

In this section we consider equation (2.1) where the piecewise constant functions $f(x)$ and $a(t)$ of Section 3 are replaced by close to them continuous nonlinearities. The basic idea is to make functions f and a continuous (or even smooth) in a small neighborhood of every discontinuity point by connecting the respective two constant values by a line segment.

We start first with the case $f(x) = f_0(x)$ and $a(t) = A_0(t)$. Let $\delta_0 > 0$ be small, and for every $\delta \in (0, \delta_0]$ introduce the continuous functions $f_\delta(x)$ and $A_0^\delta(t)$ by:

$$f(x) = f_\delta(x) = \begin{cases} +1 & \text{if } x \leq -\delta \\ -1 & \text{if } x \geq \delta \\ -(1/\delta)x & \text{if } x \in [-\delta, \delta]\,, \end{cases} \tag{4.1}$$

and
(4.2)

$$a(t) = A_0^\delta(t) = \begin{cases} a_2 + \frac{a_1-a_2}{2\delta}(t+\delta) \quad \text{if} \ \ t \in [-\delta, \delta] \\ a_1 \quad \text{if} \ \ t \in [\delta, p_1 - \delta) \\ a_1 + \frac{a_2-a_1}{2\delta}[t-(p_1-\delta)] \quad \text{if} \ \ t \in [p_1-\delta, p_1+\delta] \\ a_2 \quad \text{if} \ \ t \in [p_1+\delta, p_1+p_2-\delta) \\ a_2 + \frac{a_1-a_2}{2\delta}[t-(p_2-\delta)] \quad \text{if} \ \ t \in [p_1+p_2-\delta, p_1+p_2+\delta] \\ \text{periodic extension on } \mathbb{R} \text{ outside interval } [0,T), T = p_1+p_2\,. \end{cases}$$

Note that in the above definition of $A_0^\delta(t)$ there is an intentional overlap in values of the function on the intervals $[-\delta, \delta]$ and $[p_2-\delta, p_2+\delta]$ (where they are the same due to the intended periodicity). Likewise to (4.2), we define the continuous functions $A_1^\delta(t)$ and $A_2^\delta(t)$ based on the earlier defined piecewise constant coefficients $A_1(t)$ and $A_2(t)$ and with the same respective periods.

It is a well known fact that such small δ-perturbation of the nonlinearity f and the coefficient a lead to small smooth perturbations of the map F (away from its discontinuity point $h = 0$) (see e.g. [7, 11] for more relevant details). Below we outline the justification of this fact by showing the continuous dependence on δ and smoothness of the corresponding map for the value $x_1(\delta)$.

For $\delta \geq 0$ the value $x_1(\delta) = x(p_1; h)$ is explicitly calculated by direct integration as

$$\begin{aligned} x_1(\delta) &= h - \int_0^\delta a_\delta(t)\,dt - a_1(p_1 - 2\delta) - \int_{p_1-\delta}^{p_1} a_\delta(t)\,dt \\ &= h - a_1 p_1 + 2a_1\delta - \int_0^\delta a_\delta(t)\,dt - \int_{p_1-\delta}^{p_1} a_\delta(t)\,dt \\ &= x_1(0) + \tilde{x}_1(\delta)\,, \end{aligned}$$

where $\tilde{x}_1(\delta)$ is continuous in δ with $\tilde{x}_1(0) = 0$.

Similar calculations for the next two values $x_2(\delta)$ and $x_3(\delta)$ lead to the expression $x_3(\delta) = F_1(h,\delta) = F_1(h) + \tilde{F}_1(h,\delta)$ where $F_1(h)$ is as in (3.2) and $\tilde{F}_1(h,\delta)$ is continuous in h, δ and continuously differentiable in h with $\tilde{F}_1(h,0) = 0$ and $\partial(\tilde{F}_1(h,\delta))/\partial h \leq M$, where positive constant M is independent of $\delta \geq 0$. Analogous calculations are valid for $F_2(h,\delta)$ (we omit those calculations and particular details of the expressions). Therefore, by the continuity for small $\delta > 0$ map $F(h,\delta)$ as in (3.2) has an attracting two-cycle close to that when $\delta = 0$.

The above considerations give us the following statement:

Theorem 4.1. *There exist $T_0 > 2$ and $\delta_0 > 0$ such that for arbitrary T with $T_0 < T < \infty$ and any $0 < \delta < \delta_0$ differential delay equation (3.1) with $f(x) = f_\delta(x)$ and T-periodic $a(t) = A_1^\delta(t)$ (or $A_2^\delta(t)$) has an asymptotically stable slowly oscillating solution with the period T.*

5. Discussion and conclusions

The results of Sections 3 and 4 derived for differential delay equation (3.1) can be extended to the more general equation (2.1) with $\mu > 0$ and piecewise constant or smoothed functions $f(x)$ and $a(t)$. The calculations become more involved and complex, however, as the solutions are now piecewise exponential of the form $x(t) = A\exp\{-\mu t\} + B, A, B$ - constant. The resulting dynamics can become more complicated as well: besides the stability and periodicity they can exhibit the chaotic behaviors. The basic idea of the analysis is the same as for equation (3.1): a reduction of the dynamics to that of interval maps. A review paper [7] provides examples of such analyses as well as references to other publications.

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