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UNIQUE SOLVABILITY OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION ON THE BASIS OF VALLÉE-POUSSIN THEOREM

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Abstract. We propose explicit tests of unique solvability of two-point and focal boundary value problems for fractional functional differential equations with Riemann-Liouville derivative.

1. Introduction

In this paper we consider the fractional functional differential equation

$$(D_{0+}^{\alpha}x)(t)+\sum_{i=0}^{m}(T_i x^{(i)})(t)=f(t)\,,\quad t\in[0,1],\ m\le n-2,\ n\ge 2\,, \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of the order $n-1<\alpha\le n$ (see [11], [14]), n is integer, the operators $T_i\colon C\to L_\infty$ are linear continuous operators acting from the space of the continuous functions C to the space of essentially bounded functions L_∞, $i=0,\dots,m$, and $f\in L_\infty$.

We consider also the auxiliary equation

$$(D_{0+}^{\alpha}x)(t)+\sum_{i=0}^{m}(|T_i|x^{(i)})(t)=f(t)\,,\quad t\in[0,1],\ m\le n-2,\ n\ge 2\,, \tag{1.2}$$

where the positive operator $|T_i|$ is such that the following inequalities hold:

$$-(|T_i|1)(t)\le (T_i 1)(t)\le (|T_i|1)(t)\,,\qquad t\in[0,1]\,. \tag{1.3}$$

Of course, it will be clear below, that we are interested in the operators $|T_i|$ with the minimal norms in the space of continuous functions C.

The operators $T_i\colon C\to L_\infty$ and $|T_i|\colon C\to L_\infty$ can be, for example, of the following forms:

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1) Operators with deviations

$$(T_i x^{(i)})(t) = \sum_{j=0}^{m_i} q_{ij}(t) x^{(i)}\big(t - \tau_{ij}(t)\big)\,, \tag{1.4}$$

$$(|T_i| x^{(i)})(t) = \sum_{j=0}^{m_i} |q_{ij}(t)| x^{(i)}\big(t - \tau_{ij}(t)\big)\,,$$

where $\tau_{ij}\colon [0,1] \to \mathbb{R}$, $q_{ij}\colon [0,1] \to \mathbb{R}$, are measurable bounded functions, $\mathbb{R} = (-\infty, +\infty)$. To complete the description of these operators, we have to define what has to be substituted into (1.4) instead of $x^{(i)}(t - \tau_{ij}(t))$ in the case of $t - \tau_{ij}(t) \notin [0,1]$. Let us assume that

$$x^{(i)}(\xi) = 0 \quad \text{for} \quad \xi \notin [0,1],\ i = 0, \dots, m\,, \tag{1.5}$$

that allows us to preserve the n-dimensional fundamental system for the homogeneous equation

$$(D^{\alpha}_{0+} x)(t) + \sum_{j=0}^{m_i} q_{ij}(t) x^{(i)}(t - \tau_{ij}(t)) = 0\,. \tag{1.6}$$

2) Integral operators

$$(T_i x^{(i)})(t) = \int_0^1 K_i(t,s) x^{(i)}(s)\, ds\,, \tag{1.7}$$

$$(|T_i| x^{(i)})(t) = \int_0^1 |K_i(t,s)| x^{(i)}(s)\, ds\,,$$

under the standard assumptions on the kernels $K_i(t,s)$ implementing that $T_i\colon C \to L_\infty$, for example, $K_i(t,s)$ is a continuous function $[0,1] \times [0,1] \to \mathbb{R}$ (see, [12]).

3) Linear combinations and superpositions of the deviations and integral operators, for example, the operators

$$(T_i x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} K_{ij}(t,s) x^{(i)}\big(s - \tau_{ij}(s)\big)\, ds\,. \tag{1.8}$$

$$(|T_i| x^{(i)})(t) = \int_0^1 \sum_{j=1}^{m_i} |K_{ij}(t,s)| x^{(i)}\big(s - \tau_{ij}(s)\big)\, ds\,.$$

We consider the boundary value problem consisting of equation (1.1) and the boundary conditions

$$x^{(i)}(0) = 0 \quad \text{for} \quad i = 0, 1, \dots, n-2,\ x^{(k)}(1) = 0\,, \tag{1.9}$$

where k is an integer which is between 0 and $n-1$. In the case of $k=0$, we have the classical two-point $(n-1,1)$- problem. In the case of $k \leq n-1$, we have the sort of focal problems. We assume below that $m \leq k$.

We consider equation (1.1) in the space D of functions $x\colon [0,1] \to \mathbb{R}$ such that $x^{(n-1)}$ is absolutely continuous on every interval $[\varepsilon, 1]$, where $\varepsilon > 0$ and summable on $[0,1]$ and $x^{(n)}$ such that $tx^{(n)}$ is summable. The norm in the space D define as $\|x\|_D = \sum\limits_{i=0}^{n-2} \max\limits_{0 \leq t \leq 1} \left|x^{(i)}(t)\right| + \int\limits_0^1 \left|x^{(n-1)}(t)\right| dt + \int\limits_0^1 t \left|x^{(n)}(t)\right| dt$. Considering this space D looks naturally when fractional equations with the Riemann-Liouville derivatives and the boundary conditions (1.9) are considered. We say that $x \in D$ is a solution of (1.1) if it satisfies this equation for almost every $t \in [0,1]$. If the problem consisting of the homogeneous equation $(D_{0+}^{\alpha}x)(t) + \sum\limits_{i=0}^{m} (T_i x^{(i)})(t) = 0$ and condition (1.9) has only the trivial solution, then problem (1.1), (1.9) has a unique solution which can be represented in the form [2]

$$x(t) = \int_0^1 G(t,s) f(s) ds \,. \tag{1.10}$$

For applications of fractional differential equations in various field of science and engineering one can refer the classical books [11, 14].

The main reason for the study of fractional functional differential equations could be, in our opinion, around the following idea for the study of systems of fractional equations. Consider a boundary value problem consisting, for example, of a system of two "ordinary fractional differential equations". For its analysis, we can use the integral representations of solutions of the first equation and obtain $x_1(t)$ through $x_2(t)$. Then we substitute this representation instead of $x_1(t)$ into the second equation and obtain a scalar fractional functional differential equation. In the simplest case of a system of "ordinary" fractional equations, the equation, we get, includes the integral operator of type 2). If we start with a system of delay fractional differential equations, the equation, we get after the substitution into the second equation, is a fractional functional differential equation that includes the superpositions of deviation and integral operators. Thus, operators of type 3) appear. Examples of such systems can be found in [7, 8, 9].

Positivity of solutions is one of the most important properties in applications (see, for example, the book by Henderson and Luca [7]). Concerning problem (1.4),(1.9), in the case of so called ordinary linear equations, (i.e. $\tau_{ij}(t) \equiv 0$, $t \in [0,1]$, $j = 0, \ldots, m_i$, $i = 1, \ldots, m$ in (1.4)) and its nonlinear generalizations, we can note the following papers [3, 8, 9, 10, 13, 15].

One of the motivations for our research is Lyapunov's inequalities for fractional differential equations which have been presented in Chapter 5 of the recent book by Agarwal, Bohner, and Ozbekler [1]. Note the following assertion was presented for the first time in [5]. Actually, the result in [5] is more general than Theorem 1.1 as the solution need not be assumed to be different from zero on $(0,1)$.

α	In inequality (1.13)	In inequality (1.15)
1.6	2.052759111	4.120246548
1.5999	2.05244883	4.119533208
1.5998	2.052138367	4.11819636
1.597	2.043474592	4.098884212
1.58	1.991943084	3.97506386
1.5	1.7724538	3.45372767

Tab. 1

Theorem 1.1 ([1, 5]). *Let $1 < \alpha \leq 2$ and x be a solution of the boundary value problem*

$$\begin{cases} (D^{\alpha}_{0+}x)(t) + q_0(t)x(t) = 0 \quad \text{on} \quad [0,1]\,, \\ x(0) = x(1) = 0\,. \end{cases} \tag{1.11}$$

If $x(t) \neq 0$ for all $t \in (0,1)$, then the inequality

$$\int_0^1 |q_0(t)|\,dt > \Gamma(\alpha)4^{\alpha-1} \tag{1.12}$$

holds.

Note that in [5], it was not assumed that $x(t) \neq 0$ for $t \in (0,1)$. For (1.11) with a constant coefficient $q_0(t) = q_0$, we have (1.12) in the form

$$|q_0| \geq \Gamma(\alpha)4^{\alpha-1}\,. \tag{1.13}$$

Using Corollary 2.3 (one can refer [4] for proof), we get that the inequality

$$|q_0| < \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}}\Gamma(\alpha+1) \tag{1.14}$$

guarantees that the problem (1.11) has only the trivial solution. Note that the part on unique solvability coincides with the known result of [6]. Inequality (1.14) means that in the case of zeros of solution $x(t)$ at the points 0 and 1, we obtain that

$$|q_0(t)| \geq \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}}\Gamma(\alpha+1) \tag{1.15}$$

since in the case of the coefficient q_0 satisfying inequality (1.11) we exclude the existence of zero at the point 1, i.e. $x(1) \neq 0$. Let us compare (1.13) and (1.15), computing the right-hand sides in them, we have values in Table 1.

Table 1 demonstrates the advances of our results if we compare the results of [1, 5] and ours.

2. Main Results

Lemma 2.1. *Using the technique of* [13], *one can obtain the uniqueness of solution to the problem*

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t)\,, \\ x(0) = x'(0) = \ldots = x^{(n-2)}(0) = 0\,, \\ x^{(k)}(1) = 0\,, \end{cases} \tag{2.1}$$

where k is an integer number which is between 0 and $n-1$, in the form

$$x(t) = \int_0^1 G_k(t,s) f(s)\, ds\,, \tag{2.2}$$

where $G_k(t,s)$ is Green's function of problem (2.1) *defined by*

$$G_k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1-k}\,, & 0 \le s \le t \le 1\,, \\ -t^{\alpha-1}(1-s)^{\alpha-1-k}\,, & 0 \le t < s \le 1 \end{cases} \tag{2.3}$$

and its j-th derivative is defined by

$$\frac{\partial^j}{\partial t^j} G_k(t,s) = \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-j)}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-j-1} - t^{\alpha-j-1}(1-s)^{\alpha-1-k}\,, & 0 \le s \le t \le 1\,, \\ -t^{\alpha-j-1}(1-s)^{\alpha-1-k}\,, & 0 \le t < s \le 1\,. \end{cases} \tag{2.4}$$

Let us define the operator $K\colon L_\infty \to L_\infty$ and $|K|\colon L_\infty \to L_\infty$ by the equalities

$$(Kz)(t) = -\sum_{i=0}^{m} T_i \left[\int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s) z(s)\, ds \right](t) = f(t)\,, \tag{2.5}$$

$$(|K|z)(t) = -\sum_{i=0}^{m} |T_i| \left[\int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s) z(s)\, ds \right](t) = f(t)\,.$$

We use the notation $T_i[\gamma(t)]$, $(|T_i|[\gamma(t)])$ meaning that the operator T_i and $|T_i|$ acts on the continuous function $\gamma(t)$, i.e. $T_i[\gamma(t)] = (T_i\gamma)(t)$, $|T_i|[\gamma(t)] = (|T_i|\gamma)(t)$.

Theorem 2.2. *Assume that there exist a function $v \in D$ such that $v(t) > 0$, $v'(t) > 0$, $\cdots$, $v^{(k)}(t) > 0$ for $t \in (0,1)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0$ and*

$$(D_{0+}^{\alpha} v)(t) + \sum_{i=0}^{m} (|T_i| v^{(i)})(t) \equiv \psi(t) \le -\varepsilon < 0 \quad \textit{for} \quad t \in (0,1)\,; \tag{2.6}$$

then the problem (1.1), (1.9) *is uniquely solvable for any essentially bounded f and the spectral radius of $|K|\colon L_\infty \to L_\infty$ is less than one.*

Proof. Consider the auxiliary problem

$$\begin{cases} (D_{0+}^{\alpha} x)(t) = z(t)\,, \\ x^{(i)}(0) = v^{(i)}(0),\ x^{(k)}(1) = v^{(k)}(1)\,, \quad i = 0,1,\ldots,n-2\,, \end{cases} \tag{2.7}$$

where $z(t)$ is a function in L_∞ and such that there exists a positive number δ such that $z(t) \le -\delta$ for $t \in [0,1]$. It is clear that

$$
\begin{cases}
x(t) = \int_0^1 G_k(t,s)z(s)\,ds + u_k(t)\,,\\
x'(t) = \int_0^1 \frac{\partial}{\partial t} G_k(t,s)z(s)\,ds + u_k'(t)\,,\\
x''(t) = \int_0^1 \frac{\partial^2}{\partial t^2} G_k(t,s)z(s)\,ds + u_k''(t)\,,\\
\vdots\\
x^{(m)}(t) = \int_0^1 \frac{\partial^m}{\partial t^m} G_k(t,s)z(s)\,ds + u_k^{(m)}(t)\,,
\end{cases}
\tag{2.8}
$$

where $u(t)$ is a solution of the homogeneous equation $D^\alpha_{0+}u(t) = 0$ satisfying the conditions $u^{(i)}(0) = v^{(i)}(0)$, $i = 0, \dots, n-2$, $u^{(k)}(1) = v^{(k)}(1)$. Let us substitute these representations instead of $v(t)$ and its derivatives into inequality (2.6):

$$
z(t) + \sum_{i=0}^m T_i \left[\int_0^1 \frac{\partial^i}{\partial t^i} G_k(t,s)z(s)\,ds \right] + \sum_{i=0}^m (T_i u^i)(t)) = \psi(t)\,. \tag{2.9}
$$

It is clear that $|T_i| \colon C \to L_\infty$ are positive operators for $i = 0, 1, \dots, m$, and this imply that the operator $|K| \colon L_\infty \to L_\infty$ defined by equality (2.5) is positive. Thus, we have the equation

$$
z(t) - (|K|z)(t) = \Psi(t)\,, \quad t \in [0,1]\,, \tag{2.10}
$$

where

$$
\Psi(t) \equiv \psi(t) - \sum_{i=0}^m (|T_i| u^{(i)})(t)\,. \tag{2.11}
$$

It is clear that $u^{(i)}(t) > 0$ for $t \in (0,1]$. This implies that $\Psi(t) \le -\varepsilon < 0$. The function $w(t) = -z(t)$ satisfies the inequality $w(t) - (|K|w)(t) = -\Psi(t) > 0$ for $t \in [0,1]$. From equality (2.10), according to [12, Theorem 5.3 on page 76] it follows that $\rho(|K|) < 1$. This completes the proof of the theorem. □

Corollary 2.3. *If* $n-1 < \alpha \le n$ *and the following inequality is fulfilled*

$$
\begin{aligned}
&|T_0| \left[t^{\alpha-1} \left(\frac{\alpha}{\alpha - k} - t \right) \right] \\
&+ \sum_{i=1}^m \alpha(\alpha-1)\cdots(\alpha-i+1)|T_i| \left[t^{\alpha-i-1} \left(\frac{\alpha - i}{\alpha - k} - t \right) \right] < \Gamma(\alpha+1),\ t \in [0,1]\,,
\end{aligned}
\tag{2.12}
$$

then problem (1.1), (1.9) *is uniquely solvable for any* $f \in L_\infty$.

Proof. The proof follows from Corollary 4 of [4]. □

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