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AROUND CERTAIN CRITICAL CASES IN STABILITY STUDIES IN HYDRAULIC ENGINEERING

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Abstract. It is considered the mathematical model of a benchmark hydroelectric power plant containing a water reservoir (lake), two water conduits (the tunnel and the turbine penstock), the surge tank and the hydraulic turbine; all distributed (Darcy-Weisbach) and local hydraulic losses are neglected,the only energy dissipator remains the throttling of the surge tank. Exponential stability would require asymptotic stability of the difference operator associated to the model. However in this case this stability is "fragile" i.e. it holds only for a rational ratio of the two delays, with odd numerator and denominator also. Otherwise this stability is critical (non-asymptotic and displaying an oscillatory mode).

1. Introduction. Problem statement

This paper has two starting points and the outcome is twofold. The first statement is explained below, the second one will be revealed towards the final part. Starting with the papers of A. D. Myshkis and his co-workers e.g. [1] and also with the papers of K. L. Cooke and his co-workers e.g. [3] the following methodology was established to deal with qualitative theory for non-standard BVP (Boundary Value Problems) for $1D$ hyperbolic PDEs (Partial Differential Equations). Integrating along the characteristics, a system of FDE (Functional Differential Equations) was associated to the BVP with initial conditions and the Cauchy problem (with initial conditions) for the FDEs. Consequently, any result obtained for one of the aforementioned mathematical objects is automatically projected back on the other one. Along almost half-century (starting from 1973-74) the author of this paper promoted this approach throughout his publications with reference to applications arising from Physics and Engineering, the most comprehensive presentation of the approach being given in [9], where the theorem of Cooke in [3] is proven completely.

Now we can turn to the qualitative problem of interest to us: (asymptotic) stability of the steady states (equilibria) for the BVPs mentioned above. This problem is reduced (equivalently) to the problem of the stability for the associated system of FDEs with deviated argument. Worth mentioning that in most applications the

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FDEs turn to be of neutral type. However the N(eutral)FDEs display a peculiar aspect of the stability problem. More precisely, if we consider even the simplest scalar NFDE, it is known - see e.g. [7], Corollary 1.7, p. 30 - that if the roots of its characteristic polynomial are located in $\mathbb{C}^-$ and its difference operator is stable, the stability is exponential. If the difference operator is unstable it is possible to have unbounded solutions while if the difference operator is in a critical case, the stability is at most non-exponential. Following the seminal papers of Hale and his co-workers (see [7] for complete references), the assumption on (strong) stability of the difference operator accompanied almost all development on NFDEs.

The present paper starts from the finding that, in spite of the aforementioned basic assumption on the difference operator, there exist important applications where it is not fulfilled. Various applications in Mechanical Engineering are modeled by NFDEs with the difference operator displaying critical stability [9]. Also Hydraulic Engineering (water hammer quenching, surge tank stability) is a source of such critically stable difference operators, but with even more interesting mathematical aspects. For this reason our choice went towards applications arising from Hydraulic Engineering.

2. Application description. The basic mathematical model

It is considered the standard hydroelectric plant composed of the water reservoir (lake), two water conduits (the tunnel and the penstock), the surge tank and the hydraulic turbine. The technological diagram can be seen in [10]. The mathematical model, considering distributed parameters of the water conduits, are as follows

$$
\begin{aligned}
&\partial_{\xi_i}\left(h_i+\tfrac{1}{2}\tfrac{T_{wi}}{T_i}q_i^2\right)+T_{wi}\partial_t q_i+\tfrac{1}{2}\tfrac{\lambda_i L_i}{D_i}\tfrac{T_{wi}}{T_i}q_i|q_i|=0\,,\\
&\delta_i^2 T_{wi}\partial_t h_i+\partial_{\xi_i}q_i=0\ (\delta_i=T_{pi}/T_{wi}\ i=1,2)\ ;\ h_1(0,t)\equiv 1\,,\\
&h_1(1,t)-\tfrac{1}{2}R_1\tfrac{T_{w1}}{T_1}q_1(1,t)|q_1(1,t)|=1+z(t)+R_s\tfrac{\mathrm{d}z}{\mathrm{d}t}\\
&\qquad=h_2(0,t)+\tfrac{1}{2}R_2\tfrac{T_{w2}}{T_2}q_2(0,t)|q_2(0,t)|\,,\\
&T_s\tfrac{\mathrm{d}z}{\mathrm{d}t}=q_1(1,t)-q_2(0,t)\ ,\ q_2(1,t)=f_\theta\sqrt{h_2(1,t)}\,,\\
&T_a\tfrac{\mathrm{d}\varphi}{\mathrm{d}t}=q_2(1,t)h_2(1,t)-\nu_g\,.
\end{aligned}
\tag{2.1}
$$

The model contains rated state variables: the piezometric heads h_i $(i=1,2)$ are rated to the lake head H_0; the water flows q_i are rated to $\bar{Q}=\alpha_q F_{\theta\,\max}\sqrt{H_0}$ - the maximal available flow at the turbine wicket gates; here $F_{\theta\,\max}$ is the maximal cross section area of the wicket gates and α_q - a flow coefficient; the rotating speed of the turbine is rated to the synchronous speed Ω_c and the available mechanical power to a resulting nominal power. The various time constants T_i, T_{wi}, T_{pi}, T_s, T_a are a result of the state variables rating (scaling) and they define (2.1) as a system with several time scales. The terms in λ_i define the so called Darcy-Weisbach hydraulic losses which are distributed along the water conduits. The terms in R_i define local hydraulic losses and the term in R_s defines the dynamic hydraulic losses due to the throttling of the surge tank. It has to be mentioned that the space coordinates along the water conduits are also rated to the lengths of the conduits $\xi_i = x_i/L_i$.

This basic model is rather complete and allows obtaining other models *via* various simplifying assumptions, which in most cases arise either from neglecting small terms or from engineering inferences suggested by the practical experience. In what follows we shall discuss the model adopted in [5], obtained from (2.1) by taking into account the following assumptions: i) the space variations of the dynamic heads $(1/2)(T_{wi}/T_i)q_i^2$ are negligible in comparison to the variations of the piezometric heads mainly during water hammer: according to [2], the variation of the piezometric head can reach several dozens of meters, while the variation of the dynamic head is at most 1 meter; this assertion is documented in [2] with exploitation data from hundreds of hydraulic power plants of the former USSR; ii) all distributed and local hydraulic losses are neglected, except the losses due to the throttling of the surge tank. Under these assumptions (2.1) become

$$
\begin{aligned}
&\partial_{\xi_i} h_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t h_i + \partial_{\xi_i} q_i = 0\,,\\
&h_1(0,t) \equiv 1\ ;\ h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t)\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = f_\theta\sqrt{h_2(1,t)}\,,\\
&T_a\frac{\mathrm{d}\varphi}{\mathrm{d}t} = q_2(1,t)h_2(1,t) - \nu_g\,.
\end{aligned}
\tag{2.2}
$$

This model is considered under water hammer: the water hammer is an abnormal regime generated by sudden load discharge at the hydraulic turbine level. Here, following [5], we shall consider the total turbine shutdown by complete instantaneous closing of the turbine wicket gates: $f_\theta \equiv 0$. Consequently the boundary condition of the penstock at $\xi_2 = 1$ becomes $q_2(1,t) \equiv 0$ and the turbine equation is “cut” (decoupled) from the rest of the model. The model is thus completely linear and represented by a non-standard BVP. We call it non-standard since the boundary conditions are coupled to an ODE and this ODE at its turn is controlled by the boundary conditions. It thus appears some kind of internal feedback which can either stabilize or destabilize the dynamic process of the water hammer.

3. The associated system of functional differential equations for stability analysis

We shall start from the model resulting from (2.2) and the condition $q_2(1,t) \equiv 0$

$$
\begin{aligned}
&\partial_{\xi_i} h_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t h_i + \partial_{\xi_i} q_i = 0\,,\\
&h_1(0,t) \equiv 1\ ;\ h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t)\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = 0\,,
\end{aligned}
\tag{3.1}
$$

and compute firstly its steady state by letting the time derivatives to 0

$$
\begin{aligned}
&\bar{h}_i(\xi_i) \equiv \text{const}\ ;\ \bar{h}_1(0) = \bar{h}_1(1) = \bar{h}_2(0) = 1 + \bar{z} = 1\,,\\
&\bar{q}_i(\xi_1) \equiv \text{const}\ ;\ \bar{q}_1(1) = \bar{q}_2(0) = \bar{q}_2(1) = 0\,,
\end{aligned}
$$

thus obtaining $\bar{h}_i = 1$, $\bar{z} = 0$, $\bar{q}_i = 0$; introduce the deviations $\chi_i(\xi_i, t) := h_i(\xi_i, t) - 1$; the variables $q_i(\xi_i, t)$, $z(t)$ obviously coincide with their deviations. The system in deviations reads

$$\begin{aligned}
&\partial_{\xi_i}\chi_i + T_{wi}\partial_t q_i = 0\ ,\ \delta_i^2 T_{wi}\partial_t\chi_i + \partial_{\xi_i} q_i = 0\,,\\
&\chi_1(0,t) \equiv 0\ ;\ h_1(1,t) = z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = \chi_2(0,t)\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t)\ ,\ q_2(1,t) = 0\,.
\end{aligned} \tag{3.2}$$

To this system we associate the energy identities

$$\frac{1}{2}T_{wi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [q_i^2(\xi_i,t) + \delta_i^2\chi_i^2(\xi_i,t)]\mathrm{d}\xi_i + q_i(\xi_i,t)\chi_i(\xi_i,t)|_0^1 \equiv 0\,, \tag{3.3}$$

and the Riemann invariants (the forward and backward waves)

$$r_i^{\pm} = \frac{1}{2}(\delta_i\chi_i \pm q_i)\ \Leftrightarrow\ q_i = r_i^+ - r_i^-\ ,\ \chi_i = \frac{1}{\delta_i}(r_i^+ + r_i^-)\,. \tag{3.4}$$

Rewrite (3.2) in the Riemann invariants as follows

$$\begin{aligned}
&\delta_i T_{wi}\partial_t r_i^{\pm} \pm \partial_{\xi_i} r_i^{\pm} = 0\ ;\ r_1^+(0,t) + r_1^-(0,t) \equiv 0\,,\\
&\frac{1}{\delta_1}(r_1^+(1,t) + r_1^-(1,t)) = z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{\delta_2}(r_2^+(0,t) + r_2^-(0,t))\,,\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = r_1^+(1,t) - r_1^-(1,t) - r_2^+(0,t) + r_2^-(0,t)\,,\\
&r_2^+(1,t) - r_2^-(1,t) \equiv 0\,.
\end{aligned} \tag{3.5}$$

From now on we follow the methodology of [3, 9]. Consider the two characteristic lines crossing some point (ξ_i, t) of the half plane $\{\xi_i, t | 0 \leq \xi_i \leq 1, t > 0\}$

$$\tau_i^{\pm}(\sigma; \xi_i, t) = t \pm \delta_i T_{wi}(\sigma - \xi_i)\ ,\ i = 1, 2\,. \tag{3.6}$$

Since the Riemann invariants are constant along the characteristics (r_i^+ along τ_i^+ and r_i^- along τ_i^-), the following representation formulae are deduced

$$\begin{aligned}
r_i^+(\xi_i,t) &= r_i^+(1, t + \delta_i T_{wi}(1 - \xi_i))\,,\\
r_i^-(\xi_i,t) &= r_i^-(0, t + \delta_i T_{wi}\xi_i)\,.
\end{aligned} \tag{3.7}$$

Let consider firstly those characteristics which can be extended on the entire interval $0 \leq \sigma \leq 1$. Defining $y_i^+(t) := r_i^+(1,t)$, $y_i^-(t) := r_i^-(0,t)$, we obtain

$$\begin{aligned}
r_i^+(0,t) &= r_i^+(1, t + \delta_i T_{wi}) = y_i^+(t + \delta_i T_{wi})\,,\\
r_i^-(1,t) &= r_i^-(0, t + \delta_i T_{wi}) = y_i^-(t + \delta_i T_{wi})\,.
\end{aligned} \tag{3.8}$$

These values are substituted in (3.5); we introduce further $w_i^{\pm}(t) := y_i^{\pm}(t + \delta_i T_{wi})$ in order to obtain the more "conventional" way of writing equations with deviated argument. Next, recall that stability studies are made for large $t > 0$; it is thus sufficient to take $t > \max\{\delta_1 T_{w1}, \delta_2 T_{w2}\}$ and eliminate the variables $w_1^+(t)$ and

$w_2^-(t)$. After an additional transformation requiring inversion of a 2×2 non-singular matrix, making also the following notations [5]

$$\begin{aligned}&\rho_1 := \frac{1+(\delta_2-\delta_1)R_s'}{1+(\delta_1+\delta_2)R_s'}\ ,\ \rho_2 := \frac{1+(\delta_1-\delta_2)R_s'}{1+(\delta_1+\delta_2)R_s'}\ ,\\ &R_s' := R_s/T_s\ ;\ \vartheta := 2\delta_2 T_{w2}\ ,\ \nu\vartheta := 2\delta_1 T_{w1}(\nu = (\delta_1 T_{w1})/(\delta_2 T_{w2}))\ ,\end{aligned}$$

the following system of coupled delay differential and difference equations is obtained

$$\begin{aligned}&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = (\rho_1+\rho_2)[-\frac{\delta_1+\delta_2}{2}z(t) - w_1^-(t-\nu\vartheta) + w_2^+(t-\vartheta)]\ ,\\ &w_1^-(t) = \frac{(\rho_1+\rho_2)\delta_1}{2}z(t) + \rho_1 w_1^-(t-\nu\vartheta) + (1-\rho_1)w_2^+(t-\vartheta)\ ,\\ &w_2^+(t) = \frac{(\rho_1+\rho_2)\delta_2}{2}z(t) - (1-\rho_2)w_1^-(t-\nu\vartheta) - \rho_2 w_2^+(t-\vartheta)\ .\end{aligned} \tag{3.9}$$

The solutions of (3.9) can be constructed by steps provided initial conditions are given. For details the reader is sent to [10], Section 4. The one-to-one correspondence between the solutions of (3.9) and those of (3.5) is given by Theorem 4.1 of [10]; at its turn this correspondence strongly relies on (3.8) and on the representation formulae (3.7) re-written using the functions $w_i^\pm(t)$ as follows

$$\begin{aligned}&r_i^+(\xi_i,t) = y_i^+(t+\delta_i T_{wi}(1-\xi_i)) = w_i^+(t-\delta_i T_{wi}\xi_i)\ ,\\ &r_i^-(\xi_i,t) = y_i^-(t+\delta_i T_{wi}\xi_i) = w_i^-(t+\delta_i T_{wi}(\xi_i-1))\ .\end{aligned} \tag{3.10}$$

Summarizing, the mathematical result reads as follows

Theorem 3.1. *Consider the system of the Riemann invariants* (3.5) *with the initial conditions* $\{z(0), r_{io}^\pm(\xi_i)\}$, *where*

$$r_{io}^\pm(\xi_i) = \frac{1}{2}(\delta_i\chi_i^o(\xi_i) \pm q_i^o(\xi_i))\ ,\ 0\leq \xi_i\leq 1\ . \tag{3.11}$$

If $\{z(t), r_i^\pm(\xi_i,t)\}$ *is a classical solution of* (3.5), *then* $\{z(t), w_i^\pm(t)\}$ *is a piecewise continuous solution of* (3.9) *with the initial conditions defined by* $\{z(0), w_{io}^\pm(\theta)\}$, *where*

$$w_{io}^+(\theta) = r_{io}^+(-\theta/(\delta_i T_{wi}))\ ,\ w_{io}^-(\theta) = r_{io}^-(1+\theta/(\delta_i T_{wi}))\ ;\ -\delta_i T_{wi}\leq\theta\leq 0\ . \tag{3.12}$$

Conversely, let $\{z(t), w_i^\pm(t)\}$ *be a solution of* (3.9) *with the initial conditions* $\{z(0), w_{io}^\pm(\theta)\}$. *Then* $\{z(t), r_i^\pm(\xi_i,t)\}$, *where* $r_i^\pm(\xi_i,t)$ *are given by* (3.7), *is a (possibly discontinuous) classical solution of* (3.5) *with the initial conditions* $r_{io}^\pm(\xi_i)$ *obtained by letting* $t=0$ *in* (3.7).

4. The Lyapunov functional and the stability analysis

We refer firstly to system (3.2) and to the energy identities (3.3). The energy identities suggest the following Lyapunov functional

$$\mathcal{V}\big(z,\phi_i(\cdot),\psi_i(\cdot)\big) = \frac{1}{2}\left\{T_s z^2 + \sum_1^2 T_{wi}\int_0^1 [\phi_i^2(\xi_i) + \delta_i^2\psi_i^2(\xi_i)]\,\mathrm{d}\xi_i\right\}, \tag{4.1}$$

written as a quadratic functional on the state space $\mathbb{R} \times \mathcal{L}^2(0,1;\mathbb{R}^4)$. We write down (4.1) along the solutions of (3.2), differentiate it with respect to t and take into account the energy identities and the boundary conditions in (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^\star(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\big(z(t), q_i(\cdot,t), \chi_i(\cdot,t)\big) = -R_s T_s \left(\frac{\mathrm{d}z}{\mathrm{d}t}(t)\right)^2 \leq 0\,. \tag{4.2}$$

Inequality (4.2) gives the Lyapunov stability of the zero solution of (3.2) *in the sense of the metrics induced by the Lyapunv functional itself*

$$\mathcal{V}\big(z(t), q_i(\cdot,t), \chi_i(\cdot,t)\big) \leq \mathcal{V}\big(z_0, q_i^{\mathrm{o}}(\cdot), \chi_i^{\mathrm{o}}(\cdot)\big)\,. \tag{4.3}$$

Inequality (4.2) also shows that asymptotic stability might be obtained *via* the invariance principle of Barbashin-Krasovskii-LaSalle. For this we shall turn to system (3.9). Using the representation formulae (3.10), also (3.4), the Lyapunov functional of (4.2) becomes, after some simple manipulation and with a slight abuse of notation

$$\begin{aligned}\mathcal{V}\big(z(t), w_1^-(t+\cdot), w_2^+(t+\cdot)\big) &= \frac{1}{2}T_s z^2(t) + \frac{1}{\delta_1}\int_{-\nu\vartheta}^{\mathrm{o}} w_1^-(t+\lambda)^2 \mathrm{d}\lambda \\ &+\frac{1}{\delta_2}\int_{-\vartheta}^{\mathrm{o}} w_2^+(t+\lambda)^2 \mathrm{d}\lambda\,,\end{aligned} \tag{4.4}$$

the derivative of $\mathcal{V}$ remaining unchanged. This derivative vanishes for $\mathrm{d}z/\mathrm{d}t = 0$, that is on the set where

$$-(\delta_1+\delta_2)z(t) - 2w_1^-(t-\nu\vartheta) + 2w_2^+(t-\vartheta) = 0. \tag{4.5}$$

On this set the difference subsystem of (3.9) takes the form, after substituting $z(t)$ from (4.5))

$$\begin{aligned} w_1^-(t) &= \frac{1}{\delta_1+\delta_2}[(\delta_2-\delta_1)w_1^-(t-\nu\vartheta) + 2\delta_1 w_2^+(t-\vartheta)], \\ w_2^+(t) &= \frac{1}{\delta_1+\delta_2}[-2\delta_2 w_1^-(t-\nu\vartheta) + (\delta_2-\delta_1)w_2^+(t-\vartheta)]. \end{aligned} \tag{4.6}$$

The invariant set of (4.5) is composed of the only constant solution $\{0,0\}$ and $\bar{z} = 0$. *The only invariant set included in the set where the derivative of the Lyapunov functional vanishes is the zero solution.* The theorem of Barbashin-Krasovskii-LaSalle for system (3.9) – Theorem 9.8.2 of [7] – would give asymptotic stability and, therefore, asymptotic stability for (3.5) and, *via* (3.4), for system (3.2). However, there is a certain aspect to be taken into account: in the case of NFDE (and system (3.9) *is* neutral - see [7, Section 9, p.301]) the invariance principle is proven under the assumption that the difference operator $\mathcal{D}$ is asymptotically stable. This is not quite true for (3.9). If the difference subsystem of (3.9) is considered, its asymptotic stability is equivalent to the location of the roots of the characteristic equation

$$(1-\rho_1 \mathrm{e}^{-\lambda\nu\vartheta})(1+\rho_2\mathrm{e}^{-\lambda\vartheta}) + (1-\rho_1)(1-\rho_2)\mathrm{e}^{-\lambda(\nu+1)\vartheta} = 0 \tag{4.7}$$

in $\mathbb{C}^-$. Since the two delays are, generally speaking, rationally independent (ν is a real number), (4.7) ought have its roots with $\Re e(\lambda) \leq -\alpha < 0$ for some $\alpha > 0$. Denoting $\mu := \mathrm{e}^{\lambda\vartheta}$, the condition above reduces to the location of the roots of

$$(\mu^\nu - \rho_1)(\mu+\rho_2) + (1-\rho_1)(1-\rho_2) = 0 \tag{4.8}$$

inside the unit disk $\mathbb{D}_1 \subset \mathbb{C}$. As it was rigorously and completely proven in [11], this condition is fulfilled *only for* ν *rational with both numerator and denominator – odd numbers*. For ν rational with even numerator and odd denominator, $\mu = -1$ is a simple root of (4.8). Moreover, since the spectral radius of the difference operator equals 1, *there is no irrational* ν *such that* (4.8) *should have its roots inside* $\mathbb{D}_1 \subset \mathbb{C}$ - see [7]. We call this kind of asymptotic stability *fragile* since it holds for a countable set of rational ratios ν of the propagation delays.

Summarizing, the mathematical result is as follows

Theorem 4.1. *Consider the system* (3.2) *with the associated Lyapunov functional* (4.1)*, together with systems* (3.9)*,* (3.10) *and with the rewritten Lyapunov functional* (4.4)*. Systems* (3.2) *and* (3.9)*,* (3.10) *are stable in the sense of Lyapunov with respect to the metrics defined by their associated Lyapunov functionals. If the delay ratio* $\nu = (\delta_1 T_{w1})(\delta_2 T_{w2})^{-1}$ *is rational with both numerator and denominator - odd numbers, then this stability is also asymptotic.*

5. An even more critical system

We mention here another system arising from hydraulics, describing a hydroelectric plant supplied through two independent tunnels starting from the same reservoir (lake), endowed with surge tank, under water hammer [4]. Under lumped parameters i.e. described by ODE and under the same description as our previous structure (all losses neglected except the surge tank throttling):

$$T_{wi}\frac{\mathrm{d}q_i}{\mathrm{d}t} + z + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \ (i = 1, 2)\ ,\ T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1 + q_2\ , \tag{5.1}$$

having an invariant set defined by

$$T_{w1}q_1(t) - T_{w2}q_2(t) \equiv T_{w1}q_1(0) - T_{w2}q_2(0)\,. \tag{5.2}$$

The steady state is uniquely determined on the invariant set only. In the case of the distributed parameters (PDE description) nothing is known about the invariant set while the steady state is not uniquely determined. Other considerations on this model can be seen in [10].

6. Some conclusions

It was mentioned in the Introduction that the outcome of the paper is twofold. The first outcome refers to the mathematical aspects. Here (and not only) there are displayed applications for which the difference operator associated to the NFDE is only critically stable. The assumption on the asymptotic, even the strong stability (i.e. stability with respect to the delays) turned to be very fruitful (productive) in the sense that it allowed an immediate extension to NFDE of the results of the stability theory obtained for the R(etarded)FDE. The price to be paid was that several papers dealing with the aforementioned critical cases were obscured and forgotten. We stress that returning to their results might be useful (their list is given in [10]. Another approach to be taken within the mathematical studies would be the one suggested in [12], page 341. It is specified there that the assumption on the asymptotic stability for the difference operator is necessary to obtain the compactness of the positive orbits whenever the solution is bounded.It is then suggested to embed the resulting semi-dynamical system in a space wherein the positive orbits are pre-compact. To illustrate this approach the reader is sent to an application in Chapter V, Section 4, page 252: the application there is a BVP for a hyperbolic PDE. With the one-to-one correspondence between the solutions of the BVP for the hyperbolic PDE and those of the associated system of NFDE, the problem becomes one of choosing the state space for the NFDE - other than $\mathcal{C}$ [6]. A good reference for the role of the pre-compactness is [8]. On the other hand, the aforementioned models of hydraulics are *strongly idealized* by neglecting almost all static energy dissipation. Re-introducing some of them means changing the model and restarting the entire analysis. *Too much idealization can turn harmful!.*

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