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NONLOCAL SEMILINEAR SECOND-ORDER DIFFERENTIAL INCLUSIONS IN ABSTRACT SPACES WITHOUT COMPACTNESS

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Abstract. We study the existence of a mild solution to the nonlocal initial value problem for semilinear second-order differential inclusions in abstract spaces. The result is obtained by combining the Kakutani fixed point theorem with the approximation solvability method and the weak topology. This combination enables getting the result without any requirements for compactness of the right-hand side or of the cosine family generated by the linear operator.

1. Introduction

The main goal of the paper is to investigate the existence of a solution to the following nonlocal initial value problem for semilinear second-order differential inclusion in a Banach space

$$\begin{cases} \ddot{x}(t) \in Ax(t) + F(t, x(t))\,, & \text{for a.a. } t \in [0,T]\,, \\ x(0) = g(x) \;\; \dot{x}(0) = h(x)\,. \end{cases} \tag{1.1}$$

Throughout the paper, we assume that

(i) E is a reflexive Banach space having a Schauder basis;

(ii) $A\colon D(A) \subset E \to E$ is a closed linear densely defined operator generating a cosine family $\{C(t)\}_{t\in\mathbb{R}}$;

(iii) $F\colon [0,T] \times E \multimap E$ is a multivalued mapping with nonempty, bounded, closed and convex values;

(iv) g, $h\colon C([0,T],E) \to E$.

Differential equations and inclusions in Banach spaces have been attracting quite big attention (see, e.g., [1,2,5,13,23,24]). In particular, as pointed out by Byszewski and Lakshmikantham in [11], the study of nonlocal conditions is of significance due to their applicability in many physical and engineering problems and also in other areas of applied mathematics. Since then several authors have been investigated

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problems with nonlocal initial conditions for different classes of abstract differential equations or inclusions (see, e.g., [4, 12, 14]).
One of the key tools that will be used in the paper is an approximation solvability method that was introduced in [6] to study fully nonlinear first-order problems in Hilbert spaces. Its application was afterwards extended to first-order semilinear problems in Banach spaces in [8] and to fully nonlinear second-order problems in Hilbert spaces in [7]. Recently, it was applied to Cauchy problems for semilinear second-order differential inclusions in [20].
Motivated by the above works, the main objective of this paper is proving the existence of a mild solution to the second-order semilinear differential inclusion in a Banach space satisfying nonlocal conditions without converting it into first-order problem. To obtain desired results, we will transfer the original problem into a sequence of problems in finite dimensional spaces using the approximation solvability method. Afterwards, the solvability of approximating problems will be shown by the Kakutani fixed point problem for multivalued mappings. Finally, the convergence of obtained solutions to the solution of the original problem will be proven. This procedure will enable to obtain the existence result under easily verifiable and not restrictive conditions on the cosine family generated by the linear operator or on the right-hand side and to avoid any requirement for compactness.

2. Preliminaries

In this section, the basic notions dealing with natural projections and cosine families will be mentioned.

A sequence $\{e_n\}_n$ of vectors in E is a *Schauder basis* for E if, for every $x \in E$, there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$, such that $\|x - \sum_{i=1}^n \alpha_i e_i\| \to 0$, as $n \to \infty$.
Given a Schauder basis $\{e_n\}_n$ for E, let $E_n = \operatorname{span}\{e_1, \dots, e_n\}$ denote the n-dimensional Banach space generated by the first n vectors of the basis, and let $\mathbb{P}_n \colon E \to E_n$ be the *natural projection* of E onto E_n, i.e., $\mathbb{P}_n \left(\sum_{k=1}^\infty \alpha_k e_k\right) = \sum_{k=1}^n \alpha_k e_k$. It holds that $\mathbb{P}_n$ is linear and bounded, for every $n \in \mathbb{N}$, and that the sequence $\{\|\mathbb{P}_n\|\}_n$ is bounded, i.e. that there exists $K \geq 1$ such that

$$\|\mathbb{P}_n(x)\| \leq K\|x\| \quad \forall n \in \mathbb{N}\,,\ \forall\, x \in E\,.$$

For the main properties of the projection $\mathbb{P}_n$, we remind to [8], [9] and [19]. We recall, in particular, that if $x_n \rightharpoonup x$, then $\mathbb{P}_n(x_n) \rightharpoonup x$.
A one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself is called a *strongly continuous cosine family* if

- $C(t+s) + C(s-t) = 2C(s)C(t)$, for all t, $s \in \mathbb{R}$;
- $C(0) = I$;
- the map $t \to C(t)x$ is continuous in $\mathbb{R}$, for each fixed $x \in E$.

If $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family, then there exist $M \geq 1$ and $\omega \geq 0$ such that, for all $t \in \mathbb{R}$,

$$\|C(t)\| \leq Me^{\omega|t|}\,. \tag{2.1}$$

We also recall that the map $c\colon [0,T]\times E\to E$ defined as $c(t,x)=C(t)x$ is continuous (see [20, Lemma 3]).
The one parameter family $\{S(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping the space E into itself defined, for all $t\in\mathbb{R}$ and $x\in E$, by

$$S(t)x=\int_0^t C(s)x\,ds$$

is called the *strongly continuous sine family* associate to the cosine family. It follows from the definition of $\{S(t)\}_{t\in\mathbb{R}}$ that, for every $t\in[0,T]$,

$$\|S(t)\|\le K_0\,,$$

where

$$K_0=\begin{cases} M\frac{e^{\omega T}-1}{\omega} & \text{if} \quad \omega\neq 0\\ MT & \text{if} \quad \omega=0\,. \end{cases}$$

For more information about sine and cosine families and their properties, see, e.g., [22].

The notion of a solution to (1.1) will be understood in a mild sense. Namely, by a *mild solution* of the problem (1.1) we mean a continuous function $x\colon[0,T]\to E$ such that, for all $t\in[0,T]$,

$$x(t)=C(t)\,g(x)+S(t)h(x)+\int_0^t S(t-s)f(s)\,ds\,,$$

where

$$f\in S^1_{F,\,x}=\{f\in L^1([0,T],E):f(t)\in F(t,x(t)),\ \text{for a.a. } t\in[0,T]\}\,.$$

3. Existence result

Theorem 3.1. *Consider the problem* (1.1) *and let* $F:[0,T]\times E\multimap E$ *satisfies the following assumptions:*

- $(F1)$ $F(t,x)$ *is nonempty, convex, closed, and bounded, for every* $t\in[0,T]$ *and* $x\in E$,
- $(F2)$ *for every* $x\in E$, $F(\cdot,x)$ *has a measurable selection,*
- $(F3)$ *for a.a.* $t\in[0,T]$, $F(t,\cdot)\colon E^w\multimap E^w$, *where* E^ω *denotes the space* E *endowed with the weak topology, is u.s.c.,*
- $(F4)$ *for every* $n\in\mathbb{N}$, *there exists* $\varphi_n\in L^1([0,T],\mathbb{R})$, *with*

  $$\liminf_{n\to\infty}\frac{\|\varphi_n\|_{L^1}}{n}=0\,,$$

  such that

  $$\|z\|\le\varphi_n(t),$$

  for a.a. $t\in[0,T]$, *every* $x\in E$ *with* $\|x\|\le n$ *and every* $z\in F(t,x)$.

Moreover let g *and* h *satisfy:*

- $(gh1)$ $g,\ h\colon C([0,T],E)^w\to E^w$ *are continuous;*

($gh2$)
$$\lim_{n\to\infty} \frac{L_n}{n} = R\,,$$
where
$$L_n = \max\Big\{ \sup_{\|x\|\le n} \|g(x)\|, \sup_{\|x\|\le n} \|h(x)\| \Big\},$$
with
$$R < \frac{1}{K(KMe^{\omega T} + K_0)}\,.$$

Then the problem (1.1) *has a solution.*

Proof. In order to get the existence of a solution to the problem (1.1), we will use the approximation solvability method. Thus, for each $m \in \mathbb{N}$, consider the multimap $G_m : [0,T] \times E \to E_m$ defined as $G_m = \mathbb{P}_m \circ F$ and the operator $\Sigma_m : C([0,T], E_m) \multimap C([0,T], E_m)$ defined as
$$\Sigma_m(q)(t) = \Big\{ \mathbb{P}_m C(t)\, \mathbb{P}_m g(q) + \mathbb{P}_m S(t) h(q) + \int_0^t \mathbb{P}_m S(t-s) f(s)\, ds : f \in S^1_{G_m,\,q} \Big\}\,.$$
Let us note that the existence of a selection $f \in S^1_{G_m,\,q}$ is guaranteed, e.g., by [9, Proposition 2.2].

In order to show that Σ_m has a fixed point, we will prove that it satisfies all assumptions of the Kakutani fixed point theorem ([17, Theorem 1]). For this purpose, given $n \in \mathbb{N}$, we use the following notation
$$nB_m = \{q \in C([0,T]; E_m) : \|q(t)\| \le n, \quad \text{for every } t \in [0,T]\}\,.$$
Notice that $\Sigma_m(q) = \Sigma^1_m(q) + \Sigma^2_m(q)$, where Σ^1_m is a single valued map defined as
$$\Sigma^1_m(q)(t) = \mathbb{P}_m C(t)\, \mathbb{P}_m g(q) + \mathbb{P}_m S(t) h(q)\,,$$
while Σ^2_m is a multivalued map defined as
$$\Sigma^2_m(q) = \Big\{ \int_0^t \mathbb{P}_m S(t-s) f(s)\, ds : f \in S^1_{G_m,\,q} \Big\}\,.$$
In [20, Theorem 1], we proved a result similar to the present one in the case when the non-linear term depends also on the first derivative, but the nonlocal conditions are replaced by the Cauchy conditions. In this proof, we shall outline only the differences with respect to the proof of the quoted result. In particular, it is possible to prove by using [18, Theorem 5.1.1] together with [20, Theorem 1] that Σ^2_m has convex values, a closed graph and that it maps bounded sets into relatively compact sets. Let us now prove that Σ^1_m satisfies the same properties. Clearly, Σ^1_m is convex valued, because it is single valued.
Assume that $\big(q_k, \Sigma^1_m(q_k)\big) \to (q,\, y)$ in $C([0,T], E_m) \times C([0,T], E_m)$, and let us prove that $y = \Sigma^1_m\,(q)$. According to ($gh1$) and the boundedness of $C(t), S(t)$ and $\mathbb{P}_m$, since E_m is finite dimensional, it follows that
$$\Sigma^1_m(q_k)(t) \to \mathbb{P}_m C(t)\, \mathbb{P}_m g(q) + \mathbb{P}_m S(t) h(q)\,,$$
for every $t \in [0,T]$. Since the convergence in $C([0,T], E_m)$ implies the pointwise convergence, we get that $y = \Sigma^1_m\,(q)$, i.e. that Σ^1_m has a closed graph.

Take now $n \in \mathbb{N}$. Condition $(gh2)$ implies that $\{g(x) : x \in nB_m\}$ is bounded, thus $A = \{\mathbb{P}_m g(x) : x \in nB_m\}$ is relatively compact, because E_m is finite dimensional. Since $(t, x) \to C(t)x$ is continuous, it is uniformly continuous in the compact set $[0, T] \times \overline{A}$. We then get that, for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every t, $t_0 \in [0, T]$, $x \in nB_m$

$$\|C(t)\mathbb{P}_m g(x) - C(t_0)\mathbb{P}_m g(x)\| \leq \epsilon\,.$$

Moreover, for every $x \in nB_m$ there exists $\frac{d}{dt}S(t)h(x) = C(t)h(x)$. (2.1) and $(gh2)$ then imply that

$$\left\|\frac{d}{dt}S(t)h(x)\right\| \leq Me^{\omega T}L_n\,,$$

for every $t \in [0, T]$, $x \in nB_m$. Therefore, Σ^1_m is equicontinuous in nB_m, for every $n \in \mathbb{N}$.

In order to show that Σ_m maps bounded sets into bounded sets and that there exists a bounded set $D \subset C^1([0, T]; E_m)$ such that $\Sigma_m(D) \subset D$, it is sufficient to notice that, according to $(F4)$ and $(gh2)$ for every n, $m \in \mathbb{N}$, $q \in nB_m$ and $h \in \Sigma_m(q)$, there exists $f \in S^1_{G_m,\,q}$ and $L_n \in \mathbb{R}$, $\varphi_n \in L^1([0, T], \mathbb{R})$ such that, for every $t \in [0, T]$, the following holds

$$\begin{aligned}\|h(t)\| &\leq \|\mathbb{P}_m\|^2\|C(t)\|\|g(q)\| + \|\mathbb{P}_m\|\|S(t)\|\|h(q)\| \\ &\quad + \int_0^T \|\mathbb{P}_m\|\|S(t-s)\|\|f(s)\|\,ds \\ &\leq K^2Me^{\omega T}L_n + KK_0L_n + KK_0\|\varphi_n\|_{L^1}\,.\end{aligned}$$

Therefore,

$$\|h\|_C \leq K^2Me^{\omega T}L_n + KK_0L_n + KK_0\|\varphi_n\|_{L^1}$$

for every m, $n \in \mathbb{N}$, $q \in nB_m$, $h \in \Sigma_m(q)$. In particular, Σ^1_m maps bounded sets into bounded and equicontinuous sets, i.e. relatively compact sets in the space $C([0, T], E_m)$.

Take $N > 0$ such that

$$\frac{L_N}{N} < \frac{1}{K(KMe^{\omega T} + K_0)} \quad \text{and} \quad \frac{\|\varphi_N\|_{L^1}}{N} < \frac{1}{KK_0}\left[1 - \frac{L_N}{N}K(KMe^{\omega T} + K_0)\right].$$

Such N exists because of $(F4)$ and $(gh2)$. Afterwards,

$$\frac{K^2Me^{\omega T}L_N + KK_0L_N + KK_0||\varphi_N||_{L^1}}{N} < 1\,,$$

which guarantees that $\Sigma_m(NB_m) \subset NB_m$, for all $m \in \mathbb{N}$.

Since Σ_m is closed and maps bounded sets into relatively compact sets, it has compact values; hence, it is u.s.c. Thus, $\Sigma_m : NB_m \multimap NB_m$ is a u.s.c. compact map with convex and closed values. Applying the Kakutani fixed point theorem, we obtain that, for all $m \in \mathbb{N}$, the operator Σ_m has a fixed point q_m. Because of the technique used, we are also able to localize the fixed point in the set

$$NB = \{q \in C([0, T], E) : \|q(t)\| \leq N,\ \text{for every}\ t \in [0, T]\}\,.$$

Let us now prove that the sequence $\{q_m\}_m$ found in previous step admits a subsequence pointwise weakly converging to a solution q of Problem (1.1).
The sequence $\{q_m\}_m$ satisfies, for all $m \in \mathbb{N}$ and $t \in [0,T]$,

$$q_m(t) = \mathbb{P}_m C(t)\, \mathbb{P}_m g(q_m) + \mathbb{P}_m S(t) h(q_m) + \int_0^t \mathbb{P}_m S(t-s) f_m(s)\, ds\,,$$

where $f_m \in S^1_{G_m, q_m}$, for every $m \in \mathbb{N}$.
Reasoning like in [20, Theorem 1], it is possible to prove that there exists a subsequence, still denoted as the sequence, and a function $f \in L^1([0,T], E)$ such that

$$\int_0^t \mathbb{P}_m S(t-s) f_m(s)\, ds \rightharpoonup \int_0^t S(t-s) f(s)\, ds\,,$$

for every $t \in [0,T]$.
Now, according to $(gh2)$, since $q_m \in NB$ for every $m \in \mathbb{N}$ and E is reflexive, there exists a subsequence, still denoted as the sequence, and $\overline{g}, \overline{h} \in E$ such that

$$g(q_m) \rightharpoonup \overline{g} \text{ and } h(q_m) \rightharpoonup \overline{h}\,,$$

which implies that

$$\mathbb{P}_m C(t)\, \mathbb{P}_m g(q_m) + \mathbb{P}_m S(t) h(q_m) \rightharpoonup C(t)\overline{g} + S(t)\overline{h}\,,$$

for every $t \in [0,T]$, i.e. that

$$q_m(t) \rightharpoonup q(t) = C(t)\overline{g} + S(t)\overline{h} + \int_0^t S(t-s) f(s)\, ds.$$

Thus, $q_m \rightharpoonup q$ in $C([0,T], E)$ (see [10, Theorem 4.3]). Hence, according to $(gh1)$, $\overline{g} = g(q)$ and $\overline{h} = h(q)$, while, reasoning like in the proof of [20, Theorem 1] we get that $f \in S^1_{F,\,q}$, and the proof is complete. □

Remark 3.2. Let us note that assumption $(gh2)$ is satisfied, e.g., when (cf. assumption $(gh2)$ in [12]):

$(gh2')$ there exists $Q > 0$ such that $\|g(q)\| \le Q$ and $\|h(q)\| \le Q$, for all $q \in C([0,T]; E)$.

In such a case, $R = 0$ and Theorem 1 can be proved also replacing condition $(F4)$ by the following one:

$(F4')$ There exist $\alpha \in L^1([0,T], E)$ such that

$$\|z\| \le \alpha(t)(1 + \|x\|)\,,$$

for a.a. $t \in [0,T]$, every $x \in E$ and every $z \in F(t,x)$.

The only difference with respect to the proof of Theorem 1 concerns, in this case, the existence of a bounded set $H \subset C([0,T], E)$ such that, for every $m \in \mathbb{N}$, Σ_m maps $H \cap C([0,T], E_m)$ into itself. On this purpose, it is sufficient to reason like in [20, Theorem 2], observing that, denoted, for every fixed $j \in \mathbb{N}$,

$$q_j = \max_{t \in [0,T]} \int_0^T e^{-j(t-s)} \chi_{[0,t]}(s) \alpha(s)\, ds,$$

it is possible to prove that there exists a subsequence, still denoted as the sequence, such that $q_j \to 0$. Now take

$$H = \left\{ x \in C([0,T], E) : \max_{t\in[0,T]} e^{-\bar{j}t}\|x(t)\| \le R \right\}$$

where $\bar{j} \in \mathbb{N}$ and $R \in \mathbb{R}$ are chosen such that

$$1 - KK_0 q_{\bar{j}} > 0\,,$$

and

$$R > \frac{K^2 M e^{\omega T} Q + KK_0 Q + KK_0\|\alpha\|_{L^1}}{1 - K_2 q_{\bar{j}}}\,.$$

Remark 3.3. We point out that our existence result is proved under quite weak assumptions. Indeed, similar results are obtained in literature for even more general equations and boundary conditions, but all with very strong assumptions.
In [3], an additional term $B\dot{x}$, with B linear and bounded, appears while $h(x) \equiv x_1 \in E$, but the authors assume that $C(t)$ is compact for every t. In [12], A generates a fundamental system and g and h are assumed bounded. Moreover, they have to satisfy, as well as F, a condition involving the Hausdorff measure of noncompactness. In [14,15,16], the left-hand side is of type $\frac{d}{dt}(\dot{x}(t) - p(t,x,\dot{x}))$ or the right-hand side is of type $F(t,x,x(a(t)),\dot{x},\dot{x}(b(t)))$ or $F(t,N(t)x)$ and g and h may depend also on $\dot{x}$. However, the existence results there are proved assuming that $S(t)$ is compact, for every t, or that F maps bounded sets into relatively compact ones, eventually that it satisfies a condition involving the Hausdorff measure of noncompactness. Moreover, g and h are assumed completely continuous and bounded or globally Lipschitz continuous. In [21], the nonlinear term depends also on the weighted average of the solution and g and h depends also on $\dot{x}$, but the nonlinear term, g and h are assumed globally Lipschitz continuous.

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References

[1] Andres, J., Malaguti, L., Pavlačková, M., *On second-order boundary value problems in Banach spaces: a bound sets approach*, Topol. Methods Nonlinear Anal. **37** (2) (2011), 303–341.

[2] Andres, J., Malaguti, L., Pavlačková, M., *A Scorza-Dragoni approach to second-order boundary value problems in abstract spaces*, Appl. Math. Inf. Sci. **6** (2) (2012), 29–44.

[3] Balachandran, K., Park, J.Y., *Existence of solutions of second order nonlinear differential equations with nonlocal conditions in Banach spaces*, Indian J. Pure Appl. Math. **32** (12) (2001), 1883–1891.

[4] Benchohra, M., Nieto, J.J., Rezoug, N., *Second order evolution equations with nonlocal conditions*, Demonstr. Math. **50** (2017), 309–319.

[5] Benchohra, M., Rezoug, N., Samet, B., Zhou, Y., *Second order semilinear Volterra-type integro-differential equations with non-instantaneous impulses*, Mathematics **7** (12) (2019), 20 pp., art. no. 1134.

[6] Benedetti, I., Loi, N.V., Malaguti, L., Obukhovskii, V., *An approximation solvability method for nonlocal differential problems in Hilbert spaces*, Commun. Contemp. Math. **19** (2) (2017), 34 pp.

[7] Benedetti, I., Loi, N.V., Malaguti, L., Taddei, V., *Nonlocal diffusion second order partial differential equations*, J. Differential Equations **262** (2007), 1499–1523.

[8] Benedetti, I., Loi, N.V., Taddei, V., *An approximation solvability method for nonlocal semilinear differential problems in Banach spaces*, Discrete Contin. Dyn. Syst. Ser. A **37** (6) (2017), 2977–2998.

[9] Benedetti, I., Malaguti, L., Taddei, V., *Nonlocal semilinear evolution equations without strong compactness: theory and applications*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 371–388.

[10] Bochner, S., Taylor, A.E., *Linear functionals on certain spaces of abstractly valued functions*, Ann. Math. **39** (1938), 913–944.

[11] Byszewski, L., Lakshmikantham, V., *Theorem about the existence and uniqueness of a solution of a nonlocal abstract cauchy problem in a Banach space*, Appl. Anal. **40** (1) (1991), 11–19.

[12] Cardinali, T., Gentili, S., *An existence theorem for a non-autonomous second order nonlocal multivalued problem*, Stud. Univ. Babeş-Bolyai Math. **62** (1) (2017), 101–117.

[13] Cernea, A., *A note on the solutions of a second-order evolution in non separable Banach spaces*, Comment. Math. Univ. Carolin. **58** (3) (2017), 307–314.

[14] Henríquez, H.R., Poblete, V., Pozo, J., *Mild solutions of non-autonomous second order problems with nonlocal initial conditions*, J. Math. Anal. Appl. **412** (2014), 1064–1083.

[15] Hernández, E.M., Henríquez, H.R., *Global solutions for a functional second order abstract Cauchy problem with nonlocal conditions*, Ann. Polon. Math. **83** (2) (2004), 149–170.

[16] Hernández, E.M., Henríquez, H.R., *Existence results for second order differential equations with nonlocal conditions in Banach spaces*, Funkcial. Ekvac. **52** (1) (2009), 113–137.

[17] Kakutani, S.A., *Generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 451–459.

[18] Kamenskii, M., Obukhovskii, V., Zecca, P., *Condensing multivalued maps and semilinear differential inclusions in Banach spaces*, W. de Gruyter, Berlin, 2001.

[19] Malaguti, L., Perrotta, S., Taddei, V., *L^p exact controllability of partial differential equations with nonlocal terms*, Evol. Equ. Control Theory **11** (5) (2022), 1533–1564.

[20] Pavlačková, M., Taddei, V., *Mild solutions of second-order semilinear impulsive differential inclusions in Banach spaces*, Mathematics **10** (672) (2022).

[21] Tidke, H.L., Dhakne, M.B., *Global existence of mild solutions of second order nonlinear Volterra integrodifferential equations in Banach spaces*, Differ. Equ. Dyn. Syst. **17** (4) (2009), 331–342.

[22] Travis, C.C., Webb, G.F., *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungar. **32** (1–2) (1978), 75–96.

[23] Wu, J., *Theory and application of partial functional differential equations*, Springer-Verlag, New York, 1996.

[24] Xiao, J.-Z., Zhu, X.-H., Cheng, R., *The solution sets for second order semilinear impulsive multivalued boundary value problems*, Comput. Math. Appl. **64** (2) (2012), 147–160.

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