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DELAY-DEPENDENT STABILITY CONDITIONS FOR FUNDAMENTAL CHARACTERISTIC FUNCTIONS

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Abstract. This paper is devoted to the investigation on the stability for two characteristic functions $f_1(z) = z^2 + pe^{-z\tau} + q$ and $f_2(z) = z^2 + pze^{-z\tau} + q$, where p and q are real numbers and $\tau > 0$. The obtained theorems describe the explicit stability dependence on the changing delay τ. Our results are applied to some special cases of a linear differential system with delay in the diagonal terms and delay-dependent stability conditions are obtained.

1. Introduction

We consider two characteristic functions

$$f_1(z) = z^2 + pe^{-z\tau} + q$$

and

$$f_2(z) = z^2 + pze^{-z\tau} + q,$$

where p and q are real numbers and $\tau > 0$. Equations $f_1(z) = 0$ and $f_2(z) = 0$ are the characteristic equations of linear differential equations

$$x''(t) + px(t-\tau) + qx(t) = 0 \tag{1.1}$$

and

$$x''(t) + px'(t-\tau) + qx(t) = 0 \tag{1.2}$$

with the delay τ, respectively.

A quasi-polynomial $f(z)$ is said to be *stable* if all zeros of $f(z)$ have negative real parts. In studying the stability of a characteristic function, main concern is on the stability region, the maximal region in the space of parameters for which the characteristic function is stable. Clarifying the dependence of all parameters on stability is important; however, it is not easy for a more general quasi-polynomial that contains $f_1(z)$ and $f_2(z)$. In this case, the quasi-polynomial or the zero solution of the corresponding delay differential equation may switch finite times from stability

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to instability and vice versa as a parameter increases. Such phenomena for changing parameter are often referred to as *stability switches*; see, e.g., [2, 3].

In 1966, Hsu and Bhatt [5] first presented the following stability results for $f_1(z)$ and $f_2(z)$; see also Stépán [7, Corollary 3.4 and Theorem 3.8].

Theorem A. *Let $\tau = 1$. Then function $f_1(z)$ is stable if and only if there exists a nonnegative integer m such that either*

$$p > 0\,, \quad p < q - (2m+1)^2\pi^2 \quad \textit{and} \quad p < -q + (2m+2)^2\pi^2$$

or

$$p < 0\,, \quad p > -q + 4m^2\pi^2 \quad \textit{and} \quad p > q - (2m+1)^2\pi^2\,.$$

Theorem B. *Let $\tau = 1$. Then function $f_2(z)$ is stable if and only if*

$$p > 0, \quad q > 0 \quad \textit{and} \quad p < -\frac{2}{\pi}q + \frac{\pi}{2}$$

or there exists a nonnegative integer m such that either

$$p > 0\,, \quad p < \frac{2}{(4m+3)\pi}q - \frac{(4m+3)\pi}{2} \quad \textit{and} \quad p < -\frac{2}{(4m+5)\pi}q + \frac{(4m+5)\pi}{2}$$

or

$$p < 0\,, \quad p > -\frac{2}{(4m+1)\pi}q + \frac{(4m+1)\pi}{2} \quad \textit{and} \quad p > \frac{2}{(4m+3)\pi}q - \frac{(4m+3)\pi}{2}.$$

Notice that Theorems A and B provide the stability conditions for (1.1) and (1.2), respectively, and depend on the parameters p and q with $\tau = 1$. A natural question then arises: how do the stability conditions for $f_1(z)$ and $f_2(z)$ depend on the delay τ with fixed p and q? The purpose of this paper is to answer the question. As an application, we can obtain delay-dependent stability conditions for some special cases of a linear differential system with delay in the diagonal terms.

2. Main results

Our main results are stated as follows:

Theorem 2.1. *Function $f_1(z)$ is stable if and only if either*

$$0 < 5p < 3q \quad \textit{and} \quad \tau \in (\tau_{2,0}, \tau_{1,1}) \cup (\tau_{2,1}, \tau_{1,2}) \cup \cdots \cup (\tau_{2,k_1-1}, \tau_{1,k_1}) \tag{2.1}$$

or

$$0 < -p < q \quad \textit{and} \quad \tau \in (\tau_{1,0}, \tau_{2,0}) \cup (\tau_{1,1}, \tau_{2,1}) \cup \cdots \cup (\tau_{1,k_2-1}, \tau_{2,k_2-1})\,. \tag{2.2}$$

Here $\tau_{1,n}$, $\tau_{2,n}$, k_1, and k_2 are defined as

$$\tau_{1,n} = \frac{2n\pi}{\sqrt{p+q}}, \quad \tau_{2,n} = \frac{(2n+1)\pi}{\sqrt{-p+q}}, \quad n = 0, 1, 2, \ldots,$$

$$k_1 = \left\lceil \frac{2\sqrt{-p+q} - \sqrt{p+q}}{2(\sqrt{p+q} - \sqrt{-p+q}\,)} \right\rceil, \quad k_2 = \left\lceil \frac{\sqrt{p+q}}{2(\sqrt{-p+q} - \sqrt{p+q}\,)} \right\rceil, \tag{2.3}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, namely, $\lceil x \rceil = \min\{s \in \mathbb{Z} \mid x \le s\}$.

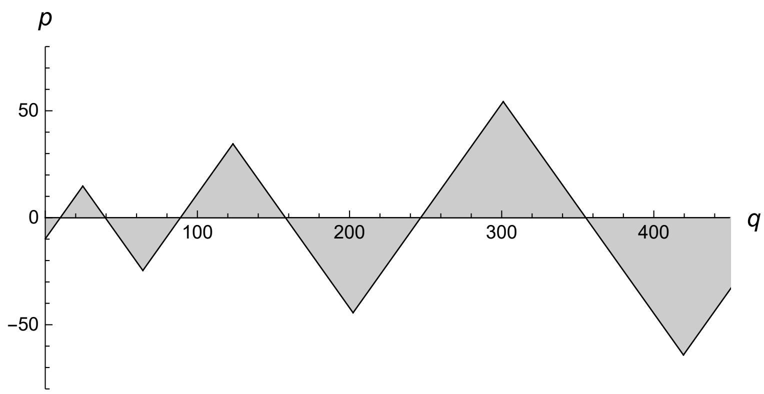

FIG. 1: Stability region of $f_1(z)$ with $\tau = 1$ (Theorem A).

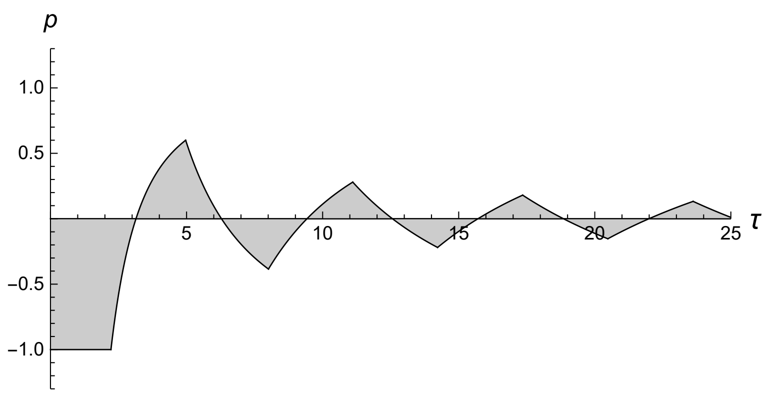

FIG. 2: Stability region of $f_1(z)$ with $q = 1$ (Theorem 2.1).

Theorem 2.2. *Function $f_2(z)$ is stable if and only if any one of the following three conditions holds:*

(2.4) $p > 0, \quad 15p^2 \geq 4q \quad \textit{and} \quad 0 < \tau < \tau_{3,0},$

(2.5) $p > 0, \quad 15p^2 < 4q \quad \textit{and} \quad \tau \in (0, \tau_{3,0}) \cup (\tau_{4,0}, \tau_{3,1}) \cup \cdots \cup (\tau_{4,k_3-1}, \tau_{3,k_3}),$

(2.6) $p < 0, \quad 3p^2 < 4q \quad \textit{and} \quad \tau \in (\tau_{3,0}, \tau_{4,0}) \cup (\tau_{3,1}, \tau_{4,1}) \cup \cdots \cup (\tau_{3,k_4-1}, \tau_{4,k_4-1}).$

Here $\tau_{3,n}$, $\tau_{4,n}$, k_3, and k_4 are defined as

$$\tau_{3,n} = \frac{(4n+1)\pi}{p + \sqrt{p^2+4q}}\,, \quad \tau_{4,n} = \frac{(4n+3)\pi}{-p + \sqrt{p^2+4q}}\,, \quad n = 0, 1, 2, \ldots,$$

$$k_3 = \left\lceil \frac{\sqrt{p^2+4q}}{4p} - 1 \right\rceil, \quad k_4 = \left\lceil -\frac{\sqrt{p^2+4q}}{4p} - \frac{1}{2} \right\rceil. \tag{2.7}$$

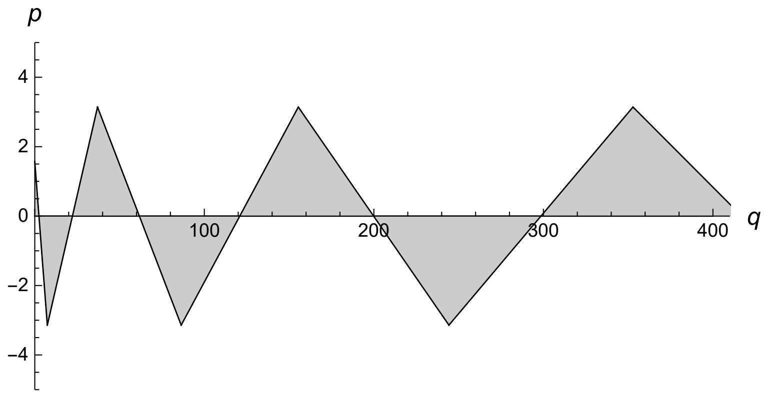

FIG. 3: Stability region of $f_2(z)$ with $\tau = 1$ (Theorem B).

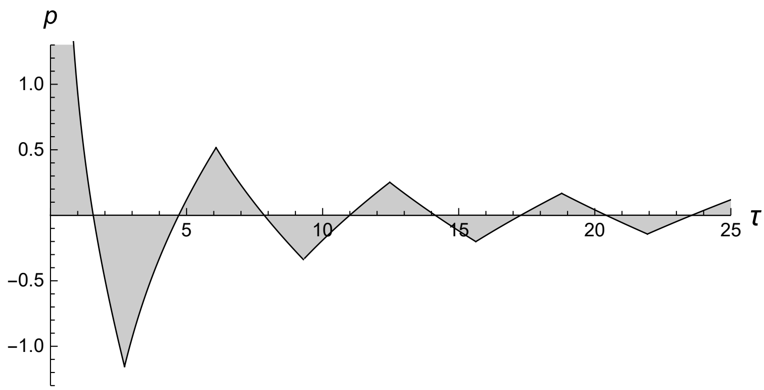

FIG. 4: Stability region of $f_2(z)$ with $q = 1$ (Theorem 2.2).

Remark 2.3. Theorems 2.1 and 2.2 show that as τ increases from 0, both $f_1(z)$ and $f_2(z)$ switch finite times from stability to instability and vice versa under suitable conditions, and they become unstable eventually; see Figs. 2 and 4.

Proof of Theorem 2.1. Let $\lambda = z\tau$ and $g_1(\lambda) = \tau^2 f_1(z)$. Then

$$g_1(\lambda) = \lambda^2 + p\tau^2 e^{-\lambda} + q\tau^2 .$$

Clearly, the stability of $f_1(z)$ is equivalent to that of $g_1(\lambda)$. Thus, we will prove that function $g_1(\lambda)$ is stable if and only if either (2.1) or (2.2) holds.

By Theorem A, function $g_1(\lambda)$ is stable if and only if there exists a nonnegative integer m such that either

$$p > 0\,, \quad p\tau^2 < q\tau^2 - (2m+1)^2\pi^2 \quad \text{and} \quad p\tau^2 < -q\tau^2 + (2m+2)^2\pi^2 \tag{2.8}$$

or

$$(2.9)\qquad p<0\,,\quad p\tau^2>-q\tau^2+4m^2\pi^2 \quad\text{and}\quad p\tau^2>q\tau^2-(2m+1)^2\pi^2\,.$$

It follows that

$$\begin{aligned}
(2.8) \iff{}& p>0\,,\quad (-p+q)\tau^2>(2m+1)^2\pi^2\,,\quad (p+q)\tau^2<(2m+2)^2\pi^2\\
\iff{}& q>p>0\,,\quad \tau>\frac{(2m+1)\pi}{\sqrt{-p+q}}\,,\quad \tau<\frac{(2m+2)\pi}{\sqrt{p+q}}\\
\iff{}& q>p>0\,,\quad \tau_{2,m}<\tau<\tau_{1,m+1}.
\end{aligned}$$

Notice that k_1 defined by (2.3) is the smallest nonnengative integer that satisfies $\tau_{2,k_1}>\tau_{1,k_1+1}$ because $\tau_{2,k}>\tau_{1,k+1}$ is equivalent to

$$k>\frac{2\sqrt{-p+q}-\sqrt{p+q}}{2(\sqrt{p+q}-\sqrt{-p+q}\,)}\ (>-1).$$

Suppose that

$$\frac{2\sqrt{-p+q}-\sqrt{p+q}}{2(\sqrt{p+q}-\sqrt{-p+q}\,)}\le 0\,,$$

namely, $3q\le 5p$. Then we obtain $k_1=0$ and $\tau_{2,k}>\tau_{1,k+1}$ for $k=0,1,2,\dots$. In this case, no nonnegative integer m that satisfies (2.8) exist. Hence, if $0<5p<3q$, then $k_1\ge 1$ and

$$0<\tau_{2,0}<\tau_{1,1}<\tau_{2,1}<\tau_{1,2}<\cdots<\tau_{2,k_1-1}<\tau_{1,k_1}<\tau_{1,k_1+1}<\tau_{2,k_1},$$

which indicates that (2.8) holds if and only if (2.1) holds.

Similarly, we observe that

$$\begin{aligned}
(2.9) \iff{}& p<0\,,\quad (p+q)\tau^2>4m^2\pi^2\,,\quad (-p+q)\tau^2<(2m+1)^2\pi^2\\
\iff{}& -q<p<0\,,\quad \tau>\frac{2m\pi}{\sqrt{p+q}}\,,\quad \tau<\frac{(2m+1)\pi}{\sqrt{-p+q}}\\
\iff{}& -q<p<0,\quad \tau_{1,m}<\tau<\tau_{2,m}.
\end{aligned}$$

Notice that k_2 defined by (2.3) is the smallest positive integer that satisfies $\tau_{1,k_2}>\tau_{2,k_2}$ because $\tau_{1,k}>\tau_{2,k}$ is equivalent to

$$k>\frac{\sqrt{p+q}}{2(\sqrt{-p+q}-\sqrt{p+q}\,)}\ (>0)\,.$$

Therefore, we obtain

$$0=\tau_{1,0}<\tau_{2,0}<\tau_{1,1}<\tau_{2,1}<\cdots<\tau_{1,k_2-1}<\tau_{2,k_2-1}<\tau_{2,k_2}<\tau_{1,k_2},$$

which implies that (2.9) holds if and only if (2.2) holds. This completes the proof. □

Proof of Theorem 2.2. Let $\lambda=z\tau$ and $g_2(\lambda)=\tau^2 f_2(z)$. Then

$$g_2(\lambda)=\lambda^2+p\tau\lambda e^{-\lambda}+q\tau^2.$$

Clearly, the stability of $f_2(z)$ is equivalent to that of $g_2(\lambda)$. Thus, we will prove that function $g_2(\lambda)$ is stable if and only if (2.4), (2.5), or (2.6) holds.

From Theorem B, function $g_2(\lambda)$ is stable if and only if

$$p > 0, \quad q > 0\,, \quad p\tau < -\frac{2q\tau^2}{\pi} + \frac{\pi}{2} \tag{2.10}$$

or there exists a nonnegative integer m such that either

$$p > 0, \quad p\tau < \frac{2q\tau^2}{(4m+3)\pi} - \frac{(4m+3)\pi}{2}\,, \quad p\tau < -\frac{2q\tau^2}{(4m+5)\pi} + \frac{(4m+5)\pi}{2} \tag{2.11}$$

or

$$p < 0\,, \quad p\tau > -\frac{2q\tau^2}{(4m+1)\pi} + \frac{(4m+1)\pi}{2}, \quad p\tau > \frac{2q\tau^2}{(4m+3)\pi} - \frac{(4m+3)\pi}{2}\,. \tag{2.12}$$

It is easy to see that

$$(2.10) \iff p > 0\,, \quad q > 0\,, \quad 0 < \tau < \frac{-p + \sqrt{p^2 + 4q}}{4q}\pi = \tau_{3,0},$$

which coincides with (2.4). We observe that

$$\begin{aligned} (2.11) &\iff p > 0, \quad \begin{cases} 4q\tau^2 - 2(4m+3)p\pi\tau - (4m+3)^2\pi^2 > 0 \\ 4q\tau^2 + 2(4m+5)p\pi\tau - (4m+5)^2\pi^2 < 0 \end{cases} \\ &\iff p > 0\,, \quad q > 0\,, \quad \tau_{4,m} < \tau < \tau_{3,m+1}. \end{aligned}$$

Notice that k_3 defined by (2.7) is the smallest nonnegative integer that satisfies $\tau_{4,k_3} > \tau_{3,k_3+1}$ because $\tau_{4,k} > \tau_{3,k+1}$ is equivalent to

$$k > \frac{\sqrt{p^2 + 4q}}{4p} - 1 \ (> -1).$$

Suppose that $\sqrt{p^2 + 4q}/(4p) - 1 \leq 0$, namely, $15p^2 \geq 4q$. Then we obtain $k_3 = 0$ and $\tau_{4,k} > \tau_{3,k+1}$ for $k = 0, 1, 2, \ldots$. In this case, no nonnegative integer m that satisfies (2.11) exist. Hence, if $p > 0$ and $15p^2 < 4q$, then $k_3 \geq 1$ and

$$0 < \tau_{3,0} < \tau_{4,0} < \tau_{3,1} < \cdots < \tau_{4,k_3-1} < \tau_{3,k_3} < \tau_{3,k_3+1} < \tau_{4,k_3}.$$

These facts indicate that (2.10) or (2.11) holds if and only if (2.4) or (2.5) holds.

Similarly, we observe that

$$\begin{aligned} (2.12) &\iff p < 0, \quad \begin{cases} 4q\tau^2 + 2(4m+1)p\pi\tau - (4m+1)^2\pi^2 > 0 \\ 4q\tau^2 - 2(4m+3)p\pi\tau - (4m+3)^2\pi^2 < 0 \end{cases} \\ &\iff p < 0, \quad q > 0, \quad \tau_{3,m} < \tau < \tau_{4,m}. \end{aligned}$$

Notice that k_4 defined by (2.7) is the smallest nonnegative integer that satisfies $\tau_{3,k_4} > \tau_{4,k_4}$ because $\tau_{3,k} > \tau_{4,k}$ is equivalent to

$$k > -\frac{\sqrt{p^2 + 4q}}{4p} - \frac{1}{2} \ (> -1).$$

Suppose that $-\sqrt{p^2 + 4q}/(4p) - 1/2 \leq 0$, namely, $3p^2 \geq 4q$. Then we obtain $k_4 = 0$ and $\tau_{3,k} > \tau_{4,k}$ for $k = 0, 1, 2, \ldots$. In this case, no nonnegative integer m that satisfies (2.12) exist. Therefore, if $p < 0$ and $3p^2 < 4q$, then $k_4 \geq 1$ and

$$0 < \tau_{3,0} < \tau_{4,0} < \tau_{3,1} < \tau_{4,1} < \cdots < \tau_{3,k_4-1} < \tau_{4,k_4-1} < \tau_{4,k_4} < \tau_{3,k_4},$$

which implies that (2.12) holds if and only if (2.6) holds. This completes the proof. □

3. Application

In this section, we investigate the asymptotic stability of the zero solution of a linear delay differential system

$$\begin{cases} x'(t) = -ax(t-r) - by(t), \\ y'(t) = -cx(t) - dy(t-r), \end{cases} \tag{3.1}$$

where a, b, c, d are real numbers and $r > 0$. The characteristic equation of (3.1) is given by

$$\det \begin{pmatrix} z + ae^{-zr} & b \\ c & z + de^{-zr} \end{pmatrix} = 0\,,$$

that is,

$$z^2 + (a+d)e^{-zr} + ade^{-2zr} - bc = 0. \tag{3.2}$$

When $a = d$, equation (3.2) is reduced to

$$\begin{cases} \left(z + ae^{-zr} + \sqrt{bc}\,\right)\left(z + ae^{-zr} - \sqrt{bc}\,\right) = 0 & (bc \geq 0)\,, \\ \left(z + ae^{-zr} + i\sqrt{-bc}\,\right)\left(z + ae^{-zr} - i\sqrt{-bc}\,\right) = 0 & (bc < 0)\,. \end{cases} \tag{3.3}$$

In 2009, the author [6] presented delay-dependent stability conditions for (3.1) with $a = d$ by using root analysis of (3.3). Consequently, let us treat other two special cases $a + d = 0$ and $ad = 0$.

Consider first the case $a + d = 0$. Then equation (3.2) becomes

$$z^2 - a^2e^{-2zr} - bc = 0. \tag{3.4}$$

By applying Theorem 2.1 to equation (3.4) with $p = -a^2$, $q = -bc$, $\tau = 2r$ and $\tau_{j,n} = 2r_{j,n}$ $(j = 1, 2)$, we obtain the following corollary.

Corollary 3.1. *Let $d = -a$. Then the zero solution of* (3.1) *is asymptotically stable if and only if*

$$a^2 + bc < 0 \quad \text{and} \quad r \in (r_{1,0}, r_{2,0}) \cup (r_{1,1}, r_{2,1}) \cup \cdots \cup (r_{1,\ell_2-1}, r_{2,\ell_2-1})\,.$$

Here $r_{1,n}$, $r_{2,n}$, and ℓ_2 are defined as

$$r_{1,n} = \frac{n\pi}{\sqrt{-a^2 - bc}}\,, \quad r_{2,n} = \frac{(n + 1/2)\pi}{\sqrt{a^2 - bc}}\,, \quad n = 0, 1, 2, \ldots,$$

$$\ell_2 = \left\lceil \frac{\sqrt{-a^2 - bc}}{2(\sqrt{a^2 - bc} - \sqrt{-a^2 - bc})} \right\rceil.$$

Next, consider the case $ad = 0$. Without loss of generality, we may assume $d = 0$. Then equation (3.2) becomes

$$z^2 + aze^{-zr} - bc = 0\,. \tag{3.5}$$

By applying Theorem 2.2 to equation (3.5) with $p = a$, $q = -bc$, $\tau = r$ and $\tau_{j,n} = r_{j,n}$ $(j = 3, 4)$, we obtain the following corollary.

Corollary 3.2. *Let $d = 0$. Then the zero solution of* (3.1) *is asymptotically stable if and only if any one of the following three conditions holds:*

(i) $a > 0, \quad 15a^2 \geq -4bc > 0 \quad$ *and* $\quad 0 < r < r_{3,0}$,

(ii) $a > 0, \quad 15a^2 < -4bc \quad$ *and* $\quad r \in (0, r_{3,0}) \cup (r_{4,0}, r_{3,1}) \cup \cdots \cup (r_{4,\ell_3-1}, r_{3,\ell_3})$,

(iii) $a < 0, \quad 3a^2 < -4bc \quad$ *and* $\quad r \in (r_{3,0}, r_{4,0}) \cup (r_{3,1}, r_{4,1}) \cup \cdots \cup (r_{3,\ell_4-1}, r_{4,\ell_4-1})$.

Here $r_{3,n}$, $r_{4,n}$, ℓ_3, and ℓ_4 are defined as

$$r_{3,n} = \frac{(4n+1)\pi}{a + \sqrt{a^2 - 4bc}}, \quad r_{4,n} = \frac{(4n+3)\pi}{-a + \sqrt{a^2 - 4bc}}, \quad n = 0, 1, 2, \ldots,$$

$$\ell_3 = \left\lceil \frac{\sqrt{a^2 - 4bc}}{4a} - 1 \right\rceil, \quad \ell_4 = \left\lceil -\frac{\sqrt{a^2 - 4bc}}{4a} - \frac{1}{2} \right\rceil.$$

In the remainder case $ad \neq 0$ and $a \neq \pm d$, although two delays r and $2r$ make the distribution of roots of (3.2) much more complicated, some new explicit stability conditions for (3.1) have been obtained; see [4] for details.

Finally, it turns out that Čermák and Kisela [1] have extended some parts of Theorem 2.1 to a stability criterion of fractional delay differential equations.

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